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Positive solutions for a class of generalized quasilinear Schrödinger equations involving concave and convex nonlinearities in Orlicz space

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Abstract. In this paper, we study the following generalized quasilinear Schrödinger equation

$$-\mathrm{div}(g^2(u)\nabla u) + g(u)g'(u)|\nabla u|^2 + V(x)u = \lambda f(x,u) + h(x,u), \qquad x \in \mathbb{R}^N,$$

where $\lambda > 0$, $N \geq 3$, $g \in \mathcal{C}^1(\mathbb{R}, \mathbb{R}^+)$. By using a change of variable, we obtain the existence of positive solutions for this problem with concave and convex nonlinearities via the Mountain Pass Theorem. Our results generalize some existing results.

Keywords: generalized quasilinear Schrödinger equation, positive solutions, concave and convex nonlinearities.

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1 Introduction

In this paper, we are concerned with a class of generalized quasilinear Schrödinger equation

$$-\mathrm{div}(g^2(u)\nabla u) + g(u)g'(u)|\nabla u|^2 + V(x)u = \lambda f(x,u) + h(x,u), \tag{1.1}$$

where $\lambda > 0$, $N \geq 3$, $2^* = \frac{2N}{N-2}$, and g satisfies:

(g) $g \in \mathcal{C}^1(\mathbb{R}, (0, +\infty))$ is even with $g'(t) \geq 0$, for all $t \in [0, +\infty)$, $g(0) = 1$ and satisfies

$$g_\infty := \lim_{t\to\infty} \frac{g(t)}{t} \in (0, \infty), \tag{1.2}$$

and

$$\beta := \sup_{t\in\mathbb{R}} \frac{tg'(t)}{g(t)} \leq 1. \tag{1.3}$$

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Mathematically, it is also a hot issue in nonlinear analysis to study the existence of solitary wave solutions for the following quasi-linear Schrödinger equation

$$i\partial_t z = -\Delta z + W(x)z - k(x, |z|) - \Delta l(|z|^2)l'(|z|^2)z, \tag{1.4}$$

where $z : \mathbb{R} \times \mathbb{R}^N \to \mathbb{C}$, $W : \mathbb{R}^N \to \mathbb{R}$ is a given potential, $l : \mathbb{R} \to \mathbb{R}$ and $k : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ are suitable functions.

The quasilinear equation of the form (1.4) appears naturally in mathematical physics and has been derived as models of several physical phenomena corresponding to various types of nonlinear terms l. Kurihara [23] considered the case where $l(s) = s$ in (1.4), and this kind of equation was used for the superfluid film [23,24] equation in fluid mechanics. [29–31] studied the equation which corresponds to the case $l(t) = t^\alpha$ for some $\alpha \geq 1$. For more details see [2–4,21,25,32,34,38] and references therein. Moreover, many conclusions about the equation (1.4) with $l(s) = 1$ have been studied, see [35–37] and the references therein.

Cuccagna [11] was interested in the existence of standing wave solutions, that is, solutions of type $z(t,x) = \exp(-iEt)u(x)$, where $E \in \mathbb{R}$ and u is a real function. It is well known that z satisfies (1.4) if and only if the function $u(x)$ solves the following equation of elliptic type with the formal variational structure

$$-\Delta u + V(x)u - \Delta l(u^2)l'(u^2)u = a(x,u), \qquad x \in \mathbb{R}^N. \tag{1.5}$$

If we take

$$g^2(u) = 1 + \frac{[(l^2(u))']^2}{2},$$

then (1.5) turns into quasilinear elliptic equations (see [39])

$$-\operatorname{div}(g^2(u)\nabla u) + g(u)g'(u)|\nabla u|^2 + V(x)u = a(x,u), \qquad x \in \mathbb{R}^N. \tag{1.6}$$

The existence of solutions for (1.6) have been extensively investigated in the literature over the past several decades (see [5,9,12,13,26,27,39–42]). For example, Shen et al. studied the existence of positive solutions for two types of quasilinear elliptic equations with degenerate coerciveness and slightly superlinear growth in [40]. By introducing a new variable replacement, Cheng et al. proved the existence of positive and soliton solutions to a class of relativistic nonlinear Schrödinger equations in [12,13]. In [9], Chen et al. proved existence and asymptotic behavior of standing wave solutions for a class of generalized quasilinear Schrödinger equations with critical Sobolev exponents. In [42], Shi et al. proved the positive solutions for generalized quasilinear Schrödinger equations with potential vanishing at infinity. Besides, Li et al. in [26] considered a class of generalized quasilinear Schrödinger equations with a Kirchhoff-type perturbation. Under the assumption that the potential may be vanishing at infinity, the existence of both the ground state and the ground state sign-changing solutions is established. Furthermore, the behavior of these solutions is studied when the perturbation vanishes. More concretely, Deng et al. [17,18] proved the existence of positive solutions with critical exponents, where critical exponents are 2^* and $\alpha 2^*$, respectively. Moreover, the existence of nodal solutions have been proved by Deng et al. in [15,16]. Very recently, in [27], Li et al. via Nehari manifold method proved the existence of ground state solutions and geometrically distinct solutions. In [28], the authors by using symmetric mountain theorem, considered the existence of a positive solution, a negative solution and infinitely many solutions. For generalized quasilinear Schrödinger equation of Kirchhoff type and generalized quasilinear Schrödinger–Maxwell system, we can refer to [6,7,26,44] and references therein.

If we set

$$g^2(u) = 1 + 2u^2,$$

then (1.6) reduces to the following well-known quasilinear Schrödinger equation

$$-\Delta u + V(x)u - u\Delta(u^2) = h(x,u). \tag{1.7}$$

For (1.7), Liu–Wang–Wang [30] and Colin–Jeanjean [14] made the change of variable by $v = f^{-1}(u)$, where f is defined by

$$f'(t) = \frac{1}{(1+2f^2(t))^{\frac{1}{2}}} \quad \text{on } [0,\infty) \quad \text{and} \quad f(t) = -f(-t) \text{ on}(-\infty,0], \tag{1.8}$$

and then equation (1.7) in form can be transformed into a semilinear equation. Afterwards, many recent studies has focused on the above quasilinear equation via the variable f, see for example [8, 14, 33] and references therein. Especially, in [33], the authors considered the existence of positive solutions for (1.7) with concave and convex nonlinearities.

To our knowledge, there are few papers studying the existence of positive solutions for (1.6) with concave and convex nonlinearities. Motivated by the previously mentioned papers, especially [33], we study the existence of positive solutions with concave and convex nonlinearities. Next, we give the following conditions on V:

(V_1) $V \in C(\mathbb{R}^N, \mathbb{R})$ and $0 < V_0 \leq \inf_{x\in\mathbb{R}^N} V(x)$;

(V_2) $[V(x)]^{-1} \in L^1(\mathbb{R}^N)$.

Moreover, the nonlinearities term f and h should satisfy the following assumptions:

(FH) $f, h \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ and $f(x,s) = 0$, $h(x,s) = 0$ for all $s \leq 0$ and $x \in \mathbb{R}^N$;

(FH_1) there exist constant $c_1 > 0$ and $q \in (1,2)$ such that

$$0 \leq f(x,s) \leq c_1|s|^{q-1}, \quad \text{for all } (x,s) \in \mathbb{R}^N \times \mathbb{R};$$

(FH_2) $\lim\limits_{s\to 0} \frac{h(x,s)}{s} = 0$ uniformly in $x \in \mathbb{R}^N$;

(FH_3) there exist $c_2 > 0$ and $4 < p < 2 \cdot 2^*$ such that

$$h(x,s) \leq c_2(1 + |s|^{p-1}), \quad \text{for all } (x,s) \in \mathbb{R}^N \times \mathbb{R};$$

(FH_4) there exists $\mu \geq 2$ such that

$$0 < 2\mu g(s)H(x,s) \leq G(s)h(x,s),$$

where $H(x,s) = \int_0^s h(x,t)dt$ and $G(s) = \int_0^s g(t)dt$;

(FH_5) there exist $c_3 > 0$ and $q_1 \in (1,2)$ such that

$$h(x,s) \geq c_3 g(s)|G(s)|^{q_1-1} \quad \text{for all } (x,s) \in \mathbb{R}^N \times \mathbb{R}.$$

Next, we recall some basic notions. Let

$$H^1(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N) : \nabla u \in L^2(\mathbb{R}^N) \right\},$$

endowed with the norm

$$\|u\|_{H^1} = \left(\int_{\mathbb{R}^N} (|\nabla u|^2 + u^2) \right)^{\frac{1}{2}}.$$

In the study of the elliptic equations, it is well known that the potential function V plays an important role in choosing of a right working space and some suitable compactness methods. Generally speaking, many papers study (1.1) under the following working space:

$$X = \left\{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^3} V(x)u^2 < \infty \right\},$$

endowed with the norm

$$\|u\|_X = \left(\int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2) \right)^{\frac{1}{2}}, \qquad u \in X.$$

But in this paper, we define the following working space:

$$E = \left\{ v \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)|G^{-1}(v)|^2 < \infty \right\},$$

which is called Orlicz–Sobolev space. Then E is a Banach space endowed with the following norm

$$\|v\| := \|\nabla v\|_2 + \inf_{\xi>0} \frac{1}{\xi} \left\{ 1 + \int_{\mathbb{R}^N} V(x)|G^{-1}(\xi v)|^2 \right\}. \tag{1.9}$$

To resolve the equation (1.1), due to the appearance of the nonlocal term $\int_{\mathbb{R}^N} g^2(u)|\nabla u|^2$, the right working space seems to be

$$E_0 = \left\{ u \in E : \int_{\mathbb{R}^N} g^2(u)|\nabla u|^2 < \infty \right\}.$$

But under the assumption of (g), it is easy to see that E_0 is not a linear space. To overcome this difficulty, we follow the idea developed by Shen and Wang in [39], that is, we make the change of variable substitution

$$u = G^{-1}(v) \quad \text{and} \quad G(u) = \int_0^u g(t)\mathrm{d}t, \qquad v \in E,$$

then

$$\int_{\mathbb{R}^N} g^2(u)|\nabla u|^2 = \int_{\mathbb{R}^N} g^2(G^{-1}(v))|\nabla G^{-1}(v)|^2 := |\nabla v|_2^2 < +\infty, \qquad v \in E.$$

In such a case, we obtain the following Euler–Lagrange functional associated with the equation (1.1)

$$J_\lambda(u) = \frac{1}{2} \int_{\mathbb{R}^N} [g^2(u)|\nabla u|^2 + V(x)u^2] - \lambda \int_{\mathbb{R}^N} F(x,u) - \int_{\mathbb{R}^N} H(x,u).$$

Therefore, after this change of variable, E can be used as the working space and the equation (1.1) in form can be transformed into the following functional

$$\mathcal{J}_\lambda(v) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla v|^2 + V(x)|G^{-1}(v)|^2) - \Psi_\lambda(v), \tag{1.10}$$

where

$$\Psi_\lambda(v) = \lambda \int_{\mathbb{R}^N} F(x, G^{-1}(v)) + \int_{\mathbb{R}^N} H(x, G^{-1}(v)). \tag{1.11}$$

Because g is a nondecreasing positive function, we get $|G^{-1}(v)| \leq \frac{1}{g(0)}|v|$. From this and our hypotheses, it is clear that $\mathcal{J}$ is well defined in E and $\mathcal{J} \in \mathcal{C}^1$.

