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Pullback attractor for a nonlocal discrete nonlinear Schrödinger equation with delays

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Abstract. We consider a nonlocal discrete nonlinear Schrödinger equation with delays. We prove that the process associated with the non-autonomous model possesses a pullback attractor. As a consequence of our discussion, the existence of a global attractor for the autonomous model is derived.

Keywords: pullback attractor, discrete nonlinear Schrödinger equation, delay terms, global attractor differential equations, difference equations.

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1 Introduction

Discrete Schrödinger equations are widely used as models in Physics and other branches of science (see, e.g., [3, 6, 11, 12, 14, 19] and the references therein). These discrete equations belong to a large class of lattice dynamical systems which has been the object of extensive research (see, for example, [4,5,7,9,12,13,19,22] and the references therein). Various properties related to the dynamics of such systems have been studied. Among them, the existence of global attractors is a theme which attracts a great deal of attention. However, most of the contributions in this line of research addressed to discrete Schrödinger models are concerned the discrete nonlinear Schrödinger equation (DNLS). In this paper, our main aim is to prove the existence of a pullback attractor for a nonlocal discrete nonlinear Schrödinger equation when delay terms are considered. The model is written as follows

$$\begin{aligned} &i\dot{u}_n(t) + \sum_{m=-\infty}^{+\infty} J(n-m)u_m(t) + g_n(t, u_{nt}) + i\gamma u_n(t) = f_n(t), \quad t > \tau, n \in \mathbb{Z}, \\ &u_n(s) = \psi_n(s-\tau), \quad \forall s \in [\tau - h, \tau], \end{aligned} \tag{1.1}$$

where τ, h, and γ are real numbers with $h > 0$ and $\gamma > 0$. In (1.1), $u_n(t)$, $f_n(t)$, and ψ_n are complex functions and u_{nt} denotes the translation of u_n at time t, defined by $u_{nt}(s) = u_n(t+s), \forall s \in [-h, 0]$. The dispersive coupling parameters $J(m)$ are assumed to be real numbers, symmetric (i.e., $J(-m) = J(m)$, for all positive integer m) and $\sum_{m=1}^{+\infty} |J(m)| < +\infty$.

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This includes important special cases as $J(m) = J_0 e^{-\beta|m|}$ and $J(m) = J_0|m|^{-s}$, where J_0, β, and s are positive real constants suitably chosen [8].

We assume that the nonlinear term $g_n(t, u_{nt})$ in (1.1) includes delay terms as follows

$$g_n(t, u_{nt}) = g_{0,n}(u_n(t)) + g_{1,n}(u_n(t - \rho(t)) + \int_{-h}^{0} b_n(s, u_n(t+s))\, ds. \tag{1.2}$$

Appropriate hypotheses on the functions $\rho : \mathbb{R} \to [0, h]$, $g_{i,n} : \mathbb{C} \to \mathbb{C}, i = 0, 1$, $b_n : [0, h] \times \mathbb{C} \to \mathbb{C}$, and $f_n(t)$ are stated in Section 2.

Specific deterministic cases of equation (1.1) have been used in the study of physical phenomena in which long-range dispersive interactions cannot be disregarded (see the physical discussions in [8]). An example is the model proposed in [17] for the description of the nonlinear dynamics of the DNA molecule.

A class of discrete Schrödinger equations of great importance is

$$i\dot{u}_n(t) + \Delta_d^p u_n(t) + g_n(t, u_{nt}) + i\gamma u_n(t) = f_n(t), \tag{1.3}$$

where $\Delta_d^p = \Delta_d \circ \cdots \circ \Delta_d$, p times, and Δ_d is the one-dimensional discrete Laplace operator defined by $\Delta_d u_n = u_{n+1} + u_{n-1} - 2u_n$. Equation (1.3) can be derived from (1.1) by choosing the coupling parameters $J(m)$ as

$$J(m) = \sum_{j=0}^{2p} \binom{2p}{j} (-1)^j \delta_{m,j-p},$$

where p is any positive integer and $\delta_{m,k}$ is the Kronecker delta.

Many contributions on existence and properties of solutions of the DNLS equation (i.e, (1.3) with $p = 1$, $g_{1,n} = b_n = 0$) and f_n independent of time can be found in the literature (see, e.g., [3, 4, 11, 19] and references therein). For example, the existence and approximation of attractors for the DNLS equation were investigated in [11] while the existence of attractors for the DNLS with retarded terms was studied in [4]. Concerning equation (1.1), in [19], the authors studied the existence of localized solutions for the homogeneous case without delays. Later, also for the autonomous deterministic model, the existence of a global attractor in weighted spaces was established in [20]. For the existence of attractors for some non-autonomous lattice dynamical systems with retarded terms of the type (1.2) and references about related works we refer the reader to the article [2]. Still concerning lattice models with nonlocal terms, we would like to mention the papers [1, 10, 15, 18, 21].

In this paper, under suitable conditions on the functions ρ, $g_{i,n}, i = 0, 1$, b_n, and f_n, we prove the existence of a pullback attractor for the *process* associated with problem (1.1). As a consequence of our discussion, the existence of a global attractor for the autonomous model is derived.

The paper is organized as follows. In Section 2, we prove that the initial value problem (1.1) is globally well posed. In Section 3, we establish the existence of a pullback attractor for the *process* associated with problem (1.1) using the results in [16]. Finally, in Section 4, we briefly show how the same ideas of the previous sections can be adapted to prove the existence of a global attractor for the autonomous model.

2 Existence of solutions

In this section, we discuss the existence of solutions for the problem (1.1). We denote by ℓ^p the usual space of complex sequences $u = (u_n)_{n\in\mathbb{Z}}$ such that $\|u\|_{\ell^p} < \infty$, where

$$\|u\|_{\ell^p} = \left(\sum_{n=-\infty}^{+\infty} |u_n|^p\right)^{\frac{1}{p}}, \quad \text{if } 1 \le p < \infty \text{ and } \|u\|_{\ell^\infty} = \sup_{n\in\mathbb{Z}} |u_n|, \text{ if } p = \infty.$$

When $p = 2$, ℓ^2 is a Hilbert space with the inner product given by

$$(u,v)_{\ell^2} = \sum_{n=-\infty}^{+\infty} u_n \overline{v}_n, \quad u,v \in \ell^2,$$

and, in this case, we denote by $\|\cdot\|$ the corresponding norm.

For $1 \le p < \infty$, $L^p(-h,0)$ denotes the usual Banach space of (class of) real functions f defined on $[-h,0]$ such that $|f|^p$ is integrable in sense of Lebesgue and we recall that for the ℓ^p spaces the following embedding relation holds:

$$\ell^q \subset \ell^p, \quad \|u\|_{\ell^p} \le \|u\|_{\ell^q}, \quad 1 \le q \le p \le \infty.$$

Regarding the functions $g_{i,n} : \mathbb{C} \to \mathbb{C}, i = 0,1$, $b_n : [-h,0] \times \mathbb{C} \to \mathbb{C}$, $f = (f_n(t))_{n\in\mathbb{Z}}$, and $\rho(t)$ in (1.1) and (1.2) we assume that

(A1) $\overline{z} g_{0,n}(z)$ is real for all $z \in \mathbb{C}$ and $n \in \mathbb{Z}$.

(A2) There exist a function $\kappa \in L^2(-h,0)$ and functions $b_{0,n} : \mathbb{C} \to \mathbb{C}$ such that

$$|b_n(s,z_1) - b_n(s,z_2)| \le \kappa(s)|b_{0,n}(z_1) - b_{0,n}(z_2)|,$$

$\forall s \in [-h,0]$ and $\forall z_1, z_2 \in \mathbb{C}$. We set $\kappa_0^2 := \int_{-h}^0 |\kappa(s)|^2\, ds$.

(A3) For every $R > 0$ there exist positive constants $L_j(R), j = 1,2$, such that

$$|g_{i,n}(z_1) - g_{i,n}(z_2)| \le L_1(R)\, |z_1 - z_2|, \quad i = 0,1,$$
$$|b_{0,n}(z_1) - b_{0,n}(z_2)| \le L_2(R)\, |z_1 - z_2|,$$

for any $n \in \mathbb{Z}$ and any $z_1, z_2 \in \mathbb{C}$ such that $|z_j| \le R$, $j = 1,2$. Moreover, $(g_{0,n}(0))_{n\in\mathbb{Z}} \in \ell^2$.

