Existence of global solutions to chemotaxis fluid system with logistic source

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Abstract. We establish the existence of global solutions and L^q time-decay of a three dimensional chemotaxis system with chemoattractant and repellent. We show the existence of global solutions by the energy method. We also study L^q time-decay for the linear homogeneous system by using Fourier transform and finding Green's matrix. Then, we find L^q time-decay for the nonlinear system using solution representation by Duhamel's principle and time-weighted estimate.

Keywords: chemotaxis system, energy method, a priori estimates, Fourier transform, time-decay rates.

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1 Introduction

Chemotaxis is the oriented movement of biological cells or microscopic organisms toward or away from the concentration gradient of certain chemicals in their environment. We may use cells to denote the biological objects whose movement we are interested in and chemo attractants or repellents to denote chemicals which attract or repel the cells. This type of movement exists in many biological phenomena, such as the movement of bacteria toward certain chemicals [1], or the movement of endothelial cells toward the higher concentration of chemoattractant that cancer cells produce [4].

Keller and Segel [11, 12] derived a mathematical model to describe the aggregation of certain types of bacteria, which consists of the equations for the cell density $n = n(x, t)$ and the concentration of chemical attractant $c = c(x, t)$ and is given by

$$\begin{cases} n_t = \Delta n - \nabla \cdot (n\chi\nabla c), \\ \alpha c_t = \Delta c + f(c, n), \end{cases}$$

where χ is the sensitivity of the cell movement to the density gradient of the attractant, α is a positive constant, and the reaction term f is a smooth function of the arguments. Since then,

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many mathematical approaches to describe chemotaxis using systems of partial differential equations have emerged, some of which will be discussed later in this section.

In this paper, we use the equations for continuum mechanics to describe the movement of cells and for the chemoattractant and repellent, we use diffusion equations. The combined effects of chemoattractant and repellent for chemotaxis are studied in diseases such as Alzheimer's disease [2].

We consider the initial value problem for the system in $\mathbb{R}^3$ given by

$$
\begin{cases}
\partial_t n + \nabla \cdot (nu) = n(n_\infty - n) \\
\partial_t u + u \cdot \nabla u + \frac{\nabla p(n)}{n} = \chi_1 \nabla c_1 - \chi_2 \nabla c_2 + \delta \Delta u \\
\partial_t c_1 = \Delta c_1 - a_{12} c_1 + a_{11} c_1 n \\
\partial_t c_2 = \Delta c_2 - a_{22} c_2 + a_{21} c_2 n,
\end{cases}
\tag{1.1}
$$

where $n(x,t), u(x,t), c_1(x,t), c_2(x,t)$ for $t > 0$, $x \in \mathbb{R}^3$, are the cell concentration, velocity of cells, chemoattractant concentration, and chemorepellent concentration, respectively. The initial data is given by

$$
(n, u, c_1, c_2)|_{t=0} = (n_0, u_0, c_{1,0}, c_{2,0})(x), \qquad x \in \mathbb{R}^3,
\tag{1.2}
$$

where it is supposed to hold that

$$
(n_0, u_0, c_{1,0}, c_{2,0})(x) \to (n_\infty, 0, 0, 0) \quad \text{as } |x| \to \infty,
$$

for some constant $n_\infty > 0$.

In this model the cells follow a convective logistic equation, the velocity is given by the compressible Navier–Stokes type equations with the added effects of chemoattractants and -repellents. The pressure for the cells $p(n)$ is a smooth function of n and $p'(n) > 0$, a positive constant δ is the coefficient for the viscosity term, and χ_1 and χ_2 express the sensitivity of the cell movement to the density gradients of the attractants and repellents, respectively. Usually $\chi_i, (i = 1, 2)$ are functions of c_i and in this paper we consider the case $\chi_i = K_i c_i$, where K_i are positive constants, so that the sensitivity is proportional to the concentration of the attractants and repellents. We choose $K_i = 2$ for simplicity. We may equally use $\chi_i = K_i c_i^{\alpha_i}$, where α_i are positive constants. For chemical substances, we use the reaction diffusion equations. The reaction terms are based on a Lotka–Volterra type model in which the nonnegative regions of c_i are invariant in the sense that if the initial conditions for c_i are nonnegative, they are nonnegative for positive t. This can be verified by the maximum principle. The couplings between c_i and n are given as nonlinear terms.

The main goal of this paper is to establish the local and global existence of smooth solutions in three dimensions around a constant state $(n_\infty, 0, 0, 0)$ and the decay rate of global smooth solutions for the above system (1.1). The main result of this paper is stated as follows.