Moreover, we can easily derive that if $v \in \mathcal{C}^2(\mathbb{R}^N)$ is a critical point of (1.10), then $u = G^{-1}(v) \in \mathcal{C}^2(\mathbb{R}^N)$ is a classical solution to the equation (1.1). In order to obtain a critical point of (1.10), we only need to find the weak solution to the following equation

$$-\Delta v + V(x)\frac{G^{-1}(v)}{g(G^{-1}(v))} = \lambda \frac{f(x, G^{-1}(v))}{g(G^{-1}(v))} + \frac{h(x, G^{-1}(v))}{g(G^{-1}(v))}, \qquad x \in \mathbb{R}^N. \tag{1.12}$$

Here, we call that $v \in E$ is a weak solution to the equation (1.12) if it holds that

$$\langle \mathcal{J}_\lambda'(v), \varphi \rangle = \int_{\mathbb{R}^N} \nabla v \cdot \nabla \varphi + \int_{\mathbb{R}^N} V(x)\frac{G^{-1}(v)}{g(G^{-1}(v))}\varphi - \langle \Psi_\lambda(v), \varphi \rangle,$$

where

$$\langle \Psi_\lambda(v), \varphi \rangle = \lambda \int_{\mathbb{R}^N} \frac{f(x, G^{-1}(v))}{g(G^{-1}(v))}\varphi + \int_{\mathbb{R}^N} \frac{h(x, G^{-1}(v))}{g(G^{-1}(v))}\varphi, \qquad \varphi \in E.$$

Then it is standard to obtain that $v \in E$ is a weak solution to the equation (1.12) if and only if v is a critical point of the functional $\mathcal{J}$ in E. All in all, if we find a critical point of the functional $\mathcal{J}$ in E, then we will get a classical solution to the equation (1.1).

Now, we state the results by the following theorems.

Theorem 1.1. *Suppose that (g), (V_1), (V_2) and (FH)–(FH_4) are satisfied. Then there exist $\lambda_0, C_0 > 0$ such that for all $\lambda \in [0, \lambda_0]$, problem (1.1) has one positive solution $u_{\lambda,1} \in H^1(\mathbb{R}^N)$ such that $\|u_{\lambda,1}\|_{H^1} \leq C_0$. Moreover, if $\lambda = 0$, then there exist constants $M, \zeta > 0$ such that*

$$u_{0,1} \leq M \exp\left(-\zeta |x|\right), \quad \text{for all } x \in \mathbb{R}^N.$$

Theorem 1.2. *Suppose that (g), (V_1), (V_2) and (FH)–(FH_5) are satisfied. Then for all $\lambda > 0$, (1.1) possesses a positive solution $v_{\lambda,2} \in H^1(\mathbb{R}^N)$, which is different of $v_{\lambda,1}$ when $\lambda \in (0, \lambda_0]$.*

Remark 1.3. Condition (g) originates from [19]. In fact, as [10], there are many functions satisfying (g). For example:

$$g(s) = \begin{cases} \sqrt{1+s^2}, & \text{if } 0 \leq s \leq 1, \\ \frac{\sqrt{2}}{2}(s+1), & \text{if } s > 1, \\ g(-s), & \text{if } s < 0, \end{cases}$$

and

$$g(s) = \sqrt{1+2s^2}.$$

Note that if we choose $g(s) = \sqrt{1+2s^2}$ in (1.1), then (1.1) will become the classical quasilinear Schrödinger equation

$$-\Delta u + V(x)u - \Delta(u^2)u = \lambda f(x,u) + h(x,u).$$

Remark 1.4. By Remark 1.3, our results extend [33].

Remark 1.5. Is easy to check that the following function satisfies $(V1)$ and $(V2)$:

$$V(x) = \begin{cases} 2, & \text{if } |x| < 1, \\ |x|^{N-1}(1+|x|^2), & \text{if } |x| \geq 1, \end{cases}$$

where $N \geq 3$.

For the above problem, there are many difficulties in treating this class of generalized quasilinear Schrödinger equations in $\mathbb{R}^N$. The first difficulty is the possible lack of compactness besides the concave term. The second difficulty is lack of natural functions space for the associated energy functional to be well defined. The function space $H^1(\mathbb{R}^N)$ cannot be applied directly to handle with this class of generalized quasilinear Schrödinger equations. To overcome these difficulties, we refer [33] and establish a different approach based on an appropriate Orlicz space. It was crucial in our argument the fact that this function space considered in our approach can be embedded into the usual Lebesgue spaces $L^r(\mathbb{R}^N)$ for all $1 \leq r < 2^*$.

Motivated by the argument used in [39], we use a change of variable to turn the problem into a semilinear one so that it has the associated functional well defined and Gateaux differentiable in a suitable Orlicz space. We prove that the energy functional satisfies the geometric hypotheses of the mountain-pass theorem and the Palais–Smale condition. In this paper, the first result is proved by using a version of the mountain-pass theorem and the second solution is obtained as a consequence of a minimization argument based on the Ekeland variational principle.

The paper is organized as follows. In Section 2, we give some preliminary lemmas. In Section 3, we give the proof of Theorem 1.1 and Theorem 1.2, respectively.

Notations. Throughout this paper, we make use of the following notations:

- $\int_{\mathbb{R}^N} \clubsuit$ denotes $\int_{\mathbb{R}^N} \clubsuit \, dx$.
- C will denote a positive constant, not necessarily the same one.
- $L^r(\mathbb{R}^N)$ denotes the Lebesgue space with norm

  $$\|u\|_r = \left(\int_{\mathbb{R}^N} |u|^r \right)^{1/r},$$

  where $1 \leq r < \infty$.
- For any $z \in \mathbb{R}^N$ and $R > 0$, $B_R(z) := \{x \in \mathbb{R}^N : |x - z| < R\}$.
- The weak convergence in $H^1(\mathbb{R}^N)$ is denoted by $\rightharpoonup$, and the strong convergence by $\to$.

2 Some preliminary lemmas

In this section, we present some useful lemmas and corollaries. Now, let us recall the following lemma which has been proved in [19].

Lemma 2.1 ([19]). *For the function g, G, and G^{-1}, the following properties hold:*

(g_1) *the functions $G(\cdot)$ and $G^{-1}(\cdot)$ are strictly increasing and odd;*

(g_2) $0 < \frac{d}{dt}\left(G^{-1}(t)\right) = \frac{1}{g(G^{-1}(t))} \le \frac{1}{g(0)}$ *for all* $t \in \mathbb{R}$;

(g_3) $|G^{-1}(t)| \le \frac{1}{g(0)}|t|$ *for all* $t \in \mathbb{R}$;

(g_4) $\lim\limits_{|t|\to 0} \frac{G^{-1}(t)}{t} = \frac{1}{g(0)}$;

(g_5) $\lim\limits_{|t|\to +\infty} \frac{G^{-1}(t)}{g(G^{-1}(t))} = \pm\frac{1}{g_\infty}$;

(g_6) $1 \le \frac{tg(t)}{G(t)} \le 2$ *and* $1 \le \frac{G^{-1}(t)g(G^{-1}(t))}{t} \le 2$ *for all* $t \ne 0$;

(g_7) $\frac{G^{-1}(t)}{\sqrt{t}}$ *is non-decreasing in* $(0,+\infty)$ *and* $|G^{-1}(t)| \le (2/g_\infty)^{1/2}\sqrt{|t|}$ *for all* $t \in \mathbb{R}$;

(g_8)
$$|G^{-1}(t)| \ge \begin{cases} G^{-1}(1)|t|, & \textit{for all } \ |t| \le 1, \\ G^{-1}(1)\sqrt{|t|}, & \textit{for all } \ |t| \ge 1; \end{cases}$$

(g_9) $\frac{t}{g(t)}$ *is increasing and* $\left|\frac{t}{g(t)}\right| \le \frac{1}{g_\infty}$ *for all* $t \in \mathbb{R}$;

(g_{10}) *the function* $[G^{-1}(t)]^2$ *is convex. In particular,* $[G^{-1}(\theta t)]^2 \le \theta[G^{-1}(t)]^2$ *for all* $t \in \mathbb{R}$, $\theta \in [0,1]$;

(g_{11}) $[G^{-1}(\theta t)]^2 \le \theta^2[G^{-1}(t)]^2$ *for all* $t \in \mathbb{R}$, $\theta \ge 1$;

(g_{12}) $[G^{-1}(t_1 - t_2)]^2 \le 4([G^{-1}(t_1)]^2 + [G^{-1}(t_2)]^2)$ *for all* $t_1, t_2 \in \mathbb{R}$;

(g_{13}) *the function* $G^{-1}(t)$ *is concave. In particular,* $G^{-1}(\theta t) \le \theta G^{-1}(t)$ *for all* $t \in \mathbb{R}$, $\theta \in [1,+\infty)$;

(g_{14}) $G^{-1}(\theta t) \ge \theta G^{-1}(t)$ *for all* $t \in \mathbb{R}$, $\theta \in [0,1]$.