(A4) There exist sequences of real numbers $k_1 = (k_{1,n})_{n\in\mathbb{Z}} \in \ell^\infty$, $k_2 = (k_{2,n})_{n\in\mathbb{Z}} \in \ell^2$ and non-negative real functions $\beta_{1,n}(\cdot) \in L^2(-h,0)$ and $\beta_{2,n}(\cdot) \in L^1(-h,0)$ such that

$$|g_{1,n}(z)| \le k_{1,n}|z| + k_{2,n} \quad \text{and} \quad |b_n(s,z)| \le \beta_{1,n}(s)|z| + \beta_{2,n}(s),$$

for all $n \in \mathbb{Z}$, $s \in [-h,0]$, and $z \in \mathbb{C}$. We set $K_1 = \|k_1\|_{\ell^\infty}$, $K_2 = \|k_2\|$, and

$$B_1 = \sup_{n\in\mathbb{Z}} \left(\int_{-h}^0 \beta_{1,n}^2(s)\, ds\right)^{1/2} < \infty,\ B_2 = \left[\sum_{n=-\infty}^{+\infty} \left(\int_{-h}^0 b_{2,n}(s)\, ds\right)^2\right]^{1/2} < \infty.$$

(A5) $f \in C(\mathbb{R}; \ell^2)$.

(A6) $\rho \in C(\mathbb{R}; [0,h])$.

(A7) $\int_{-\infty}^t \|f(s)\|^2\, ds < \infty$, $\forall t \in \mathbb{R}$.

Example 2.1. Let $0 \neq \chi = (\chi_n)_{n\in\mathbb{Z}} \in \ell^p$, for some $1 \leq p \leq \infty$, and $\varphi_1 : \mathbb{R} \to \mathbb{R}$ defined by $\varphi_1(t) = \frac{t^2}{a+bt^2}$, where a and b are positive real constants. Also define the functions $g_{1,n} : \mathbb{C} \to \mathbb{C}$, $b_{0,n} : \mathbb{C} \to \mathbb{C}$ and $b_n : [-h,0] \times \mathbb{C} \to \mathbb{C}$ by

$$g_{1,n}(z) = b_{0,n}(z) = \chi_n\, \varphi_1(|z|)\, z,$$

$$b_n(s,z) = \chi_n\, \varphi_1(|z|)\, z\frac{1}{h}(s+h), \quad \forall n \in \mathbb{Z},\ s \in [-h,0] \text{ and } z \in \mathbb{C}.$$

Then, the hypotheses (A2)–(A4) are satisfied with

$$L_1(R) = L_2(R) = \left(\frac{1}{b} + \frac{R}{\sqrt{ab}}\right) \|\chi\|_{\ell^p},$$

$$\kappa(s) = \frac{1}{h}(s+h), \quad k_{1,n} = \frac{1}{b}|\chi_n|, \quad k_{2,n} = 0, \quad \beta_{1,n}(s) = \frac{1}{bh}|\chi_n|(s+h), \quad \text{and} \quad \beta_{2,n} = 0.$$

Conditions (A1) and (A3) concerning $g_{0,n}$ are satisfied, for example, if $g_{0,n}(z) = \chi_n\, \varphi_2(|z|)z$, with χ_n as before and any $\varphi_2 \in C^1(\mathbb{R}^+;\mathbb{R})$, such that $\varphi_2(0) = 0$.

Now let us write (1.1) as an evolution equation with a retarded term in ℓ^2. For any $u = (u_n)_{n\in\mathbb{Z}}$ we define $(Au)_n = \sum_{m=-\infty}^{+\infty} J(n-m)u_m$, $\forall n \in \mathbb{Z}$.

Lemma 2.2. *$A : \ell^2 \to \ell^2$ is a bounded operator and $\|Au\| \leq 4\|J\|_{\ell^1}\|u\|$, $\forall u \in \ell^2$.*

Proof. See Lemma 2.1 in [20]. □

We consider the space $E_h = C([-h,0];\ell^2)$ with the usual norm given by $\|u\|_{E_h} = \max_{s\in[-h,0]} \|u(s)\|$ and define the map $g : \mathbb{R} \times E_h \to \ell^2$ by $(g(t,v))_{n\in\mathbb{Z}} = g_n(t,v_n)$, where $v(s) = (v_n(s))_{n\in\mathbb{Z}}$, for any $s \in [-h,0]$, and

$$g_n(t,v_n) = g_{0,n}(v_n(0)) + g_{1,n}(v_n(-\rho(t))) + \int_{-h}^{0} b_n(s,v_n(s))\, ds.$$

If we set $u_t = (u_{nt})_{n\in\mathbb{Z}}$ for any $t \geq \tau$, then we can write the initial value problem (1.1) in ℓ^2 as

$$\begin{aligned} &i\dot{u}(t) + Au(t) + g(t,u_t) + i\gamma u(t) = f(t), \quad t > \tau, \\ &u(s) = \psi(s-\tau), \quad \forall s \in [\tau-h,\tau], \end{aligned} \tag{2.1}$$

where $\psi(s) = (\psi_n(s))_{n\in\mathbb{Z}}$, for any $s \in [-h,0]$.

We now define the map $\mathcal{B} : \mathbb{R} \times E_h \to \ell^2$ by

$$\mathcal{B}(t,v) = -i\big[Av(0) + g(t,v) + i\gamma v(0) - f(t)\big].$$

Then, problem (2.1) can be rewritten as the following functional equation in ℓ^2

$$\begin{aligned} &\frac{du}{dt} + \mathcal{B}(t,u_t) = 0, \quad t > \tau \\ &u_\tau = \psi. \end{aligned} \tag{2.2}$$

The following two lemmas are sufficient to ensure the existence of a local solution for (2.1).

Lemma 2.3. *Assume that (A2)–(A6) hold. Then the map $\mathcal{B}$ is continuous and satisfies the local Lipschitz condition: For any $v, w \in E_h$, with $\|v\|_{E_h} \leq R$ and $\|w\|_{E_h} \leq R$, there exists a positive constant $L = L(R)$ such that*

$$\|\mathcal{B}(t,v) - \mathcal{B}(t,w)\| \leq L\,\|v - w\|_{E_h}, \quad \forall t \in \mathbb{R}.$$

Proof. Using (A2)–(A6) we see that $\mathcal{B}$ is well defined. Fix $(t,v) \in \mathbb{R} \times E_h$ and consider $t^m \to t$ in $\mathbb{R}$ and $v^m \to v$ in E_h. We have that

$$\begin{aligned}\|\mathcal{B}(t^m,v^m) - \mathcal{B}(t,v)\| \leq \|A(v^m(0) - v(0))\| + \|g(t^m,v^m) - g(t,v)\| \\ + \gamma\|v^m(0) - v(0)\| + \|f(t^m) - f(t)\|.\end{aligned} \tag{2.3}$$

Since the sequence $(v^m)_{m\in\mathbf{N}}$ is bounded in E_h, then using the assumptions (A2), (A3), and (A6) we can find a positive constant L depending only on $\|v\|_{E_h}$ such that

$$\begin{aligned}\|g(t^m,v^m) - g(t,v)\|^2 \leq 4 \sum_{n=-\infty}^{+\infty} |g_{0,n}(v_n^m(0)) - g_{0,n}(v_n(0))|^2 \\ + 4 \sum_{n=-\infty}^{+\infty} |g_{1,n}(v_n^m(-\rho(t^m))) - g_{1,n}(v_n(-\rho(t)))|^2 \\ + 4 \sum_{n=-\infty}^{+\infty} \left(\int_{-h}^{0} |b_n(s,v_n^m(s)) - b_n(s,v_n(s))|\,ds \right)^2 \\ \leq 8\,L^2\|v^m - v\|_{E_h}^2 + 4\,L^2 \sum_{n=-\infty}^{+\infty} \left(\int_{-h}^{0} |\kappa(s)|\,|v_n^m(s) - v_n(s)|\,ds \right)^2.\end{aligned} \tag{2.4}$$

Using the Cauchy–Schwarz inequality and the fact that $\|v^m - v\|_{E_h} < \infty$ we can estimate the last term in (2.4) as follows

$$\begin{aligned}\sum_{n=-\infty}^{+\infty} \left(\int_{-h}^{0} |\kappa(s)|\,|v_n^m(s) - v_n(s)|\,ds \right)^2 ds \leq \kappa_0^2 \sum_{n=-\infty}^{+\infty} \int_{-h}^{0} |v_n^m(s) - v(s)|^2\,ds \\ \leq \kappa_0^2 \int_{-h}^{0} \sum_{n=-\infty}^{+\infty} |v_n^m(s) - v(s)|^2\,ds \leq \kappa_0^2\|v^m - v\|_{E_h}^2\,h.\end{aligned} \tag{2.5}$$