Proposition 2.2 ([19]). *Assume that* V *satisfies* (V_1)–(V_2). *Then the space* E *has the following properties:*

(1) if $\{v_n\} \subset E$ *is such that* $v_n(x) \to v(x)$ *a.e. in* $\mathbb{R}^N$ *and*
$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} V(x)[G^{-1}(v_n)]^2 = \int_{\mathbb{R}^N} V(x)[G^{-1}(v)]^2,$$
then
$$\inf_{\xi>0} \frac{1}{\xi}\left\{1 + \int_{\mathbb{R}^N} V(x)|G^{-1}(\xi(v_n - v))|^2\right\} \to 0;$$

(2) the embedding $E \hookrightarrow D^{1,2}(\mathbb{R}^N)$, $E \hookrightarrow H^1(\mathbb{R}^N)$ *and* $X \hookrightarrow E$ *are continuous;*

(3) the map $v \mapsto G^{-1}(v)$ *from* E *to* $L^r(\mathbb{R}^N)$ *is continuous for each* $r \in [2, 2\cdot 2^*]$;

(4) if $v \in E$ *and* $u = G^{-1}(v)$, *then*
$$\|ug(u)\| \le 4\|v\|;$$

(5) if $v_n \rightharpoonup v$ *in* $D^{1,2}(\mathbb{R}^N)$ *and* $\left\{\int_{\mathbb{R}^N} V(x)[G^{-1}(v_n)]^2 dx\right\}$ *is bounded then, up to a subsequence,* $G^{-1}(v_n) \to 0$ *strongly in* $L^r(\mathbb{R}^N)$ *for any* $2 \le r < 2\cdot 2^*$.

Proposition 2.3 ([10]). *Assume that V satisfies (V_1)–(V_2). Then the space E has the following properties:*

(i) E is a normed linear space with respect to the norm given in (1.9);

(ii) there exists a positive constant $C > 0$ such that for all $v \in E$,

$$\frac{\int_{\mathbb{R}^N} V(x)|G^{-1}(v)|^2}{\left[1 + \left(\int_{\mathbb{R}^N} V(x)|G^{-1}(v)|^2\right)^{1/2}\right]} \leq C\|v\|; \tag{2.1}$$

(iii) if $v_n \to v$ in E, then

$$\int_{\mathbb{R}^N} V(x)\left||G^{-1}(v_n)|^2 - |G^{-1}(v)|^2\right| \to 0,$$

and

$$\int_{\mathbb{R}^N} V(x)\left|G^{-1}(v_n) - G^{-1}(v)\right|^2 \to 0.$$

Lemma 2.4. *Suppose that (V_1)–(V_2) holds. Then the embedding*

$$X \hookrightarrow L^r(\mathbb{R}^N)$$

is continuous for $1 \leq r \leq 2^*$ and compact for $1 \leq r < 2^*$.

Proof. Similar to the proof of [33], by (V_1), the embedding $X \hookrightarrow H^1(\mathbb{R}^N)$ is continuous. Thus $X \hookrightarrow L^r(\mathbb{R}^N)$ is continuous for $2 \leq r \leq 2^*$. Moreover, if $u \in X$ we get

$$\int_{\mathbb{R}^N} |u| \leq \left(\int_{\mathbb{R}^N} V(x)^{-1}\right)^{1/2} \|u\|_X.$$

Therefore, by interpolation the first part the lemma is proved. Next, let $\{u_n\} \subset X$ be a bounded sequence. Hence up to a subsequence, $u_n \rightharpoonup u_0$ in X. Given $\varepsilon > 0$, for enough large $R > 0$, we have

$$\int_{|x|>R} V(x)^{-1} \leq \left[\frac{\varepsilon}{2(C + \|u_0\|_X)}\right]^2,$$

which shows that

$$\int_{|x|>R} |u_n - u_0| \leq \left(\int_{|x|>R} V(x)^{-1}\right)^{1/2} \|u_n - u_0\|_X \leq \frac{\varepsilon}{2},$$

and since $X \hookrightarrow L^1(B_R)$ is compact, it follows that there exists $\mathbf{N}$ such that for all $n \geq \mathbf{N}$

$$\int_{B_R} |u_n - u_0| \leq \frac{\varepsilon}{2}.$$

Thus $u_n \to u_0$ in $L^1(\mathbb{R}^N)$. Now, if $r \in [1, 2^*)$, by interpolation inequality, for some $0 < \sigma \leq 1$, we have

$$\|u_n - u_0\|_r \leq \|u_n - u_0\|_1^{\sigma} \|u_n - u_0\|_{2^*}^{1-\sigma} \leq C\|u_n - u_0\|_1^{\sigma} \to 0,$$

and this completes the proof. □

Lemma 2.5. *The map $v \mapsto G^{-1}(v)$ from E to $L^r(\mathbb{R}^N)$ is continuous for each $r \in [1, 2 \cdot 2^*]$. Moreover, under assumption (V_2), the above map is compact for $r \in [1, 2 \cdot 2^*)$.*

Proof. Let $v \in E$. By definition, we know that $G^{-1}(v) \in X$, which together with Lemma 2.4 and (g_2) in Lemma 2.1 implies that

$$\|G^{-1}(v)\|_r \leq C\|G^{-1}(v)\|_X \leq C\left[\int_{\mathbb{R}^N}(|\nabla v|^2 + V(x)|G^{-1}(v)|^2)\right]^{1/2}, \tag{2.2}$$

for all $1 \leq r \leq 2^*$. Moreover, by the Gagliardo–Nirenberg inequality and (g_9) in Lemma 2.1, we have

$$\begin{aligned} \|G^{-1}(v)\|_{2\cdot 2^*} &= \|G^{-1}(v)^2\|_{2^*}^{1/2} \\ &\leq C\|\nabla(G^{-1}(v)^2)\|_2^{1/2} \\ &= C\left[\int_{\mathbb{R}^N}\left|\frac{G^{-1}(v)}{g(G^{-1}(v))}\right|^2|\nabla v|^2\right]^{1/2} \\ &\leq \frac{C}{g_\infty}\|v\|. \end{aligned} \tag{2.3}$$

Thus for all $v \in E$, we know $G^{-1}(v) \in L^{2\cdot 2^*}(\mathbb{R}^N)$.

Let $\{v_n\}$ be a sequence in E such that $v_n \to v$ in E. Thus

$$\frac{\partial v_n}{\partial x_i} \to \frac{\partial v}{\partial x_i} \quad \text{in } L^2(\mathbb{R}^N),$$

for $i = 1, 2, \ldots, N$. By (iii) in Proposition 2.3, we have

$$\int_{\mathbb{R}^N} V(x)\left||G^{-1}(v_n)| - |G^{-1}(v)|\right|^2 dx \to 0. \tag{2.4}$$

Therefore, by Lemma A.1 in [43], up to a subsequence, there exists $U_i \in L^2(\mathbb{R}^N)$ for $i = 1, 2, \ldots, N$ such that

$$\left|\frac{\partial v_n}{\partial x_i}\right| \leq U_i(x), \quad \text{a.e. } x \in \mathbb{R}^N.$$

Hence

$$\left|\frac{\partial G^{-1}(v_n)}{\partial x_i}\right| = \left|\frac{1}{g(G^{-1}(v_n))}\frac{\partial v_n}{\partial x_i}\right| \leq \frac{1}{g(0)}U_i(x),$$

and

$$\frac{\partial G^{-1}(v_n)}{\partial x_i} = \frac{1}{g(G^{-1}(v_n))}\frac{\partial v_n}{\partial x_i} \to \frac{1}{g(G^{-1}(v))}\frac{\partial v}{\partial x_i} = \frac{\partial G^{-1}(v)}{\partial x_i} \quad \text{a.e. } x \in \mathbb{R}^N,$$

for $i = 1, 2, \ldots, N$. So by the Lebesgue Dominated Converge Theorem, we have

$$G^{-1}(v_n) \to G^{-1}(v) \quad \text{in } D^{1,2}(\mathbb{R}^N),$$

which together with (2.4), we have

$$G^{-1}(v_n) \to G^{-1}(v) \quad \text{in } X.$$

Moreover, by Lemma 2.4, one has

$$G^{-1}(v_n) \to G^{-1}(v) \quad \text{in } L^r(\mathbb{R}^N) \quad \text{for } 1 \leq r \leq 2^*.$$

By (2.3), we have

$$|G^{-1}(v_n - v)|^2 \to 0 \quad \text{in } L^{2^*}(\mathbb{R}^N).$$

Again by Lemma A.1 in [43], there exists $W \in L^{2^*}(\mathbb{R}^N)$ such that

$$|G^{-1}(v_n - v)|^2 \le W(x) \quad \text{a.e. } x \in \mathbb{R}^N.$$

By the convexity of $G^{-1}(v)^2$, we get

$$\begin{aligned}
|G^{-1}(v_n)^{2\cdot 2^*}| &\le \left|\frac{1}{2}G^{-1}(2(v_n - v))^2 + \frac{1}{2}G^{-1}(2v)^2\right|^{2^*} \\
&\le \left|\frac{C_0}{2}G^{-1}(v_n - v)^2 + \frac{C_0}{2}G^{-1}(v)^2\right|^{2^*} \\
&\le \frac{C_0 2^{2^*-1}}{2}\left(|G^{-1}(v_n - v)^2|^{2^*} + |G^{-1}(v)^2|^{2^*}\right) \\
&\le \frac{C_0 2^{2^*-1}}{2}\left(W(x)^{2^*} + |G^{-1}(v)^2|^{2^*}\right) \in L^1(\mathbb{R}^N).
\end{aligned}$$

Hence by the Lebesgue dominated converge theorem, one has

$$G^{-1}(v_n) \to G^{-1}(v) \quad \text{in } L^{2\cdot 2^*}(\mathbb{R}^N).$$

Therefore, this completes the proof of continuity.

Next, we will prove the compactness. Let $\{v_n\} \subset E$ be a bounded sequence. Then $\{v_n\}$ is bounded in $D^{1,2}(\mathbb{R}^N)$ and by (2.1), we conclude that there exists a constant $C > 0$ such that

$$\int_{\mathbb{R}^N} V(x)G^{-1}(v_n)^2 \le C.$$

Form (2.2) and (2.3), we can know that $\{G^{-1}(v_n)\}$ is bounded in X and in $L^{2\cdot 2^*}(\mathbb{R}^N)$. The compact embedding $X \hookrightarrow L^1(\mathbb{R}^N)$ implies that, up to a subsequence, there is $w \in L^1(\mathbb{R}^N)$ such that $G^{-1}(v_n) \to w$ in $L^1(\mathbb{R}^N)$ and almost everywhere in $\mathbb{R}^N$. Thus, by the Brezis–Lieb Lemma we conclude that $w \in L^{2\cdot 2^*}(\mathbb{R}^N)$ and according to interpolation inequality, given any $1 \le q < 2 \cdot 2^*$, there exists $0 < \varsigma \le 1$ such that

$$\|G^{-1}(v_n) - w\|_q \le \|G^{-1}(v_n) - w\|_1^{\varsigma}\|G^{-1}(v_n) - w\|_{2\cdot 2^*}^{1-\varsigma} \le C\|G^{-1}(v_n) - w\|_1^{\varsigma},$$

which shows that $G^{-1}(v_n) \to w$ in $L^r(\mathbb{R}^N)$ for $1 \le r < 2 \cdot 2^*$. This completes the proof. □

Lemma 2.6. *The embedding $E \hookrightarrow L^r(\mathbb{R}^N)$ is continuous for $1 \le r \le 2^*$.*

Proof. Firstly, by (g_8) in Lemma 2.1, we can get

$$|t| \le \frac{1}{G^{-1}(0)}|G^{-1}(t)| + \frac{1}{G^{-1}(0)^2}|G^{-1}(t)|^2. \tag{2.5}$$

Moreover, by Lemma 2.5, if $v \in E$, then $v \in L^1(\mathbb{R}^N)$. That is to say that if $v_n \to 0$ in E, then we have $G^{-1}(v_n) \to 0$ in $L^1(\mathbb{R}^N)$ and $L^2(\mathbb{R}^N)$. Thus by (2.5), we know $v_n \to 0$ in $L^1(\mathbb{R}^N)$. Thus $E \hookrightarrow L^1(\mathbb{R}^N)$ is continuous. Using one more time (2.5), we have

$$|t|^{2^*} \le \frac{1}{G^{-1}(0)}|G^{-1}(t)|^{2^*} + \frac{1}{G^{-1}(0)^2}|G^{-1}(t)|^{2\cdot 2^*}.$$

It follows from Lemma 2.5 that $v_n \to 0$ in $L^{2^*}(\mathbb{R}^N)$. Finally, by interpolation the results obviously holds. □

Proposition 2.7 ([10]). *E is a Banach space. Moreover, $C_0^\infty(\mathbb{R}^N)$ is dense in E.*

Proposition 2.8. *The functional $\mathcal{J}_\lambda$ is well defined, continuous and Gateaux-differentiable in E with*

$$\langle \mathcal{J}_\lambda'(v), \varphi\rangle = \int_{\mathbb{R}^N} \nabla v \nabla \varphi + \int_{\mathbb{R}^N} V(x)\frac{G^{-1}(v)}{g(G^{-1}(v))}\varphi - \langle \Psi_\lambda(v), \varphi\rangle,$$

where

$$\langle \Psi_\lambda(v), \varphi\rangle = \lambda \int_{\mathbb{R}^N} \frac{f(x, G^{-1}(v))}{g(G^{-1}(v))}\varphi + \int_{\mathbb{R}^N} \frac{h(x, G^{-1}(v))}{g(G^{-1}(v))}\varphi, \qquad v, \varphi \in E.$$

Moreover, for $v \in E$ we know that $\mathcal{J}_\lambda'(v) \in E^*$ and if $v_n \to v$ in E then

$$\langle \mathcal{J}_\lambda'(v_n), \varphi\rangle \to \langle \mathcal{J}_\lambda'(v), \varphi\rangle,$$

for each $\varphi \in E$, that is, $\mathcal{J}_\lambda'(v_n) \to \mathcal{J}_\lambda'(v)$ in the weak $*$ topology of E^*.

Proof. By (FH_1)–(FH_3), for each $v \in E$, we have

$$\int_{\mathbb{R}^N} F(x, G^{-1}(v)) \le \frac{c_1}{q}\int_{\mathbb{R}^N} |G^{-1}(v)|^q, \tag{2.6}$$

and

$$\int_{\mathbb{R}^N} H(x, G^{-1}(v)) \le C\int_{\mathbb{R}^N} (|G^{-1}(v)|^2 + |G^{-1}(v)|^p). \tag{2.7}$$

Hence, by Lemma 2.6, $\Psi_\lambda(v)$ is well defined.

Let $v_n \to v$ in E, then by the continuous embedding $E \hookrightarrow D^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$, we have $v_n \to v$ in $D^{1,2}(\mathbb{R}^N)$, $v_n \to v$ in $L^r(\mathbb{R}^N)$ for $1 \le r \le 2^*$ and

$$\int_{\mathbb{R}^N} V(x)G^{-1}(v_n)^2 \to \int_{\mathbb{R}^N} V(x)G^{-1}(v)^2.$$

It follows from (2.6), (2.7) and Lebesgue's Dominated Converge Theorem implies that

$$\int_{\mathbb{R}^N} F(x, G^{-1}(v_n)) \to \int_{\mathbb{R}^N} F(x, G^{-1}(v)),$$
$$\int_{\mathbb{R}^N} H(x, G^{-1}(v_n)) \to \int_{\mathbb{R}^N} H(x, G^{-1}(v)).$$

Thus $\mathcal{J}_\lambda$ is continuous.

Next, we prove that $\mathcal{J}_\lambda$ is Gateaux-differentiable in E. Note that for any fixed $v, \varphi \in E$, by the mean value theorem, there exists $0 < \theta < 1$ such that

$$\frac{1}{2}\int_{\mathbb{R}^N} \frac{V(x)(|G^{-1}(v+t\varphi)|^2 - |G^{-1}(v)|^2)}{t}dx = \int_{\mathbb{R}^N} V(x)\frac{G^{-1}(v+\theta t\varphi)}{g(G^{-1}(v+\theta t\varphi))}\varphi dx.$$

For any $|t| \le 1$, we have

$$\begin{aligned}\left|V(x)\frac{G^{-1}(v+\theta t\varphi)}{g(G^{-1}(v+\theta t\varphi))}\varphi\right| &\le CV(x)|(v+\theta t\varphi)\varphi| \\ &\le CV(x)|v\varphi + \varphi^2| \\ &\le CV(x)(|v\varphi| + |\varphi|^2) \in L^1(\mathbb{R}^N).\end{aligned}$$

Since

$$\frac{G^{-1}(v+\theta t\varphi)}{g(G^{-1}(v+\theta t\varphi))}\varphi \to \frac{G^{-1}(v)}{g(G^{-1}(v))}\varphi, \quad a.e. \text{ on } \mathbb{R}^N, \quad \text{as } t \to 0,$$

then by the Lebesgue Dominated Convergence Theorem, we have

$$\frac{1}{2}\int_{\mathbb{R}^N}\frac{V(x)(|G^{-1}(v+t\varphi)|^2-|G^{-1}(v)|^2)}{t}\to\int_{\mathbb{R}^N}V(x)\frac{G^{-1}(v)}{g(G^{-1}(v))}\varphi,\quad \text{as } t\to 0,$$

Similar to the proof and using (FH)–(FH_1), we can conclude that

$$\int_{\mathbb{R}^N}\frac{F(x,G^{-1}(v+t\varphi))-F(x,G^{-1}(v))}{t}\to\int_{\mathbb{R}^N}\frac{f(x,G^{-1}(v))}{g(G^{-1}(v))}\varphi,\quad \text{as } t\to 0,$$

and

$$\int_{\mathbb{R}^N}\frac{H(x,G^{-1}(v+t\varphi))-F(x,G^{-1}(v))}{t}\to\int_{\mathbb{R}^N}\frac{h(x,G^{-1}(v))}{g(G^{-1}(v))}\varphi,\quad \text{as } t\to 0,$$

Based on the above discussion, we have that $\mathcal{J}\in\mathcal{C}^1(E,\mathbb{R})$.

To prove $\mathcal{J}'\in E^*$ for $v\in E$, we only need to check the term $\int_{\mathbb{R}^N}V(x)\frac{G^{-1}(v)}{g(G^{-1}(v))}\varphi$. In fact, let $\omega_n\to 0$ in E. By Proposition 2.3-(iii), we have

$$\int_{\mathbb{R}^N}V(x)|G^{-1}(\omega_n)|^2\to 0,\quad \text{as } n\to\infty.$$

Moreover, it follows from (g_2), (g_9) in Lemma 2.1 and (2.5) that

$$\begin{aligned}
\left|\int_{\mathbb{R}^N}V(x)\frac{G^{-1}(v)}{g(G^{-1}(v))}\omega_n\right| &\le \int_{\mathbb{R}^N}V(x)\left|\frac{G^{-1}(v)}{g(G^{-1}(v))}\right||\omega_n|\\
&\le \frac{1}{G^{-1}(0)}\int_{\mathbb{R}^N}V(x)\left|\frac{G^{-1}(v)}{g(G^{-1}(v))}\right||G^{-1}(\omega_n)|\\
&\qquad+\frac{1}{G^{-1}(0)^2}\int_{\mathbb{R}^N}V(x)\left|\frac{G^{-1}(v)}{g(G^{-1}(v))}\right||G^{-1}(\omega_n)|^2\\
&\le \frac{1}{G^{-1}(0)}\int_{\mathbb{R}^N}V(x)\left|G^{-1}(v)\right||G^{-1}(\omega_n)|\\
&\qquad+\frac{1}{g_\infty G^{-1}(0)^2}\int_{\mathbb{R}^N}V(x)|G^{-1}(\omega_n)|^2\\
&\le \frac{1}{G^{-1}(0)}\left[\int_{\mathbb{R}^N}V(x)\left|G^{-1}(v)\right|^2\right]^{1/2}\left[\int_{\mathbb{R}^N}V(x)\left|G^{-1}(\omega_n)\right|^2\right]^{1/2}\\
&\qquad+\frac{1}{g_\infty G^{-1}(0)^2}\int_{\mathbb{R}^N}V(x)|G^{-1}(\omega_n)|^2,
\end{aligned}$$

which implies that

$$\int_{\mathbb{R}^N}V(x)\frac{G^{-1}(v)}{g(G^{-1}(v))}\omega_n\to 0.$$

Thus $\mathcal{J}'\in E^*$ for any $v\in E$.