From (2.3), (2.4), (2.5), (A5), and Lemma 2.2 we deduce the continuity of $\mathcal{B}$. In a similar manner we prove the Lipschitz condition. □

Lemma 2.4. *Assume that (A2)–(A6) hold. Then the map $\mathcal{B}$ is bounded, i.e., it takes bounded subsets of $\mathbb{R} \times E_h$ onto bounded subsets of ℓ^2.*

Proof. Let $\mathcal{O}$ be a bounded subset of $\mathbb{R} \times E_h$. Then, there exists a positive constant R such that $|t|^2 + \|v\|_{E_h}^2 \leq R^2$, $\forall (t,v) \in \mathcal{O}$. Using Lemma 2.3 we find a positive constant $L = L(R)$ such that

$$\begin{aligned}\|\mathcal{B}(t,v)\| &\leq \|\mathcal{B}(t,v) - \mathcal{B}(t,0)\| + \|\mathcal{B}(t,0)\| \\ &\leq LR + \max_{|t|\leq R} \|\mathcal{B}(t,0)\| < \infty, \quad \forall (t,v) \in \mathcal{O}.\end{aligned}$$ □

Using Lemmas 2.3, 2.4 and applying the Theory of Functional Equations to problem (2.2) we deduce the following result of existence of local solution for (2.1).

Theorem 2.5. *Assume that (A2)–(A6) hold. Then, for each $\psi \in E_h$, the initial value problem (2.1) has a unique solution $u = u(t)$ defined in $[\tau - h, T)$ such that $u \in C([\tau - h, T); \ell^2) \cap C^1([\tau, T); \ell^2)$. Moreover, if $T < \infty$ then $\lim_{t \to T^-} \|u(t)\| = \infty$.*

Next let us show that the local solution obtained in Theorem 2.5 can be extended globally.

Lemma 2.6. *Assume that (A1)–(A6) hold. Then the solution u of (2.1) with $u_\tau = \psi \in E_h$ satisfies*

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u(t)\|^2 + \frac{\gamma}{2}\|u(t)\|^2 &\le \frac{1}{2\gamma}\|f(t)\|^2 + (K_1\|u(t-\rho(t))\| + K_2)\|u(t)\| \\ &\quad + \left[B_1\left(\int_{-h}^0 \|u(t+s)\|^2\,ds\right)^{1/2} + B_2\right]\|u(t)\|, \quad \tau \le t < T.\end{aligned} \tag{2.6}$$

Proof. Taking the imaginary part of the inner product of equation (2.1) with u in ℓ^2, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2 + \mathrm{Im}(Au(t), u(t))_{\ell^2} + \gamma\|u(t)\|^2 + \mathrm{Im}(g(t, u_t), u(t))_{\ell^2} = \mathrm{Im}(f(t), u(t))_{\ell^2},$$

for all $\tau \le t < T$. Since

$$\mathrm{Im}(f(t), u(t))_{\ell^2} \le \frac{1}{2\gamma}\|f(t)\|^2 + \frac{\gamma}{2}\|u(t)\|^2,$$

$$(Au(t), u(t))_{\ell^2} = J(0)\|u(t)\|^2 + 2\sum_{m=1}^{+\infty}\sum_{n=-\infty}^{+\infty} J(m)\,\mathrm{Re}(\overline{u_{n+m}(t)}u_n(t)),$$

then, using (A1), we get the inequality

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u(t)\|^2 + \frac{\gamma}{2}\|u(t)\|^2 &\le \frac{1}{2\gamma}\|f(t)\|^2 - \mathrm{Im}\sum_{n=-\infty}^{+\infty} g_{1,n}(u_n(t-\rho(t)))\bar{u}_n \\ &\quad - \mathrm{Im}\sum_{n=-\infty}^{+\infty}\int_{-h}^0 b_n(s, u_n(t+s))\,ds\,\bar{u}_n, \quad \tau \le t < T.\end{aligned} \tag{2.7}$$

Let us estimate the last two terms in (2.7) using the assumption (A4) and the fact that $\|u_t\|_{E_h} < \infty$, $\forall \tau \le t < T$. We have that

$$\begin{aligned}-\mathrm{Im}\sum_{n=-\infty}^{+\infty} g_{1,n}(u_n(t-\rho(t)))\bar{u}_n &\le \sum_{n=-\infty}^{+\infty}[k_{1,n}|u_n(t-\rho(t))| + k_{2,n}]|u_n| \\ &\le (K_1\|u(t-\rho(t))\| + K_2)\|u\|,\end{aligned} \tag{2.8}$$

$$\begin{aligned}-\mathrm{Im}\sum_{n=-\infty}^{+\infty}\int_{-h}^0 b_n(s, u_n(t+s))\,ds\,\bar{u}_n &\le \sum_{n=-\infty}^{+\infty}\int_{-h}^0[\beta_{1,n}(s)|u_n(t+s)| + \beta_{2,n}(s)]\,ds|u_n| \\ &\le \left[\sum_{n=-\infty}^{+\infty}\left(\int_{-h}^0 \beta_{1,n}^2(s)\,ds\right)\left(\int_{-h}^0 |u_n(t+s)|^2\,ds\right)\right]^{1/2}\|u\| + B_2\|u\| \\ &\le \left[B_1\left(\int_{-h}^0 \|u(t+s)\|^2\,ds\right)^{1/2} + B_2\right]\|u\|.\end{aligned} \tag{2.9}$$

From (2.7)–(2.9) we obtain (2.6). □

We now make the following assumptions on the constants B_1, K_1, γ, h, and a suitable positive parameter μ, which will be used in Section 3 to define the universe where the pullback attractor will lie in.

(A8) We assume that there exists a positive real number μ such that

(i) If $K_1 > 0$ and $B_1 \geq 0$ then

$$4B_1^2 h < e^{-\mu h}\gamma\left(\frac{\gamma}{2} - \mu\right) \tag{2.10}$$

and

$$\mu > 2K_1 e^{\mu h}. \tag{2.11}$$

(ii) If $K_1 = 0$ and $B_1 > 0$ then

$$\mu < \frac{\gamma}{2} \quad \text{and} \quad \mu > \frac{4}{\gamma} B_1^2 e^{2\mu h} h. \tag{2.12}$$

(iii) If $K_1 = B_1 = 0$ then $\mu = \frac{\gamma}{2}$ and h is arbitrary.

Remark 2.7. Conditions in (A8) will be used in the next theorem to prove an estimate for the solution of (2.1) that allows us to extend it globally and that will be used in the proofs of Lemmas 3.1, 3.2 and 3.3 in Section 3. It is clear from (2.10) that $\mu < \frac{\gamma}{2}$. We also observe that (2.11) holds if and only if $0 < 2K_1 < \frac{1}{he}$, where $\frac{1}{he}$ is the maximum value of the real function $\phi(s) = se^{-hs}$, $s \geq 0$. From this we see that $2K_1 eh < 1$ and $\mu \in (\mu_1, \mu_2)$, where $\mu_j, j = 1, 2$, are the two positive solutions of the equation $\mu e^{-\mu h} = 2K_1$.

Theorem 2.8. *Assume that (A1)–(A8) hold. Then, the solution $u = u(t)$ of (2.1) with $u_\tau = \psi \in E_h$ exists globally. Moreover, for each $\tau < T < \infty$, the map $\Im : E_h \to C([\tau, T]; E_h)$, defined by $\Im(\psi)(t) = u_t$, $\forall \tau \leq t \leq T$, is continuous.*

Proof. Assume that (A8)(i) holds. Multiplying (2.6) by $e^{\mu t}$ and integrating the resulting inequality over $[\tau, t]$ we have, for any positive real constants ε and ε',

$$\begin{aligned} e^{\mu t}\|u(t)\|^2 \leq{}& e^{\mu\tau}\|\psi\|_{E_h}^2 + (\mu - \gamma + \varepsilon + \varepsilon')\int_\tau^t e^{\mu s}\|u(s)\|^2\, ds + \frac{1}{\gamma}\int_\tau^t e^{\mu s}\|f(s)\|^2\, ds \\ &+ \left(\frac{2B_2^2}{\varepsilon} + \frac{K_2^2}{\varepsilon'}\right)\frac{e^{\mu t}}{\mu} + 2K_1\int_\tau^t e^{\mu s}\|u_s\|_{E_h}^2\, ds \\ &+ \frac{2B_1^2}{\varepsilon}\int_\tau^t\int_{-h}^0 e^{\mu t'}\|u(t'+s)\|^2\, ds\, dt'. \end{aligned} \tag{2.13}$$