Similar to the proof of the first part in this proposition, we can prove that if $v_n\to v$ in E, then

$$\langle\mathcal{J}_\lambda'(v_n),\varphi\rangle\to\langle\mathcal{J}_\lambda'(v),\varphi\rangle,$$

for each $\varphi\in E$, that is, $\mathcal{J}_\lambda'(v_n)\to\mathcal{J}_\lambda'(v)$ in the weak $*$ topology of E^*. □

3 Proofs of Theorem 1.1 and Theorem 1.2

This section is devoted to prove Theorem 1.1 and Theorem 1.2. To this end, we will present two lemmas to show that the functional $\mathcal{J}_\lambda$ verifies the mountain pass geometry. Before proving the two lemmas, we need to the following version mountain pass theorem, which is a consequence of the Ekeland variational principle as developed in [1].

Theorem 3.1 ([1]). *Let E be a Banach space and $\Phi \in \mathcal{C}(E, R)$, Gateaux-differentiable for all $v \in E$, with G-derivative $\Phi'(v) \in E^*$ continuous from the norm topology of E to the weak $*$ topology of E^*, Φ satisfies (PS) condition and $\Phi(0) = 0$. Let $\mathcal{S}$ be a closed subset of E which disconnects (archwise) E. Let $v_0 = 0$ and $v_1 \in E$ be points belonging to distinct connected components of $E \backslash S$. Suppose that*

$$\inf_{\mathcal{S}} \Phi \geq \eta > 0 \quad \textit{and} \quad \Phi(v_1) \leq 0.$$

Then

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \Phi(\gamma(t)) \geq \alpha,$$

and there exists a $(PS)_c$ sequence for Φ. c is critical value of Φ.

Next, we prove that there exists $\lambda_0 > 0$ such that for all $\lambda \in [0, \lambda_0]$, $\mathcal{J}_\lambda$ satisfies all the conditions of Theorem 3.1. To this end, for $\rho > 0$, let us define the following set

$$\mathcal{S}(\rho) = \left\{ x \in \mathbb{R}^N : \mathcal{P}(v) = \rho^2 \right\},$$

where $\mathcal{P} : E \to \mathbb{R}$ is given by

$$\mathcal{P}(v) = \int_{\mathbb{R}^N} \left(|\nabla v|^2 + V(x) |G^{-1}(v)|^2 \right).$$

Since $\mathcal{P}(v)$ is continuous then $\mathcal{S}(\rho)$ is a closed subset and disconnects the space E for $\rho > 0$.

Lemma 3.2. *Suppose that (V_1)–(V_2) and (FH)–(FH_4) are satisfied. Then there exist $\lambda_0, \eta, \rho > 0$ such that for all $\lambda \in [0, \lambda_0]$, $\mathcal{J}_\lambda(v) \geq \eta$ for all $v \in \mathcal{S}(\rho)$.*

Proof. By (FH_2) and (FH_3), for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$|H(x, G^{-1}(s))| \leq \varepsilon |G^{-1}(s)|^2 + C_\varepsilon |G^{-1}(s)|^p \quad \text{for all } (x, s) \in \mathbb{R}^N \times \mathbb{R}.$$

Thus for $v \in \mathcal{S}_\rho$, by (V_1) and Hölder's inequality, we get

$$\begin{aligned}
\int_{\mathbb{R}^N} H(x, G^{-1}(v)) &\leq \varepsilon \int_{\mathbb{R}^N} |G^{-1}(v)|^2 + C_\varepsilon \int_{\mathbb{R}^N} |G^{-1}(v)|^p \\
&\leq \frac{\varepsilon}{V_0} \int_{\mathbb{R}^N} V(x) |G^{-1}(v)|^2 + C_\varepsilon \int_{\mathbb{R}^N} |G^{-1}(v)^2|^{p/2} \\
&\leq \frac{\varepsilon}{V_0} \rho^2 + C_\varepsilon \left[\int_{\mathbb{R}^N} |G^{-1}(v)|^2 \right]^{\frac{p\kappa}{2}} \left[\int_{\mathbb{R}^N} |G^{-1}(v)^2|^{2^*} \right]^{1 - \frac{p\kappa}{2}} \\
&\leq \frac{\varepsilon}{V_0} \rho^2 + C_\varepsilon \rho^{p\kappa} \left[\int_{\mathbb{R}^N} |\nabla (G^{-1}(v)^2)|^2 \right]^{\frac{2^*}{2}\left(1 - \frac{p\kappa}{2}\right)},
\end{aligned}$$

where

$$\kappa = \frac{2 \cdot 2^* - p}{p(2^* - 1)}.$$

Moreover, since $v \in \mathcal{S}_\rho$, then

$$\int_{\mathbb{R}^N} |\nabla(G^{-1}(v)^2)|^2 = 4\int_{\mathbb{R}^N} \left|\frac{G^{-1}(v)}{g(G^{-1}(v))}\right|^2 |\nabla v|^2 \le \frac{4}{g_\infty^2}\int_{\mathbb{R}^N} |\nabla v|^2 \le \frac{4}{g_\infty^2}\rho^2.$$

Therefore, one has

$$\begin{aligned}\int_{\mathbb{R}^N} H(x, G^{-1}(v)) &\le \frac{\varepsilon}{V_0}\rho^2 + C_\varepsilon \rho^{p\kappa}\left[\frac{4}{g_\infty^2}\rho^2\right]^{\frac{2^*}{2}\left(1-\frac{p\kappa}{2}\right)} \\ &\le \frac{\varepsilon}{V_0}\rho^2 + C\rho^{\frac{2(p+N)}{N+2}}.\end{aligned} \tag{3.1}$$

Next, by (FH_1) and (2.2), we conclude that

$$\begin{aligned}\int_{\mathbb{R}^N} F(x, G^{-1}(v)) &\le c_1\int_{\mathbb{R}^N} |G^{-1}(v)|^q \\ &\le \|G^{-1}(v)\|_X^q \\ &\le C\left[\int_{\mathbb{R}^N}(|\nabla v|^2 + V(x)|G^{-1}(v)|^2)\right]^{q/2},\end{aligned}$$

and so $v \in \mathcal{S}_\rho$, we get

$$\int_{\mathbb{R}^N} F(x, G^{-1}(v)) \le C\rho^q. \tag{3.2}$$

It follows from (3.1) and (3.2) that for $v \in \mathcal{S}_\rho$,

$$\begin{aligned}\mathcal{J}_\lambda(v) &\ge \left(\frac{1}{2} - \frac{\varepsilon}{V_0}\right)\rho^2 - C\lambda\rho^q - C\rho^{\frac{2(p+N)}{N+2}} \\ &= \rho^2\left(\frac{1}{2} - \frac{\varepsilon}{V_0} - C\rho^{\frac{2(p-2)}{N+2}}\right) - C\lambda\rho^q.\end{aligned}$$

Next, we choose $0 < 2\varepsilon < V_0$ and $\rho_0 > 0$ such that

$$\alpha_0 := \frac{1}{2} - \frac{\varepsilon}{V_0} - C\rho_0^{\frac{2(p-2)}{N+2}} > 0,$$

where implies that

$$\mathcal{J}_\lambda(v) \ge \rho_0^q(\alpha_0\rho_0^{2-q} - \lambda C).$$

In the above inequality, choosing $\lambda_0 = \frac{\alpha_0\rho_0^{2-q}}{4C}$ and $\eta := \frac{3\alpha_0\rho_0^2}{4} > 0$ such that for all $\lambda \in [0, \lambda_0]$,

$$\mathcal{J}_\lambda(v) \ge \eta > 0.$$

This completes the proof. □

Lemma 3.3. *Suppose that (V_1)–(V_2) and (FH)–(FH_4) are satisfied. Then for $\lambda \in [0, \lambda_0]$, there exists $v \in E$ such that $\mathcal{P}(v) > \rho_0$ and $\mathcal{J}_\lambda(v) < 0$.*

Proof. To this end, we prove that for fixed $\psi \in E\backslash\{0\}$,

$$\mathcal{J}_\lambda(t\psi) \to -\infty \quad \text{as } t \to +\infty.$$

By (FH_4), there exist $C_1, C_2 > 0$ such that $H(x, G^{-1}(s)) \geq C_1 s^{2\mu} - C_2$ for all $(x,s) \in \mathbb{R}^N \times \mathbb{R}$. Choosing $\psi \in (\mathcal{C}_0^\infty(\mathbb{R}^N), [0,1])$ such that $\operatorname{supp}\psi = \overline{\Omega}$, we have

$$\begin{aligned}
\mathcal{J}_\lambda(t\psi) &\leq \frac{1}{2}\int_{\mathbb{R}^N}\left(|t\nabla\psi|^2 + V(x)|G^{-1}(t\psi)|^2\right) - \int_{\mathbb{R}^N} H(x, G^{-1}(t\psi)) \\
&\leq t^2 \int_{\overline{\Omega}} (|\nabla\psi|^2 + V(x)\psi^2) - C_1\int_{\overline{\Omega}} |G^{-1}(t\psi)|^{2\mu} + C_2|\overline{\Omega}| \\
&\leq t^2\left[\int_{\overline{\Omega}} (|\nabla\psi|^2 + V(x)\psi^2) - C_1\int_{\overline{\Omega}} \frac{|G^{-1}(t\psi)|^{2\mu}}{t^2} + C_2\frac{|\overline{\Omega}|}{t^2}\right],
\end{aligned}$$

where $|\overline{\Omega}|$ denotes the Lebesgue measure of $\overline{\Omega}$. Moreover, by (g_7) in Lemma 2.1, we have

$$\int_{\overline{\Omega}} \frac{|G^{-1}(t\psi)|^{2\mu}}{t^2} = \int_{\overline{\Omega}} \left(\frac{G^{-1}(t\psi)}{\sqrt{t|\psi|}}\right)^4 |G^{-1}(t\psi)|^{2\mu-4}\psi^2 \to +\infty \quad \text{as } t \to +\infty.$$

Hence, we take $v = t\psi$ with t large enough. This completes the proof. □

A sequence $\{v_n\} \subset E$ is said to be a $(PS)_c$-sequence if $\mathcal{J}_\lambda(v_n) \to c$ and $\mathcal{J}'_\lambda(v_n) \to 0$. $\mathcal{J}_\lambda$ is said to satisfy the $(PS)_c$-condition if any $(PS)_c$-sequence has a convergent subsequence. Now, we will prove that $\mathcal{J}_\lambda$ satisfies $(PS)_c$-condition.