Let us estimate the last term in (2.13) using the initial condition in (2.1). We have

$$\begin{aligned} \int_\tau^t\int_{-h}^0 e^{\mu t'}\|u(t'+s)\|^2\, ds\, dt' &= \int_{-h}^0\int_\tau^t e^{-\mu s}e^{\mu(t'+s)}\|u(t'+s)\|^2\, dt'\, ds \\ &\leq e^{\mu h}\int_{-h}^0\int_{\tau-h}^t e^{\mu\sigma}\|u(\sigma)\|^2 d\sigma\, ds \\ &= e^{\mu h}h\left[\int_{\tau-h}^\tau e^{\mu\sigma}\|u(\sigma)\|^2 d\sigma + \int_\tau^t e^{\mu\sigma}\|u(\sigma)\|^2 d\sigma\right] \\ &\leq \frac{e^{\mu(\tau+h)}h}{\mu}\|\psi\|_{E_h}^2 + e^{\mu h}h\int_\tau^t e^{\mu\sigma}\|u(\sigma)\|^2 d\sigma. \end{aligned} \tag{2.14}$$

Substituting (2.14) into (2.13) we get

$$\begin{aligned} e^{\mu t}\|u(t)\|^2 \le e^{\mu\tau}\|\psi\|_{E_h}^2 &+ \left(\mu - \gamma + \varepsilon + \varepsilon' + \frac{2B_1^2 e^{\mu h} h}{\varepsilon}\right)\int_\tau^t e^{\mu s}\|u(s)\|^2\, ds \\ &+ \frac{2B_1^2 e^{\mu h} h}{\mu\varepsilon} e^{\mu\tau}\|\psi\|_{E_h}^2 + \left(\frac{2B_2^2}{\varepsilon} + \frac{K_2^2}{\varepsilon'}\right)\frac{e^{\mu t}}{\mu} \\ &+ \frac{1}{\gamma}\int_\tau^t e^{\mu s}\|f(s)\|^2\, ds + 2K_1\int_\tau^t e^{\mu s}\|u_s\|_{E_h}^2\, ds. \end{aligned} \tag{2.15}$$

Using (2.10) we can choose $\varepsilon = \frac{\gamma}{2}$ and

$$\varepsilon' = \frac{\gamma}{2} - \mu - \frac{4B_1^2 e^{\mu h} h}{\gamma} \tag{2.16}$$

in (2.15) to obtain

$$\begin{aligned} e^{\mu t}\|u(t)\|^2 \le e^{\mu\tau}\left(1 + \frac{4B_1^2 e^{\mu h} h}{\mu\gamma}\right)\|\psi\|_{E_h}^2 &+ \left(\frac{4B_2^2}{\gamma} + \frac{K_2^2}{\varepsilon'}\right)\frac{e^{\mu t}}{\mu} \\ + \frac{1}{\gamma}\int_\tau^t e^{\mu s}\|f(s)\|^2\, ds &+ 2K_1\int_\tau^t e^{\mu s}\|u_s\|_{E_h}^2\, ds. \end{aligned} \tag{2.17}$$

Since $\|u(s)\| \le \|\psi\|_{E_h}$, $\forall s \in [\tau - h, \tau]$, then we can replace t in (2.17) by $t + \sigma$, with $\sigma \in [-h, 0]$, to deduce that

$$e^{\mu t}\|u_t\|_{E_h}^2 \le M(t) + L\int_\tau^t e^{\mu s}\|u_s\|_{E_h}^2\, ds,$$

where $L = 2K_1 e^{\mu h}$ and

$$M(t) = e^{\mu(\tau+h)}\left(1 + \frac{4B_1^2 e^{\mu h} h}{\mu\gamma}\right)\|\psi\|_{E_h}^2 + \left(\frac{4B_2^2}{\gamma} + \frac{K_2^2}{\varepsilon'}\right)\frac{e^{\mu(t+h)}}{\mu} + \frac{e^{\mu h}}{\gamma}\int_\tau^t e^{\mu s}\|f(s)\|^2\, ds.$$

The above inequality implies that

$$e^{\mu t}\|u_t\|_{E_h}^2 \le e^{L(t-\tau)}M(\tau) + e^{Lt}\int_\tau^t e^{-Ls}M'(s)\, ds. \tag{2.18}$$

Performing the calculations in (2.18) using $M(t)$ above and the fact that $\mu > L$ by (2.11), we find the following estimate for the solution of (2.1)

$$\|u_t\|_{E_h}^2 \le c_1\|\psi\|_{E_h}^2 e^{(L-\mu)t}e^{(\mu-L)\tau} + \frac{2\mu - L}{\mu - L}c_2 + \frac{e^{\mu h}}{\gamma}\int_{-\infty}^t \|f(s)\|^2\, ds, \tag{2.19}$$

where

$$c_1 = e^{\mu h}\left(1 + \frac{4B_1^2 e^{\mu h} h}{\mu\gamma}\right) \quad \text{and} \quad c_2 = \left(\frac{4B_2^2}{\gamma} + \frac{K_2^2}{\varepsilon'}\right)\frac{e^{\mu h}}{\mu}. \tag{2.20}$$

Now, assume that (A8)(ii) holds. For this case we replace (2.14) by

$$\begin{aligned} e^{\mu t}\|u(t)\|^2 \le e^{\mu\tau}\|\psi\|_{E_h}^2 &+ (\mu - \gamma + \varepsilon + \varepsilon')\int_\tau^t e^{\mu s}\|u(s)\|^2\, ds \\ &+ \frac{2B_1^2 e^{\mu h} h}{\mu\varepsilon} e^{\mu\tau}\|\psi\|_{E_h}^2 + \left(\frac{2B_2^2}{\varepsilon} + \frac{K_2^2}{\varepsilon'}\right)\frac{e^{\mu t}}{\mu} \\ &+ \frac{1}{\gamma}\int_\tau^t e^{\mu s}\|f(s)\|^2\, ds + \frac{2B_1^2 e^{\mu h} h}{\varepsilon}\int_\tau^t e^{\mu s}\|u_s\|_{E_h}^2\, ds. \end{aligned} \tag{2.21}$$

Since that $\mu < \frac{\gamma}{2}$, then we can choose $\varepsilon = \frac{\gamma}{2}$ and $\varepsilon' = \frac{\gamma}{2} - \mu$ in (2.21) and proceed as before to obtain

$$\begin{aligned} e^{\mu t}\|u(t)\|^2 \leq e^{\mu(\tau+h)}\left(1+\frac{4B_1^2 e^{\mu h}h}{\mu\gamma}\right)\|\psi\|_{E_h}^2 + \left(\frac{4B_2^2}{\gamma}+\frac{K_2^2}{\varepsilon'}\right)\frac{e^{\mu(t+h)}}{\mu} \\ + \frac{e^{\mu h}}{\gamma}\int_\tau^t e^{\mu s}\|f(s)\|^2\,ds + L\int_\tau^t e^{\mu s}\|u_s\|_{E_h}^2\,ds, \end{aligned} \tag{2.22}$$

where $L = \frac{4}{\gamma}B_1^2 e^{2\mu h}h$. By (2.12) we see that $\mu > L$. Therefore, we can deduce the estimate (2.19) with c_1 and c_2 as in (2.20), with $\varepsilon' = \frac{\gamma}{2} - \mu$. Similarly, we can treat the case (A8)(iii) to obtain the estimate

$$\|u_t\|_{E_h}^2 \leq c_1'\|\psi\|_{E_h}^2 e^{-\mu t}e^{\mu\tau} + 2c_2' + \frac{e^{\mu h}}{\gamma}\int_{-\infty}^t \|f(s)\|^2\,ds, \tag{2.23}$$

where

$$c_1' = 2e^{\mu h} \text{ and } c_2' = \frac{e^{\mu h}}{\mu^2}\left(B_2^2 + K_2^2\right). \tag{2.24}$$

From (2.19) or (2.23) and Theorem 2.5 we conclude that the solution of (2.1) exists globally. Next, let us prove that the map $\mathfrak{I}$ is continuous. Fix $\tau < T < \infty, \psi \in E_h$ and consider $\psi_1 \in E_h$ such that $\|\psi - \psi_1\|_{E_h} < 1$. Let us denote by $v = v(t)$ the solution of (2.1) with initial condition $v(s) = \psi_1(s-\tau)$, $\forall s \in [\tau - h, \tau]$. Using the estimate (2.19) or (2.23) we can find a positive constant K_0 depending on $\|\psi\|_{E_h}$ and T such that $\|u_t\|_{E_h} \leq K_0$ and $\|v_t\|_{E_h} \leq K_0$, for all $\tau \leq t \leq T$. Then, using the integral representations of u and v and Lemma 2.3 it follows that

$$\begin{aligned} \|u(t) - v(t)\| &\leq \|\psi(0) - \psi_1(0)\| + \int_\tau^t \|\mathcal{B}(s,u_s) - \mathcal{B}(s,v_s)\|\,ds \\ &\leq \|\psi - \psi_1\|_{E_h} + L(K_0)\int_\tau^t \|u_s - v_s\|_{E_h}\,ds. \end{aligned} \tag{2.25}$$

Replacing t in (2.25) by $t+\sigma$, with $\sigma \in [-h, 0]$, taking into account that $\|u(t+\sigma) - v(t+\sigma)\|_{E_h} \leq \|\psi - \psi_1\|_{E_h}$ if $t+\sigma \leq \tau$, we obtain

$$\|u_t - v_t\|_{E_h} \leq \|\psi - \psi_1\|_{E_h} + L(K_0)\int_\tau^t \|u_s - v_s\|_{E_h}\,ds, \qquad \forall \tau \leq t \leq T.$$

Then, by Gronwall's inequality, we conclude that $\|u_t - v_t\|_{E_h} \leq e^{L(K_0)(T-\tau)}\|\psi - \psi_1\|_{E_h}$, which implies the continuity of $\mathfrak{I}$. □

3 Existence of a pullback attractor

By Theorem 2.8 we can associate to the initial value problem (2.1) a process $\{U(t,\tau)\}_{t\geq\tau}$ of continuous maps $U(t,\tau)$ in E_h defined by $U(t,\tau)\psi = u_t$, where $\tau \leq t$ and $u = u(t)$ is the global solution of (2.1). In this section, we establish the existence of a pullback attractor for the process $\{U(t,\tau)\}_{t\geq\tau}$ using the results obtained in [16]. We are interested in the existence of a pullback attractor for a family of sets depending on time (see [16, Section 3]). Motivated by the estimate (2.19) we consider the set $\mathcal{R}_\mu$ of all functions $r : \mathbb{R} \to (0,\infty)$ such that

$$\lim_{t\to-\infty} e^{(\mu-L)t}r^2(t) = 0. \tag{3.1}$$

Let us denote by $\mathcal{D}_\mu$ the class of all families $\hat{D} = \{D(t); t \in \mathbb{R}\}$ of nonempty subsets of E_h such that $D(t) \subset B_{E_h}[0; r_{\hat{D}}(t)] := \{\psi \in E_h; \|\psi\|_{E_h} \le r_{\hat{D}(t)}\}$, for some radius $r_{\hat{D}} \in \mathcal{R}_\mu$. For the case (A8)(iii) we consider in (3.1) $L = 0$. In what follows, we will assume that (A8)(i) or (A8)(ii) holds. Suitable modifications will be indicated for the case (A8)(iii). We will also consider L as in the proof of Theorem 2.8 and the constants c_1, c_2, c_1' and c_2' given by (2.20) and (2.24).

Lemma 3.1. *Assume that (A1)–(A8) hold. Then, the family $\hat{B}_\mu$ of closed balls $B_\mu(t) = B_{E_h}[0; R_\mu(t)]$, where for each $t \in \mathbb{R}$, the radius $R_\mu(t)$ is defined by*

$$R_\mu^2(t) = \frac{2\mu - L}{\mu - L} c_2 + \frac{e^{\mu h}}{\gamma} \int_{-\infty}^{t} \|f(s)\|^2 \, ds + 1, \tag{3.2}$$

is pullback $\mathcal{D}_\mu$-absorbing for the process $\{U(t,\tau)\}_{t \ge \tau}$.

Proof. Since $\mu > L$, then using (A7), we have

$$\lim_{t \to -\infty} e^{(\mu - L)t} R_\mu^2(t) = \lim_{t \to -\infty} e^{(\mu - L)t} \left(\frac{2\mu - L}{\mu - L} c_2 + \frac{e^{\mu h}}{\gamma} \int_{-\infty}^{t} \|f(s)\|^2 \, ds + 1 \right) = 0,$$

which shows that $\hat{B}_\mu \in \mathcal{D}_\mu$. Now, fixed $t \in \mathbb{R}$ and $\hat{D} \in \mathcal{D}_\mu$, there exists a $\tau_0 = \tau_0(t, \hat{D}) \le t$ such that

$$e^{(\mu - L)\tau} r_{\hat{D}}^2(\tau) < c_1^{-1} e^{(\mu - L)t},$$

for any $\tau \le \tau_0$. Then, for any $\psi \in D(\tau)$, using (2.19) we obtain

$$\begin{aligned} \|U(t,\tau)\psi\|_{E_h}^2 &\le c_1 r_{\hat{D}}^2(\tau) e^{(\mu - L)\tau} e^{(L - \mu)t} + \frac{2\mu - L}{\mu - L} c_2 + \frac{e^{\mu h}}{\gamma} \int_{-\infty}^{t} \|f(s)\|^2 \, ds \\ &\le R_\mu^2(t). \end{aligned}$$

Therefore, $U(t,\tau)D(\tau) \subset B_\mu(t)$, for all $\tau \le \tau_0$, which proves that the family $\hat{B}_\mu$ is pullback $\mathcal{D}_\mu$-absorbing for the process $\{U(t,\tau)\}_{t \ge \tau}$. □

In Lemma 3.1, in the case (A8)(iii), we take $L = 0$ and replace c_2 by c_2' in (3.2). Next, let us prove an estimate for the tails of the solutions $u = u(t)$ of (2.1) when the initial conditions $u_\tau = \psi$ belong to $B_\mu(\tau)$.

Lemma 3.2. *Assume that (A1)–(A8) hold. Let $\hat{B}_\mu$ be the pullback $\mathcal{D}_\mu$-absorbing family defined in Lemma 3.1. Then, for any $\varepsilon > 0$ and any $t' < T$, there exist $\tau_0 = \tau_0(\varepsilon, t', T, \hat{B}_\mu)$ and a positive integer $k = k(\varepsilon, T, \hat{B}_\mu)$, such that*

$$\max_{s \in [-h,0]} \sum_{|n| > 2k} |u_n(t+s)|^2 < \varepsilon, \, \forall \tau \le \tau_0, \; t \in [t', T],$$

for any solution $u = u(t)$ of (2.1) with initial condition $u_\tau \in B_\mu(\tau)$.

Proof. Assume that (A8)(i) holds. Similarly, we treat the case (A8)(ii). Let $u_\tau = \psi \in B_\mu(\tau)$ and consider the corresponding solution $u = u(t)$ of (2.1) defined in $[\tau, \infty)$. Let $\theta \in C^1(\mathbb{R}^+; \mathbb{R})$ be a function such that $\theta \equiv 0$ on $[0,1]$, $\theta \equiv 1$ on $[2, \infty)$, $0 \le \theta \le 1$, and $|\theta'(t)| \le 2, \forall t \ge 0$. Let $v = (v_n(t))_{n \in \mathbb{Z}}$, where $v_n(t) = \theta(\frac{|n|}{k}) u_n(t)$, with $k > 0$ fixed in $\mathbb{Z}$. In order to simplify notation, we will write $\theta_n = \theta(\frac{|n|}{k})$, $\|w\|_\theta = \sum_{n=-\infty}^{+\infty} \theta_n |w_n|^2$ and $\|u_t\|_{E_{h,\theta}}^2 = \max_{s \in [-h,0]} \|u_t(s)\|_\theta^2$. Taking the imaginary part of the inner product of equation (2.1) with v in ℓ^2 we find

$$\frac{1}{2} \frac{d}{dt} (u, v)_{\ell^2} + \gamma (u, v)_{\ell^2} = \operatorname{Im}(f, v)_{\ell^2} - \operatorname{Im}(Au, v)_{\ell^2} - \operatorname{Im}(g(t, u_t), v)_{\ell^2}, \quad \forall t \ge \tau. \tag{3.3}$$