Lemma 3.4. *Any $(PS)_c$ sequence for $\mathcal{J}_\lambda$ is bounded in E.*

Proof. Suppose that $\{v_n\}$ is a $(PS)_c$ for $\mathcal{J}_\lambda$, that is, $\mathcal{J}_\lambda(v_n) \to c$ and $\mathcal{J}'_\lambda(v_n) \to 0$. Using (g)

$$\begin{aligned}
&\mathcal{J}_\lambda(v_n) - \frac{1}{2\mu}\langle \mathcal{J}'_\lambda(v_n), v_n\rangle \\
&= \left(\frac{1}{2} - \frac{1}{2\mu}\right)\int_{\mathbb{R}^N} |\nabla v_n|^2 + \int_{\mathbb{R}^N} V(x)\left(\frac{1}{2}|G^{-1}(v_n)|^2 - \frac{1}{2\mu}\frac{G^{-1}(v_n)}{g(G^{-1}(v_n))}v_n\right) \\
&\quad - \frac{1}{2\mu}\int_{\mathbb{R}^N}\left[2\mu H(x, G^{-1}(v_n)) - \frac{h(x, G^{-1}(v_n))}{g(G^{-1}(v_n))}v_n\right] \\
&\quad - \lambda\int_{\mathbb{R}^N} F(x, G^{-1}(v_n)) + \frac{\lambda}{2\mu}\int_{\mathbb{R}^N} \frac{f(x, G^{-1}(v_n))}{g(G^{-1}(v_n))}v_n \\
&\geq \left(\frac{1}{2} - \frac{1}{2\mu}\right)\int_{\mathbb{R}^N} |\nabla v_n|^2 + \int_{\mathbb{R}^N} V(x)\left(\frac{1}{2}|G^{-1}(v_n)|^2 - \frac{1}{2\mu}\frac{G^{-1}(v_n)}{g(G^{-1}(v_n))}v_n\right) \\
&\quad - \frac{1}{2\mu}\int_{\mathbb{R}^N}\left[2\mu H(x, G^{-1}(v_n)) - \frac{h(x, G^{-1}(v_n))}{g(G^{-1}(v_n))}v_n\right] - \lambda\int_{\mathbb{R}^N} F(x, G^{-1}(v_n)).
\end{aligned} \tag{3.3}$$

Using the definition of G, we can get $G(t) \leq g(t)t$ for all $t \geq 0$. In fact, by $g'(t) \geq 0$ for all $t \geq 0$

$$G(t) = \int_0^t g(s)ds = sg(s)\Big|_0^t - \int_0^t sg'(s)ds \leq g(t)t. \tag{3.4}$$

By (3.4) and (g_6) in Lemma 2.1, it is easy to check that

$$\frac{1}{2}|G^{-1}(s)|^2 \leq \frac{G^{-1}(s)s}{g(G^{-1}(s))} \leq |G^{-1}(s)|^2, \quad \text{for all } s \in \mathbb{R}. \tag{3.5}$$

By (3.3) and (3.5), we have

$$c + o_n(1)\|v_n\| \geq \left(\frac{1}{2} - \frac{1}{2\mu}\right)\int_{\mathbb{R}^N}\left(|\nabla v_n|^2 + V(x)|G^{-1}(v_n)|^2\right) - \lambda C\|G^{-1}(v_n)\|_q^q. \tag{3.6}$$

It follows from (2.2) and (3.6) that

$$c + o_n(1)\|v_n\| + \lambda C \left[\int_{\mathbb{R}^N} \left(|\nabla v_n|^2 + V(x)|G^{-1}(v_n)|^2\right)\right]^{q/2} \geq \left(\frac{1}{2} - \frac{1}{2\mu}\right) \int_{\mathbb{R}^N} \left(|\nabla v_n|^2 + V(x)|G^{-1}(v_n)|^2\right),$$

which implies that there exists $C > 0$ such that

$$\int_{\mathbb{R}^N} \left(|\nabla v_n|^2 + V(x)|G^{-1}(v_n)|^2\right) \leq C, \tag{3.7}$$

due to $q \in (1,2)$. Since $s^{1/2} \leq 1 + s$ for all $s \geq 0$, then we have the following estimate

$$\begin{aligned} \|v_n\| &\leq \left(\int_{\mathbb{R}^N} |\nabla v_n|^2\right)^{1/2} + 1 + \int_{\mathbb{R}^N} V(x)|G^{-1}(v_n)|^2 \\ &\leq 2 + \int_{\mathbb{R}^N} \left(|\nabla v_n|^2 + V(x)|G^{-1}(v_n)|^2\right). \end{aligned} \tag{3.8}$$

It follows from (3.7) and (3.8) that $\{v_n\}$ is bounded in E. □

Lemma 3.5. *Any $(PS)_c$ sequence for $\mathcal{J}_\lambda$ has a converge subsequence.*

Proof. Let $\{v_n\}$ be a $(PS)_c$ for $\mathcal{J}_\lambda$. By Lemma 3.4, we know that $\{v_n\}$ is bounded in E. Since $E \hookrightarrow H^1(\mathbb{R}^N)$, $\{v_n\}$ is bounded in $H^1(\mathbb{R}^N)$. Hence up to a subsequence, there exists $v \in H^1(\mathbb{R}^N)$ such that

$$v_n \rightharpoonup v \quad \text{in } H^1(\mathbb{R}^N), \qquad v_n \rightharpoonup v \quad \text{in } L^r(\mathbb{R}^N) \quad \text{for all } 1 \leq r \leq 2^*, \qquad v_n \to v \quad \text{a.e. in } \mathbb{R}^N.$$

Using (2.1) and Fatou's Lemma, we have

$$\int_{\mathbb{R}^N} V(x)|G^{-1}(v)|^2 \leq \liminf_{n\to\infty} \int_{\mathbb{R}^N} V(x)|G^{-1}(v_n)|^2 \leq C,$$

which shows that $v \in E$. Moreover, by Lemma 2.5, one has

$$G^{-1}(v_n) \to G^{-1}(v) \quad \text{in } L^r(\mathbb{R}^N) \quad \text{for all } 1 \leq r < 2 \cdot 2^*. \tag{3.9}$$

Since $[G^{-1}(s)]^2$ is convex, then $\mathcal{P}(s)$ is also convex function. Therefore by Lemma 15.3 in [22], we have

$$\begin{aligned} \frac{1}{2}\mathcal{P}(v) - \frac{1}{2}\mathcal{P}(v_n) &\geq \frac{1}{2}\langle \mathcal{P}'(v_n), v - v_n\rangle \\ &= \int_{\mathbb{R}^N} \nabla v_n \nabla(v - v_n) + \int_{\mathbb{R}^N} V(x)\frac{G^{-1}(v_n)}{g(G^{-1}(v_n))}(v - v_n). \end{aligned}$$

Moreover,

$$\begin{aligned} &\frac{1}{2}\int_{\mathbb{R}^N}[|\nabla v| + V(x)|G^{-1}(v)|^2] - \frac{1}{2}\int_{\mathbb{R}^N}[|\nabla v_n| + V(x)|G^{-1}(v_n)|^2] \\ &\geq \lambda \int_{\mathbb{R}^N} \frac{f(x, G^{-1}(v_n))}{g(G^{-1}(v_n))}(v - v_n) + \int_{\mathbb{R}^N} \frac{h(x, G^{-1}(v_n))}{g(G^{-1}(v_n))}(v - v_n) \\ &\quad + \langle \mathcal{J}_\lambda'(v_n), v - v_n\rangle. \end{aligned} \tag{3.10}$$

Writing

$$\int_{\mathbb{R}^N} \frac{h(x,G^{-1}(v_n))}{g(G^{-1}(v_n))}(v-v_n) = \int_{\mathbb{R}^N} \left[\frac{h(x,G^{-1}(v_n))}{g(G^{-1}(v_n))} - \frac{h(x,G^{-1}(v))}{g(G^{-1}(v))}\right](v-v_n) + \int_{\mathbb{R}^N} \frac{h(x,G^{-1}(v))}{g(G^{-1}(v))}(v-v_n).$$

Due to $\frac{h(x,G^{-1}(v))}{g(G^{-1}(v))} \in L^{2N/(N+2)}(\mathbb{R}^N)$ and $v_n \rightharpoonup v$ in $L^{2^*}(\mathbb{R}^N)$, we have

$$\int_{\mathbb{R}^N} \frac{h(x,G^{-1}(v))}{g(G^{-1}(v))}(v-v_n) \to 0 \quad \text{as } n \to \infty.$$

By (FH_2)–(FH_3), (3.9) and the Lebesgue Dominated Converge Theorem, we have

$$\frac{h(x,G^{-1}(v_n))}{g(G^{-1}(v_n))} \to \frac{h(x,G^{-1}(v))}{g(G^{-1}(v))} \quad \text{in } L^{2N/(N+2)}(\mathbb{R}^N).$$

By Hölder inequality and the boundedness of $\{v_n\}$ in $L^{2^*}(\mathbb{R}^N)$, one has

$$\int_{\mathbb{R}^N} \left[\frac{h(x,G^{-1}(v_n))}{g(G^{-1}(v_n))} - \frac{h(x,G^{-1}(v))}{g(G^{-1}(v))}\right](v-v_n) \to 0 \quad \text{as } n \to \infty.$$

Thus

$$\int_{\mathbb{R}^N} \frac{h(x,G^{-1}(v_n))}{g(G^{-1}(v_n))}(v-v_n) \to 0 \quad \text{as } n \to \infty.$$

Similarly, we can prove the following

$$\int_{\mathbb{R}^N} \frac{f(x,G^{-1}(v_n))}{g(G^{-1}(v_n))}(v-v_n) \to 0 \quad \text{as } n \to \infty,$$

which dues to

$$v_n \rightharpoonup v \text{ in } L^q(\mathbb{R}^N) \quad \text{and} \quad \frac{f(x,G^{-1}(v_n))}{g(G^{-1}(v_n))} \to \frac{f(x,G^{-1}(v))}{g(G^{-1}(v))} \quad \text{in } L^{q/(q-1)}(\mathbb{R}^N).$$

By virtue of $\langle \mathcal{J}'_\lambda(v_n), v-v_n\rangle = o_n(1)$, by (3.10), we get

$$\liminf_{n\to\infty} \int_{\mathbb{R}^N} [|\nabla v_n| + V(x)|G^{-1}(v_n)|^2] \le \int_{\mathbb{R}^N} [|\nabla v| + V(x)|G^{-1}(v)|^2].$$