Let us estimate the terms on the right-hand side of (3.3). Since $\psi \in B_\mu(\tau)$ then, using (2.19), we see that

$$\|u(t)\| \leq r_0, \qquad \forall t \in [\tau, T],$$

with $r_0 = (c_1 + 1)R_\mu(T)$. Moreover, by the definition of θ, we have that $|\theta_{n+m} - \theta_n| \leq \frac{2}{k}m$ and $|\theta_{n+m} - \theta_n| \leq 2$. Then,

$$\begin{aligned}
-\operatorname{Im}(Au(t), v(t))_{\ell^2} &= -\operatorname{Im}\left\{ J(0)\|u(t)\|_\theta^2 + \sum_{n=-\infty}^{+\infty}\sum_{m=1}^{+\infty} J(m)(\theta_{n+m} - \theta_n)\overline{u_{n+m}(t)}u_n(t) \right\} \\
&\leq \sum_{n=-\infty}^{+\infty}\sum_{m=1}^{+\infty} |J(m)|\,|\theta_{n+m} - \theta_n|\,|u_{n+m}(t)|\,|u_n(t)| \leq \nu(T,k,l),
\end{aligned}$$

where $\nu(T,k,l) = \left(\frac{2}{k}\sum_{m=1}^{l} m|J(m)| + 2\sum_{m=l+1}^{+\infty}|J(m)|\right)r_0^2,\ l \geq 1.$

Using the hypotheses (A1) and (A4) and proceeding as in the proof of Lemma 2.6 we obtain the estimate

$$\begin{aligned}
-\operatorname{Im}(g(t,u_t), v(t))_{\ell^2} &\leq \sum_{n=-\infty}^{+\infty} \theta_n |g_{1,n}(t, u_n(t-\rho(t)))|\,|u_n(t)| \\
&\quad + \sum_{n=-\infty}^{+\infty} \theta_n \int_{-h}^{0} |b_n(s, u_n(t+s))|\,ds|u_n(t)| \\
&\leq (K_1\|u(t-\rho(t))\|_\theta + K_{2,\theta})\|u\|_\theta \\
&\quad + \left[B_1 \left(\int_{-h}^{0} \|u(t+s)\|_\theta^2\,ds \right)^{1/2} + B_{2,\theta} \right] \|u\|_\theta,
\end{aligned}$$

where $B_{2,\theta} = \left[\sum_{n=-\infty}^{+\infty}\theta_n\left(\int_{-h}^{0}\beta_{2,n}(s)\,ds\right)^2\right]^{1/2}$ and $K_{2,\theta} = \left(\sum_{n=-\infty}^{+\infty}\theta_n k_{2,n}^2\right)^{1/2}$.

In addition, we know that

$$-\operatorname{Im}(f(t), v(t))_{\ell^2} \leq \frac{1}{2\gamma}\|f(t)\|_\theta^2 + \frac{\gamma}{2}\|u(t)\|_\theta^2.$$

Therefore,

$$\begin{aligned}
\frac{d}{dt}\|u(t)\|_\theta^2 + \gamma\|u(t)\|_\theta^2 &\leq \frac{1}{\gamma}\|f\|_\theta^2 + 2\left(K_1\|u(t-\rho(t))\|_\theta + K_{2,\theta}\right)\|u(t)\|_\theta \\
&\quad + 2\left[B_1 \left(\int_{-h}^{0} \|u(t+s)\|_\theta^2\,ds \right)^{1/2} + B_{2,\theta} \right] \|u(t)\|_\theta + 2\nu(T,k,l),
\end{aligned} \tag{3.4}$$

for all $\tau \leq t \leq T$.

Now, we multiply (3.4) by $e^{\mu t}$ and use the inequalities

$$2\left[B_1 \left(\int_{-h}^{0} \|u(t+s)\|_\theta^2\,ds \right)^{1/2} + B_{2,\theta} \right] \|u\|_\theta \leq \frac{4B_1^2}{\gamma}\int_{-h}^{0}\|u(t+s)\|_\theta^2\,ds + \frac{4B_{2,\theta}^2}{\gamma} + \frac{\gamma}{2}\|u\|_\theta^2,$$

$$2\left(K_1\|u(t-\rho(t))\|_\theta + K_{2,\theta}\right)\|u(t)\|_\theta \leq 2K_1\|u_t\|_{E_{h,\theta}}^2 + \frac{K_{2,\theta}^2}{\varepsilon'} + \varepsilon'\|u\|_\theta^2,$$

where $\varepsilon' > 0$, to find

$$\frac{d}{dt}\left(e^{\mu t}\|u(t)\|_\theta^2\right) \le \left(\mu - \frac{\gamma}{2} + \varepsilon'\right) e^{\mu t}\|u(t)\|_\theta^2 + \frac{1}{\gamma} e^{\mu t}\|f(t)\|_\theta^2 + 2K_1 e^{\mu t}\|u_t\|_{E_{h,\theta}}^2 + \left(\frac{4B_{2,\theta}^2}{\gamma} + \frac{K_{2,\theta}^2}{\varepsilon'}\right) e^{\mu t} + 2\nu(T,k,l)e^{\mu t} + \frac{4B_1^2 e^{\mu t}}{\gamma}\int_{-h}^{0}\|u(t+s)\|_\theta^2\,ds, \qquad \forall \tau \le t \le T. \tag{3.5}$$

Integrating (3.5) over $[\tau, t]$ and using the following estimate analogous to (2.14)

$$\int_\tau^t \int_{-h}^0 e^{\mu t'}\|u(t+s)\|_\theta^2\,ds\,dt' \le \frac{e^{\mu(\tau+h)}h}{\mu}\|\psi\|_{E_h}^2 + e^{\mu h}h\int_\tau^t e^{\mu s}\|u(s)\|_\theta^2\,ds,$$

we obtain

$$e^{\mu t}\|u(t)\|_\theta^2 \le e^{\mu\tau}\left(1 + \frac{4B_1^2 e^{\mu h}h}{\mu\gamma}\right)\|\psi\|_{E_h}^2 + \left(\mu - \frac{\gamma}{2} + \varepsilon' + \frac{4B_1^2 e^{\mu h}h}{\gamma}\right)\int_\tau^t e^{\mu s}\|u(s)\|_\theta^2\,ds + \left(\frac{4B_{2,\theta}^2}{\gamma} + \frac{K_{2,\theta}^2}{\varepsilon'} + 2\nu(T,k,l)\right)\frac{e^{\mu t}}{\mu} + 2K_1\int_\tau^t e^{\mu s}\|u_s\|_{E_{h,\theta}}^2\,ds + \frac{1}{\gamma}\int_\tau^t e^{\mu s}\|f(s)\|_\theta^2\,ds.$$

By condition (2.10) we can choose ε' as in (2.16) in the above inequality to obtain

$$e^{\mu t}\|u(t)\|_\theta^2 \le e^{\mu\tau}\left(1 + \frac{4B_1^2 e^{\mu h}h}{\mu\gamma}\right)\|\psi\|_{E_h}^2 + \left(\frac{4B_{2,\theta}^2}{\gamma} + \frac{K_{2,\theta}^2}{\varepsilon'} + 2\nu(T,k,l)\right)\frac{e^{\mu t}}{\mu} + 2K_1\int_\tau^t e^{\mu s}\|u_s\|_{E_{h,\theta}}^2\,ds + \frac{1}{\gamma}\int_\tau^t e^{\mu s}\|f(s)\|_\theta^2\,ds. \tag{3.6}$$

Replacing t by $t+\sigma$, with $\sigma \in [-h, 0]$ in (3.6) and using the inequality $\|u(t+\sigma)\| = \|\psi(t+\sigma)\| \le \|\psi\|_{E_h}$, valid for $t+\sigma < \tau$, we deduce that

$$e^{\mu t}\|u_t\|_{E_{h,\theta}}^2 \le M_\theta(t) + L\int_\tau^t e^{\mu s}\|u_s\|_{E_{h,\theta}}^2\,ds, \tag{3.7}$$

where $L = 2K_1 e^{\mu h}$ and

$$M_\theta(t) = e^{\mu(\tau+h)}\left(1 + \frac{4B_1^2 e^{\mu h}h}{\mu\gamma}\right)\|\psi\|_{E_h}^2 + \left(\frac{4B_{2,\theta}^2}{\gamma} + \frac{K_{2,\theta}^2}{\varepsilon'} + 2\nu(T,k,l)\right)\frac{e^{\mu(t+h)}}{\mu} + \frac{e^{\mu h}}{\gamma}\int_\tau^t e^{\mu s}\|f(s)\|_\theta^2\,ds.$$

We know that $\mu > L$. Then, from (3.7) and $\psi \in B_\mu(\tau)$, we obtain

$$\|u_t\|_{E_{h,\theta}}^2 \le c_1 R_\mu^2(\tau)e^{(L-\mu)t}e^{(\mu-L)\tau} + \frac{2\mu - L}{\mu - L}c_{2,\theta} + \frac{2(2\mu - L)}{\mu(\mu - L)}e^{\mu h}\nu(T,k,l) + \frac{e^{\mu h}}{\gamma}\int_{-\infty}^t \|f(s)\|_\theta^2\,ds, \qquad \forall t \ge \tau, \tag{3.8}$$

where $c_{2,\theta} = \left(\frac{4B_{2,\theta}^2}{\gamma} + \frac{K_{2,\theta}^2}{\varepsilon'}\right)\frac{e^{\mu h}}{\mu}$. Similarly, if (A8)(ii) holds, we obtain (3.8) with $L = \frac{4}{\gamma}B_1^2 e^{2\mu h}h$.

To conclude the proof, let $\varepsilon > 0$ be given. Since $\hat{B}_\mu \in \mathcal{D}_\mu$ and $\sum_{m=1}^{\infty} |J(m)| < \infty$, then there exist $\tau_0 = \tau_0(t', T, \varepsilon, \hat{B}_\mu) < t'$ and a positive integer $l(\varepsilon)$ such that

$$c_1 R_\mu^2(\tau) e^{(L-\mu)t} e^{(\mu-L)\tau} < \frac{\varepsilon}{4}, \qquad \forall \tau \le \tau_0,\ t \in [t', T],$$

and

$$\frac{4(2\mu - L)e^{\mu h}}{\mu(\mu - L)} r_0^2 \sum_{m=l(\varepsilon)+1}^{+\infty} |J(m)| < \frac{\varepsilon}{4}.$$

Then, from (3.8) we have

$$\|u_t\|_{E_{h,\theta}}^2 < \frac{\varepsilon}{2} + \frac{2\mu - L}{\mu - L} c_{2,\theta} + \frac{4(2\mu - L)e^{\mu h}}{\mu(\mu - L)} \frac{r_0^2}{k} \sum_{m=1}^{l(\varepsilon)} m|J(m)| + \frac{e^{\mu h}}{\gamma} \int_{-\infty}^{T} \|f(s)\|_\theta^2 \, ds,$$

for all $\tau \le \tau_0$ and $t' \le t \le T$. Observe that the hypothesis (A7) and the Lebesgue Dominated Convergence Theorem imply that

$$\lim_{k \to +\infty} \int_{-\infty}^{T} \sum_{|n|>k} |f_n(s)|^2 \, ds = 0.$$

Using this fact and also $\sum_{n=-\infty}^{\infty} \left(\int_{-h}^{0} \beta_{2,n}(s)\, ds\right)^2 < \infty$ and $\sum_{-\infty}^{\infty} k_{2,n}^2 < \infty$ we can find a positive integer $k = k(\varepsilon, T, \hat{B}_\mu)$ such that

$$\frac{2\mu - L}{\mu - L} c_{2,\theta} + \frac{4(2\mu - L)e^{\mu h}}{\mu(\mu - L)} \frac{r_0^2}{k} \sum_{m=1}^{l(\varepsilon)} m|J(m)| + \frac{e^{\mu h}}{\gamma} \int_{-\infty}^{T} \|f(s)\|_\theta^2 \, ds < \frac{\varepsilon}{2}.$$

Therefore,

$$\max_{s \in [-h,0]} \sum_{|n|>2k} |u_n(t+s)|^2 \le \|u_t\|_{E_{h,\theta}}^2 < \varepsilon, \quad \text{if } \tau \le \tau_0,\ t' \le t \le T. \qquad \square$$

In the case A(8)(iii), in (3.8), we take $L = 0$, replace c_1 by c_1' and $c_{2,\theta}$ and $R_\mu^2(\tau)$ by

$$c_{2,\theta}' = \frac{e^{\mu h}}{\mu^2}\left(B_{2,\theta}^2 + K_{2,\theta}^2\right) \quad \text{and} \quad R_\mu^2(\tau) = c_2' + \frac{e^{\mu h}}{2\mu} \int_{-\infty}^{t} \|f(s)|^2 \, ds.$$

Lemma 3.3. *Under the assumptions (A1)–(A8), the process $\{U(t,\tau)\}_{t \ge \tau}$ is pullback $\mathcal{D}$-asymptotically compact.*

Proof. Fixed $t \in \mathbb{R}$ and $\hat{D}_\mu \in \mathcal{D}_\mu$, consider the sequences $(\tau_m)_{m\in\mathbb{N}}$ and $(u_t^m)_{m\in\mathbb{N}}$, such that $\tau_m \to -\infty$ and $u_t^m = U(t, \tau_m)\psi^m$, with $\psi^m \in D(\tau_m)$. We want to prove that $(u_t^m)_{m\in\mathbb{N}}$ has a subsequence which is relatively compact in E_h. Given $\varepsilon > 0$, by Lemma 3.2, there exist $\tau = \tau(\varepsilon, t, \hat{B}_\mu) < t - h$ and a positive integer $n_1 = n_1(\varepsilon, t, \hat{B}_\mu)$ such that

$$\max_{s \in [-h,0]} \sum_{|n|>n_1} |u_n(t+s)|^2 < \frac{\varepsilon^2}{8}, \tag{3.9}$$

where $u = u(t) = (u_n(t))$ is any solution of the initial value problem (2.1) with $u_\tau \in B_\mu(\tau)$.

Since $\hat{B}_\mu$ is pullback $\mathcal{D}_\mu$-absorbing and $\tau_m \to -\infty$, without loss of generality, we can assume that

$$U(\tau,\tau_m)\psi^m \in B_\mu(\tau), \qquad \forall m \geq 1. \tag{3.10}$$

Also, by the definition of a process, we know that

$$U(t',\tau)\,U(\tau,\tau_m)\psi^m = U(t',\tau_m)\psi^m, \qquad \forall \tau \leq t' \leq t. \tag{3.11}$$

Using (3.10), (3.11), and the estimate (2.19) we see that

$$\|U(t',\tau_m)\psi^m\|_{E_h} \leq K, \qquad \forall \tau \leq t' \leq t, \tag{3.12}$$

where $K = K(t) = (c_1+1)R^2_\mu(t)$. In particular, the sequence $(u^m_t(s))_{m\in\mathbb{N}}$ is bounded in ℓ^2, for any $s \in [-h,0]$. Therefore, for any fixed $s \in [-h,0]$, there exists a subsequence, which we will still denote by $(u^m_t(s))_{m\in\mathbb{N}}$ and $\zeta(s) \in \ell^2$, such that

$$u^m(t+s) \rightharpoonup \zeta(s) \quad \text{weakly in } \ell^2. \tag{3.13}$$

Let us show that the convergence in (3.13) is strong in ℓ^2. Since $\zeta(s) \in \ell^2$, then there exists a positive integer n_2 such that

$$\sum_{|n|>n_2} |\zeta_n(s)|^2 < \frac{\varepsilon^2}{8}. \tag{3.14}$$

Moreover, using the weak convergence (3.13), we can find a positive integer $m_1 = m_1(\varepsilon, t, \hat{B}_\mu)$ such that

$$\sum_{|n|\leq n_0} |u^m_n(t+s) - \zeta_n(s)|^2 < \frac{\varepsilon^2}{2}, \qquad \forall m \geq m_1, \tag{3.15}$$

where $n_0 = \max\{n_1, n_2\}$. From (3.14) and (3.15), for any $m \geq m_1$, we have that

$$\begin{aligned}\|u^m(t+s) - \zeta(s)\|^2 &\leq \sum_{|n|\leq n_0} |u^m_n(t+s) - \zeta_n(s)|^2 + 2\sum_{|n|>n_0} |u^m_n(t+s)|^2 \\ &\quad + 2\sum_{|n|>n_0} |\zeta_n(s)|^2 < \frac{3\varepsilon^2}{4} + 2\sum_{|n|>n_0} |u^m_n(t+s)|^2.\end{aligned} \tag{3.16}$$

Using the estimate (3.9) with $u_\tau = U(\tau,\tau_m)\psi^m$, $m \geq m_1$, from (3.16) we conclude that

$$\|u^m(t+s) - \zeta(s)\|^2 < \varepsilon^2.$$

Therefore, $(u^m_t(s))_{m\in\mathbb{N}}$ is relatively compact in ℓ^2 for each $s \in [-h,0]$.