In addition, by the semicontinuity of norm and Fatou's Lemma, we have

$$\liminf_{n\to\infty} \int_{\mathbb{R}^N} |\nabla v_n|^2 \ge \int_{\mathbb{R}^N} |\nabla v|^2,$$

$$\liminf_{n\to\infty} \int_{\mathbb{R}^N} V(x)|G^{-1}(v_n)|^2 \ge \int_{\mathbb{R}^N} V(x)|G^{-1}(v)|^2.$$

Therefore we have

$$\liminf_{n\to\infty} \int_{\mathbb{R}^N} |\nabla v_n|^2 = \int_{\mathbb{R}^N} |\nabla v|^2,$$

and

$$\liminf_{n\to\infty} \int_{\mathbb{R}^N} V(x)|G^{-1}(v_n)|^2 = \int_{\mathbb{R}^N} V(x)|G^{-1}(v)|^2.$$

By (1) in Proposition 2.2, we get

$$\inf_{\xi>0} \frac{1}{\xi}\left\{1 + \int_{\mathbb{R}^N} V(x)|G^{-1}(\xi(v_n - v))|^2\right\} \to 0,$$

which together with $\nabla v_n \to \nabla v$ in $L^2(\mathbb{R}^N)$, implies that $v_n \to v$ in E. □

Proof of Theorem 1.1. By Lemmas 3.2–3.5, all conditions of Theorem 3.1 are satisfied. Thus there exists a critical point c_λ for $\mathcal{J}_\lambda$ at mountain pass level

$$c_\lambda = \inf_{\gamma\in\Gamma_\lambda} \max_{t\in[0,1]} \mathcal{J}_\lambda(\gamma(t)) > 0,$$

where

$$\Gamma_\lambda = \{\gamma \in \mathcal{C}([0,1],E) : \gamma(0) = 0 \text{ and } \mathcal{J}_\lambda(\gamma(1)) < 0\}.$$

Therefore for all $\phi \in E$, we get

$$\int_{\mathbb{R}^N} \nabla v_\lambda \nabla\phi + V(x)\frac{G^{-1}(v_\lambda)}{g(G^{-1}(v_\lambda))}\phi = \lambda\int_{\mathbb{R}^N}\frac{f(x,G^{-1}(v_\lambda))}{g(G^{-1}(v_\lambda))}\phi + \int_{\mathbb{R}^N}\frac{h(x,G^{-1}(v_\lambda))}{g(G^{-1}(v_\lambda))}\phi. \tag{3.11}$$

Choosing $\phi = -v_\lambda^-$, where $v_\lambda^- = \max\{-v_\lambda, 0\}$, we have

$$\int_{\mathbb{R}^N}\left[|\nabla v_\lambda^-|^2 + V(x)\frac{G^{-1}(v_\lambda)}{g(G^{-1}(v_\lambda))}(-v_\lambda^-)\right] = 0.$$

Since $G^{-1}(v_\lambda)(-v_\lambda^-) \geq 0$, we conclude that

$$\int_{\mathbb{R}^N}|\nabla v_\lambda^-|^2 = 0 \quad \text{and} \quad \int_{\mathbb{R}^N} V(x)\frac{G^{-1}(v_\lambda)}{g(G^{-1}(v_\lambda))}(-v_\lambda^-) = 0.$$

Thus $v_\lambda^- = 0$ a.e. in $\mathbb{R}^N$ and we have $v_\lambda \geq 0$. It follows from the strong maximum principle that $v_\lambda > 0$ in $\mathbb{R}^N$, therefore $u_{\lambda,1} = G^{-1}(v_\lambda)$ is a positive solution for (1.1).

Next, we shall prove that there exists $C > 0$ such that $\|u_{\lambda,1}\| \leq C$ for all $\lambda \in [0,\lambda_0]$. In (3.11), taking $\phi = v_\lambda$ and using (3.5), we get

$$\begin{aligned} 2\int_{\mathbb{R}^N}|\nabla v_\lambda|^2 + 2\int_{\mathbb{R}^N}V(x)|G^{-1}(v_\lambda)|^2 &\geq 2\int_{\mathbb{R}^N}|\nabla v_\lambda|^2 + 2\int_{\mathbb{R}^N}V(x)\frac{G^{-1}(v_\lambda)v_\lambda}{g(G^{-1}(v_\lambda))} \\ &\geq \int_{\mathbb{R}^N}\frac{h(x,G^{-1}(v_\lambda))v_\lambda}{g(G^{-1}(v_\lambda))}. \end{aligned} \tag{3.12}$$

Since $\mathcal{J}_\lambda(v_\lambda) = c_\lambda$, we get

$$\begin{aligned} 2\mu c_\lambda = \mu\int_{\mathbb{R}^N}(|\nabla v_\lambda|^2 + V(x)|G^{-1}(v_\lambda)|^2)dx - 2\mu\lambda\int_{\mathbb{R}^N}F(x,G^{-1}(v_\lambda))dx \\ - 2\mu\int_{\mathbb{R}^N}H(x,G^{-1}(v_\lambda))dx, \end{aligned} \tag{3.13}$$

and $c_\lambda \leq c_0$, where

$$c_0 = \inf_{\gamma\in\Gamma_0}\max_{t\in[0,1]}\mathcal{J}_0(\gamma(t)) > 0,$$

with $\mathcal{J}_0$ is given by

$$\mathcal{J}_0(v) = \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla v_\lambda|^2 + V(x)|G^{-1}(v_\lambda)|^2)dx - \int_{\mathbb{R}^N}H(x,G^{-1}(v_\lambda))dx,$$

and

$$\Gamma_0 = \{\gamma \in \mathcal{C}([0,1],E) : \gamma(0) = 0 \text{ and } \mathcal{J}_0(\gamma(1)) < 0\}.$$

Thus, by (3.12), (3.13), (FH_4) and (2.2), one has

$$\begin{aligned}
&(\mu-2)\int_{\mathbb{R}^N}(|\nabla v_\lambda|^2+V(x)|G^{-1}(v_\lambda)|^2)\\
&\quad= -2\int_{\mathbb{R}^N}(|\nabla v_\lambda|^2+V(x)|G^{-1}(v_\lambda)|^2)+2\mu\lambda\int_{\mathbb{R}^N}F(x,G^{-1}(v_\lambda))\\
&\qquad+2\mu\int_{\mathbb{R}^N}H(x,G^{-1}(v_\lambda))dx+2\mu c_\lambda\\
&\quad\le \int_{\mathbb{R}^N}\left[2\mu H(x,G^{-1}(v_\lambda))-\frac{h(x,G^{-1}(v_\lambda))}{g(G^{-1}(v_\lambda))}v_\lambda\right]+2\mu c_1\int_{\mathbb{R}^N}|G^{-1}(v_\lambda)|^q+2\mu c_0\\
&\quad\le 2\mu c_1\left[\int_{\mathbb{R}^N}(|\nabla v_\lambda|^2+V(x)|G^{-1}(v_\lambda)|^2)\right]^{q/2}+2\mu c_0,
\end{aligned}$$

which implies that $\int_{\mathbb{R}^N}(|\nabla v_\lambda|^2+V(x)|G^{-1}(v_\lambda)|^2)$ is bounded in λ. Thus

$$\|u_{\lambda,1}\|_{H^1}=\|G^{-1}(v_\lambda)\|_{H^1}\le C\left[\int_{\mathbb{R}^N}(|\nabla v_\lambda|^2+V(x)|G^{-1}(v_\lambda)|^2)\right]^{1/2}\le C.$$

Next, we study the exponential decay property for solutions of (1.1) when $\lambda=0$. Let v_0 be a solution of (1.12) for $\lambda=0$. Now, we first prove that $v_0\in L^\infty(\mathbb{R}^N)$. Thus we conclude that

$$\int_{\mathbb{R}^N}\left(\nabla v_0\nabla\phi+V(x)\frac{G^{-1}(v_0)}{g(G^{-1}(v_0))}\phi\right)=\int_{\mathbb{R}^N}\frac{h(x,G^{-1}(v_0))}{g(G^{-1}(v_0))}\phi,\qquad \phi\in E. \tag{3.14}$$

For each $k>0$, let

$$v_k=\begin{cases}v_0, & \text{if } v_0\le k,\\ 0, & \text{if } v_0\ge k,\end{cases}$$

and

$$\varrho_k=v_k^{2(\beta-1)}v_0\quad\text{and}\quad w_k=v_k^{\beta-1}v_0,$$

where $\beta>1$. By (V_1), (FH_2) and (FH_3), we have

$$h(x,G^{-1}(v_0))\le\frac{V_0}{2}|G^{-1}(v_0)|+C_{V_0}|G^{-1}(v_0)|^{p-1}.$$

Thus choosing ϱ_k as a test function in (3.14), we know

$$\begin{aligned}
\int_{\mathbb{R}^N}v_k^{2(\beta-1)}|\nabla v_0|^2&\le\int_{\mathbb{R}^N}v_k^{2(\beta-1)}|\nabla v_0|^2+2(\beta-1)\int_{\mathbb{R}^N}v_k^{2(\beta-1)-1}v_0\nabla v_k\cdot\nabla v_0\\
&\le C_{V_0}\int_{\mathbb{R}^N}\frac{|G^{-1}(v_0)|^{p-1}}{g(G^{-1}(v_0))}v_k^{2(\beta-1)}v_0.
\end{aligned}$$

By (3.5), we have

$$\frac{v_0}{g(G^{-1}(v_0))}\le G^{-1}(v_0).$$

It follows from (g_7) in Lemma 2.1 and the above inequality that

$$\int_{\mathbb{R}^N}v_k^{2(\beta-1)}|\nabla v_0|^2\le C\int_{\mathbb{R}^N}v_k^{2(\beta-1)}v_0^{\frac{p}{2}}=C\int_{\mathbb{R}^N}v_0^{\frac{p}{2}-2}w_k^2. \tag{3.15}$$