Next, let us show that $(u^m_t)_{m\in\mathbb{N}}$ is equicontinuous in $[-h,0]$. Using the integral representation of the solution of (2.1) we obtain

$$\|u^m(t+s_1) - u^m(t+s_2)\| \leq \int_{t+s_1}^{t+s_2} \|\mathcal{B}(r,u^m_r)\|\,dr, \tag{3.17}$$

for any $-h \leq s_1 \leq s_2 \leq 0$. Using (3.12) in (3.17) and Lemma 2.4, we deduce the existence of a positive constant $L(K)$ such that $\|u^m(t+s_1) - u^m(t+s_2)\| \leq L(K)(s_2-s_1)$, $\forall m \in \mathbb{N}$, which implies the equicontinuity. By the Ascoli–Arzelà Theorem, we conclude that $(u^m_t)_{m\in\mathbb{N}}$ is relatively compact in E_h. This completes the proof of Lemma 3.3. □

As consequence of Lemmas 3.1, 3.3 and of Theorem 18 in [16] we obtain the main result of this section.

Theorem 3.4. *Assume that (A1)–(A8) hold. Then, the process $\{U(t,\tau)\}_{t\geq\tau}$ possesses a unique pullback $\mathcal{D}_\mu$-attractor $\hat{A}$ in $\mathcal{D}_\mu$.*

Proof. By Lemmas 3.1, 3.3 and Theorem 18 in [16] the process $\{U(t,\tau)\}_{t\geq\tau}$ possesses a pullback $\mathcal{D}_\mu$-attractor $\hat{A}$. Since the family $\mathcal{D}_\mu$ is inclusion-closed and each member $B_\mu(t)$ of the pullback family $\hat{B}_\mu$ is a closed subset of E_h, then $\hat{A} \in \mathcal{D}_\mu$ and it is the unique pullback $\mathcal{D}_\mu$-attractor belonging to the class $\mathcal{D}_\mu$. □

4 The autonomous model

In this section, we consider the autonomous model

$$\begin{aligned} &i\dot{u}_n(t) + \sum_{m=-\infty}^{+\infty} J(n-m)u_m(t) + g_n(u_{nt}) + i\gamma u_n(t) = f_n, \quad t>0, n\in\mathbb{Z}, \\ &u_n(s) = \psi_n(s), \quad \forall s\in[-h,0], \end{aligned} \tag{4.1}$$

where $f = (f_n)_{n\in\mathbb{Z}}$ and

$$g_n(u_{nt}) = g_{0,n}(u_n(t)) + g_{1,n}(u_n(t-\rho)) + \int_{-h}^{0} b_n(s, u_n(t+s))\,ds,$$

with $0 < \rho \leq h$. We assume that $f \in \ell^2$ and the functions $g_{0,n}$, $g_{1,n}$, and b_n satisfy the assumptions (A1)–(A4) stated in Section 2.

Defining the map $g : E_h \to \ell^2$ by $(g(v))_{n\in\mathbb{Z}} = g_n(v_n)$, where

$$g_n(v_n) = g_{0,n}(v_n(0)) + g_{1,n}(v_n(-\rho)) + \int_{-h}^{0} b_n(s, v_n(s))\,ds,$$

we can write (4.1) in ℓ^2 as

$$\begin{aligned} &i\dot{u}(t) + Au(t) + g(u_t) + i\gamma u(t) = f, \quad t>0 \\ &u(s) = \psi(s), \quad \forall s\in[-h,0], \end{aligned} \tag{4.2}$$

where, as before, $u(t) = (u_n(t))_{n\in\mathbb{Z}}$ and $\psi(s) = (\psi_n(s))_{n\in\mathbb{Z}}$.

Using the assumptions (A1)–(A4) and the Theory of Functional Equations we obtain a local solution for the problem (4.2) with $\psi \in E_h$.

In what follows, we will use the same notations of Sections 2 and 3 and, as before, we will assume that (A8)(i) or (ii) holds. Similarly, we can prove the results for (A8)(iii). Proceeding as in the proof of Theorem 2.8 we can prove the following lemma.

Lemma 4.1. *Assume that (A1)–(A4) and (A8) hold. Then, the solution $u = u(t)$ of (4.2) with initial condition $u_0 = \psi \in E_h$, defined in the maximal interval of existence $[0,T)$, satisfies*

$$\|u_t\|_{E_h}^2 \leq c_1\|\psi\|_{E_h}^2 e^{-(\mu-L)t} + \frac{2\mu-L}{\mu-L}\left(c_2 + \frac{e^{\mu h}}{\mu\gamma}\|f\|^2\right). \tag{4.3}$$

As a consequence of (4.3) we conclude that the solution $u = u(t)$ of (4.2) exists on $[0,\infty)$ and we can define a semigroup $\{S(t)\}_{t\geq 0}$ on E_h associated with (4.2) as follows

$$S(t)\psi = u_t, \quad \forall t\geq 0.$$

Moreover, from (4.3) we deduce that the closed ball $\mathcal{O}_0 = B_{E_h}[0; r_0]$ in E_h, where

$$r_0 = \left[\frac{2\mu - L}{\mu - L} \left(c_2 + \frac{e^{\mu h}}{\mu\gamma} \|f\|^2 \right) + 1 \right]^{1/2}, \tag{4.4}$$

is an absorbing set for $\{S(t)\}_{t\geq 0}$ in E_h.

Next, let us modify the proof of Lemma 3.2 to show that $\{S(t)\}_{t\geq 0}$ is asymptotically compact in E_h.

Lemma 4.2. *Assume that (A1)–(A4) and (A8) hold. Also, assume that $\psi \in \mathcal{O}_0$. Then, for any $\epsilon > 0$, there exist $T(\epsilon) \geq 0$ and a positive integer $k(\epsilon)$, such that the solution $u = u(t)$ of (4.2) satisfies*

$$\max_{s\in[-h,0]} \sum_{|n|>k(\epsilon)} |u_n(t+s)|^2 < \epsilon, \quad \forall t \geq T(\epsilon).$$

Proof. Since $\psi \in \mathcal{O}_0$, then by (4.3) and (4.4), we have

$$\|u_t\|_{E_h} \leq r_1, \quad \forall t \geq 0, \tag{4.5}$$

where $r_1 = (c_1 + 1)^{1/2} r_0$.

Using (4.5) and proceeding as in the proof of Lemma 3.2 we can prove that

$$e^{\mu t} \|u_t\|^2_{E_{h,\theta}} \leq M_\theta(t) + L \int_0^t e^{\mu s} \|u_s\|^2_{E_{h,\theta}} \, ds, \tag{4.6}$$

with

$$M_\theta(t) = e^{\mu h} \left(1 + \frac{4B_1^2}{\mu\gamma} e^{\mu h} \right) \|\psi\|^2_{E_h} + \left(\frac{4B^2_{2,\theta}}{\gamma} + \frac{K^2_{2,\theta}}{\varepsilon'} + 2\nu(k,l) + \frac{1}{\gamma}\|f\|^2_\theta \right) \frac{e^{\mu(t+h)}}{\mu},$$

where

$$\nu(k,l) = \left(\frac{2}{k} \sum_{m=1}^{l} m|J(m)| + 2 \sum_{m=l+1}^{+\infty} |J(m)| \right) r_1^2.$$

From (4.6) we obtain

$$\|u_t\|^2_{E_{h,\theta}} \leq c_1 r_1^2 e^{-(\mu - L)t} + \frac{2\mu - L}{\mu - L} c_{2,\theta} + \frac{2(2\mu - L)}{\mu(\mu - L)} e^{\mu h} \nu(k,l), \tag{4.7}$$

where

$$c_{2,\theta} = \left(\frac{4B^2_{2,\theta}}{\gamma} + \frac{K^2_{2,\theta}}{\varepsilon'} + \frac{1}{\gamma}\|f\|^2_\theta \right) \frac{e^{\mu h}}{\mu}.$$

Finally, from (4.7) we can conclude the proof of Lemma 4.2. □

Under the hypotheses of Lemma 4.1, using Lemma 4.2 and proceeding as in the proof of Lemma 3.3, we show that the semigroup $\{S(t)\}_{t\geq 0}$ is asymptotically compact in E_h. Thus, we can derive the desired result in this section.

Theorem 4.3. *Under the same hypotheses of Lemma 4.1, the semigroup $\{S(t)\}_{t\geq 0}$ possesses a unique global attractor $\mathcal{A}$ in E_h.*

Remark 4.4. When $\rho(t) \equiv \rho$ and $f(t) \equiv f$ in problem (2.1), then the constant family $\hat{A} = \{A(t) = \mathcal{A}; t \in \mathbb{R}\}$ is the pullback $\mathcal{D}$-attractor from Theorem 4.3.

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