Moreover, using the Gagliardo–Nirenberg–Sobolev inequality and (3.15), one has

$$\begin{aligned}\left(\int_{\mathbb{R}^N} w_k^{2^*}\right)^{2/2^*} &\le C\int_{\mathbb{R}^N} |\nabla w_k|^2\\ &\le C\int_{\mathbb{R}^N} v_k^{2(\beta-1)}|\nabla v_0|^2 + C(\beta-1)^2\int_{\mathbb{R}^N} v_0^2 v_k^{2(\beta-2)}|\nabla v_k|^2\\ &\le C\beta^2\int_{\mathbb{R}^N} v_k^{2(\beta-1)}|\nabla v_0|^2\\ &\le C\beta^2\int_{\mathbb{R}^N} v_0^{\frac{p}{2}-2} w_k^2,\end{aligned}$$

which dues to $v_k \le v_0$, $1 \le \beta^2$ and $(\beta-1)^2 \le \beta^2$. By Hölder's inequality,

$$\left(\int_{\mathbb{R}^N} w_k^{2^*}\right)^{2/2^*} \le C\beta^2 \left(\int_{\mathbb{R}^N} v_0^{2^*}\right)^{\frac{(\frac{p}{2}-2)}{2^*}} \left(\int_{\mathbb{R}^N} w_k^{\frac{2\cdot 2^*}{2^*-\frac{p}{2}+2}}\right)^{\frac{2^*-\frac{p}{2}+2}{2^*}}.$$

Since $w_k \le v_0^{\beta}$, using the continuity of the embedding $E \hookrightarrow L^{2^*}(\mathbb{R}^N)$, we have

$$\left(\int_{\mathbb{R}^N} [v_0 v_k^{\beta-1}]^{2^*}\right)^{2/2^*} \le C\beta^2 \|v_0\|^{(\frac{p}{2}-2)} \left(\int_{\mathbb{R}^N} v_0^{\frac{2\cdot 2^*\beta}{2^*-\frac{p}{2}+2}}\right)^{\frac{2^*-\frac{p}{2}+2}{2^*}}.$$

Taking $\beta = 1 + \frac{2^*-\frac{p}{2}}{2}$, we get $\frac{2\cdot 2^*\beta}{2^*-\frac{p}{2}+2} = 2^*$. Let $\delta := \frac{2\cdot 2^*}{2^*-\frac{p}{2}+2}$. Thus

$$\left(\int_{\mathbb{R}^N} |v_0 v_k^{\beta-1}|^{2^*}\right)^{2/2^*} \le C\beta^2 \|v_0\|^{(\frac{p}{2}-2)} \|v_0\|_{\beta\delta}^{2\beta}.$$

Using Fatou's Lemma in k, we have

$$\|v_0\|_{2^*\beta} \le \left(C\beta^2\|v_0\|^{(\frac{p}{2}-2)}\right)^{\frac{1}{2\beta}} \|v_0\|_{\beta\delta}. \tag{3.16}$$

For $m = 0, 1, 2, \ldots$, let $2^*\beta_m = \delta\beta_{m+1}$ with $\beta_0 = \beta$. Hence, similar to (3.16), for β_1, we know that

$$\begin{aligned}\|v_0\|_{2^*\beta_1} &\le \left(C\beta_1^2\|v_0\|^{(\frac{p}{2}-2)}\right)^{\frac{1}{2\beta_1}} \|v_0\|_{\beta_1\delta}\\ &\le \left(C\beta_1^2\|v_0\|^{(\frac{p}{2}-2)}\right)^{\frac{1}{2\beta_1}} \left(C\beta^2\|v_0\|^{(\frac{p}{2}-2)}\right)^{\frac{1}{2\beta}} \|v_0\|_{\beta\delta}\\ &\le \left(C\|v_0\|^{(\frac{p}{2}-2)}\right)^{\frac{1}{2\beta_1}+\frac{1}{2\beta}} \beta^{\frac{1}{\beta}} \beta_1^{\frac{1}{\beta_1}} \|v_0\|_{2^*}.\end{aligned}$$

Since $\beta_{m+1} = \beta_m \cdot \beta$, we know $\beta_m = \beta^m \cdot \beta$. Thus by iteration, we have

$$\|v_0\|_{2^*\beta_m} \le \left(C\|v_0\|^{(\frac{p}{2}-2)}\right)^{\frac{1}{2\beta}\sum_{i=0}^m \beta^{-i}} \beta^{\frac{1}{\beta}\sum_{i=0}^m \beta^{-i}} \beta^{\frac{1}{\beta}\sum_{i=0}^m i\beta^{-i}} \|v_0\|_{2^*}.$$

Using $\beta > 1$, we conclude that $\sum_{i=0}^m \beta^{-i}$ and $\sum_{i=0}^m i\beta^{-i}$. Thus letting $m \to \infty$, we have $v_0 \in L^\infty(\mathbb{R}^N)$ and

$$\|v_0\|_\infty \le C\|v_0\|^{\frac{2^*-2}{2^*-\frac{p}{2}}}.$$

By (V_1), (g_9) in Lemma 2.1 and (3.14), for all $\varphi \in C_0^\infty(\mathbb{R}^N)$, we get

$$\int_{\mathbb{R}^N} \nabla v_0 \nabla \varphi \le C\int_{\mathbb{R}^N} v_0 \varphi.$$

Thus using an elliptic estimate in [20], for $\iota > \frac{N}{2}$ and any ball $B_R(x)$ centered at any $x \in \mathbb{R}^N$, we have

$$\sup_{y\in B_R(x)} v_0(y) \le C\left[\|v_0\|_{L^2(B_{2R}(x))} + \|v_0\|_{L^\iota(B_{2R}(x))}\right].$$

Obviously,

$$v_0(x) \le C\left[\|v_0\|_{L^2(B_{2R}(x))} + \|v_0\|_{L^\iota(B_{2R}(x))}\right].$$

Since

$$\|v_0\|_{L^2(B_{2R}(x))} + \|v_0\|_{L^\iota(B_{2R}(x))} \to 0 \quad \text{as } |x| \to \infty,$$

it follows that

$$v_0(x) \to 0 \quad \text{as } |x| \to \infty.$$

At last, we give a proof of the exponential decay for v_0. By (FH_2) and since

$$\lim_{s\to 0} \frac{G^{-1}(s)}{sg(G^{-1}(s))} = 1,$$

we can choose $R_0 > 0$ such that for all $|x| \ge R_0$,

$$\frac{G^{-1}(v_0(x))}{g(G^{-1}(v_0(x)))} \ge \frac{3}{4}v_0(x), \tag{3.17}$$

and

$$\frac{h(x, G^{-1}(v_0(x)))}{g(G^{-1}(v_0(x)))} \le \frac{V_0}{2}v_0(x). \tag{3.18}$$

Now, we define

$$\chi(x) = M\exp(-\zeta|x|),$$

where ζ and M are such that $4\zeta^2 < V_0$ and for all $|x| = R_0$,

$$M\exp(-\zeta R_0) \ge v_0(x).$$

It is easy to check that for all $x \ne 0$,

$$\Delta\chi \le \zeta^2\chi. \tag{3.19}$$

Let $\vartheta = \chi - v_0$. Then it follows from (3.17)–(3.19) and

$$-\Delta v_0 + V(x)\frac{G^{-1}(v_0)}{g(G^{-1}(v_0))} = \frac{h(x, G^{-1}(v_0))}{g(G^{-1}(v_0))}, \qquad x \in \mathbb{R}^N,$$

that

$$\begin{aligned} -\Delta\vartheta + \frac{V_0}{4}\vartheta &\ge 0 \quad \text{in } |x| \ge R_0,\\ \vartheta &\ge 0 \quad \text{in } |x| = R_0,\\ \lim_{|x|\to\infty} \vartheta(x) &= 0. \end{aligned}$$

By the maximum principle, we know that $\vartheta(x) \ge 0$ for all $|x| \ge R_0$. Hence

$$\vartheta(x) \le M\exp(-\zeta|x|) \quad \text{for all } |x| \ge R_0,$$

which implies that

$$u_0 = G^{-1}(v_0) \le v_0(x) \le M\exp(-\zeta|x|) \quad \text{for all } x \in \mathbb{R}^N.$$

This completes the proof. □

Next, we shall use the Ekeland variational principle in [43] to prove Theorem 1.2. To this end, we prove the following lemma.

Lemma 3.6. *Suppose that (V_1)–(V_2) and (FH)–(FH_5) are satisfied. Then there exists $\psi \in E$ such that $\mathcal{J}_\lambda(t\psi) < 0$ for t enough small.*

Proof. To this end, by (g_{14}) in Lemma 2.1 and choosing $\psi \in (C_0^\infty(\mathbb{R}^N), [0,1]) \backslash \{0\}$ such that $\operatorname{supp} \psi = \overline{\Omega}$, we have

$$\begin{aligned}
\mathcal{J}_\lambda(t\psi) &\le \frac{1}{2}\int_{\mathbb{R}^N} \left(|t\nabla\psi|^2 + V(x)|G^{-1}(t\psi)|^2\right) - \int_{\mathbb{R}^N} H(x, G^{-1}(t\psi)) \\
&\le t^2 \int_{\overline{\Omega}} (|\nabla\psi|^2 + V(x)\psi^2) - C_1 \int_{\overline{\Omega}} |G^{-1}(t\psi)|^{q_1} \\
&\le t^2 \left[\int_{\overline{\Omega}} (|\nabla\psi|^2 + V(x)\psi^2) - C_1 t^{q_1-2} \int_{\overline{\Omega}} |G^{-1}(\psi)|^{q_1}\right] < 0,
\end{aligned}$$

where $|\overline{\Omega}|$ denotes the Lebesgue measure of $\overline{\Omega}$ and t enough small. This completes the proof. □

Proof of Theorem 1.2. By the previous proof, we know that $\mathcal{J}_\lambda$ is bounded in B_R for $R > 0$. By Lemma 3.6, we have

$$-\infty < b_\lambda := \inf_{B_R} \mathcal{J}_\lambda < 0.$$

Since $\mathcal{J}_\lambda$ satisfies (PS)-condition. By the Ekeland variational principle (see [43]) for $\mathcal{J}_\lambda$ in $\overline{B}_R$, there exists $\varpi_\lambda \in E$ such that for all $\lambda > 0$

$$\mathcal{J}_\lambda(\varpi_\lambda) = b_\lambda \quad \text{and} \quad \mathcal{J}'_\lambda(\varpi_\lambda) = 0.$$

Therefore $u_{\lambda,2} = G^{-1}(\varpi_\lambda)$ is a solution of (1.1).

Moreover, for $\lambda \in (0, \lambda_0]$, we have $\mathcal{J}_\lambda(\varpi_\lambda) < 0 < \eta \le \mathcal{J}_\lambda(v_\lambda)$, which shows that $u_{\lambda,1}$ is different from $u_{\lambda,2}$, where $(0, \lambda_0]$. □

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