Theorem 1.1. *Let $N \geq 4$ be an integer. There exists a positive numbers ϵ_0, C_0 such that if*

$$
\|[n_0 - n_\infty, u_0, c_{1,0}, c_{2,0}]\|_{H^N} \leq \epsilon_0,
$$

then, the Cauchy problem (1.1)–(1.2) has a unique solution $(n, u, c_1, c_2)(t)$ globally in time which satisfies

$$
(u, c_1, c_2)(t) \in C([0, \infty); H^N(\mathbb{R}^3)) \cap C^1([0, \infty); H^{N-2}(\mathbb{R}^3)),
$$
$$
n - n_\infty \in C([0, \infty); H^N(\mathbb{R}^3)) \cap C^1([0, \infty); H^{N-1}(\mathbb{R}^3))
$$

and there are constants $\lambda_1 > 0$ and $\lambda_2 > 0$ such that

$$\|[n - n_\infty, u, c_1, c_2]\|_{H^N}^2 + \lambda_1 \int_0^t \|\nabla[u, c_1, c_2]\|_{H^N}^2 + \lambda_2 \int_0^t \|[n - n_\infty, c_1, c_2]\|_{H^N}^2$$
$$\leq C_0 \|[n_0 - n_\infty, u_0, c_{1,0}, c_{2,0}]\|_{H^N}^2. \tag{1.3}$$

Furthermore, the global solution $[n, u, c_1, c_2]$ satisfies the following time-decay rates for $t \geq 0$:

$$\|n - n_\infty\|_{L^q} \leq C(1 + t)^{-2 + \frac{3}{2q}}, \tag{1.4}$$

$$\|u\|_{L^q} \leq C(1 + t)^{\frac{-3}{2} + \frac{3}{2q}}, \tag{1.5}$$

$$\|c_1, c_2\|_{L^q} \leq C(1 + t)^{\frac{-3}{2}}, \tag{1.6}$$

with $2 \leq q < \infty$, $C > 0$.

The proof of the existence of global solutions in Theorem 1.1 is based on the local existence and an a priori estimates. We show the local solutions by constructing a sequence of approximation functions based on iteration. To obtain the a priori estimates we use the energy method. Moreover, to obtain the time-decay rate in L^q norm of solutions in Theorem 1.1, we first find the Green's matrix for the linear system using the Fourier transform and then obtain the refined energy estimates with the help of Duhamel's principle.

To motivate our study, we present previous related work on chemotaxis models. Many of them are based on the Keller–Segel system. Wang [21] explored the interactions between the nonlinear diffusion and logistic source on the solutions of the attraction–repulsion chemotaxis system in three dimensions. E. Lankeit and J. Lankeit [13] proved the global existence of classical solutions to a chemotaxis system with singular sensitivity. Liu and Wang [14] established the existence of global classical solutions and steady states to an attraction–repulsion chemotaxis model in one dimension based on the energy methods.

Concerning the chemotaxis models based on fluid dynamics, there are two approaches, incompressible and compressible. For the incompressible case, Chae, Kang and Lee [3], and Duan, Lorz, and Markowich [8] showed the global-in-time existence for the incompressible chemotaxis equations near the constant states, if the initial data is sufficiently small. Rodriguez, Ferreira, and Villamizar-Roa [19] showed the global existence for an attraction–repulsion chemotaxis fluid model with logistic source. Tan and Zhou [20] proved the global existence and time decay estimate of solutions to the Keller–Segel system in R^3 with the small initial data. For the compressible case, Ambrosi, Bussolino, and Preziosi [2] discussed the vasculogenesis using the compressible fluid dynamics for the cells and the diffusion equation for the attractant.

Many related approaches use the Fourier transform, and we only mention that Duan [6] and Duan, Liu, and Zhu [7] proved the time-decay rate by the combination of energy estimates and spectral analysis. Also by using Green's function and Schauder fixed point theorem, one can study the existence and regularity of solution for these kinds of equations (see [9, 10, 17, 18]).

For later use in this paper, we give some notations. C denotes some positive constant and λ_i, where $i = 1, 2$, denotes some positive (generally small) constant, where both C and λ_i may take different values in different places. For any integer $m \geq 0$, we use H^m to denote the Sobolev space $H^m(\mathbb{R}^3)$. Set $L^2 = H^0$. We set $\partial^\alpha = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \partial_{x_3}^{\alpha_3}$ for a multi-index $\alpha = [\alpha_1, \alpha_2, \alpha_3]$. The length of α is $|.| = \alpha_1 + \alpha_2 + \alpha_3$; we also set $\partial_j = \partial_{x_j}$ for $j = 1, 2, 3$. For an integrable

function $f : \mathbb{R}^3 \to \mathbb{R}$, its Fourier transform is defined by $\hat{f} = \int_{\mathbb{R}^3} e^{-ix\cdot\xi} f(x)dx$, $x \cdot \xi = \sum_{i=0}^{3} x_j \xi_j$, and $x \in \mathbb{R}^3$, where $i = \sqrt{-1}$ is the imaginary unit. Let us denote the space

$$X(0, T) = \{(u, c_1, c_2) \in C([0, T]; H^N(\mathbb{R}^3)) \cap C^1([0, T]; H^{N-2}(\mathbb{R}^3)),$$
$$n - n_\infty \in C([0, T]; H^N(\mathbb{R}^3)) \cap C^1([0, T]; H^{N-1}(\mathbb{R}^3))\}.$$

This paper is organized as follows. In Section 2, we reformulate the Cauchy problem under consideration. In Section 3, we prove the global existence and uniqueness of solutions. In Section 4, we investigate the linearized homogeneous system to obtain the $L^2 - L^q$ time-decay property and the explicit representation of solutions. In Section 5, we study the L^q time-decay rates of solutions to the reformulated nonlinear system and finish the proof of Theorem 1.1.

2   Reformulation of the system (1.1)

Let $U(t) = [n, u, c_1, c_2]$ be a smooth solution to the Cauchy problem of the chemotaxis system (1.1) with initial data $U_0 = [n_0, u_0, c_{1,0}, c_{2,0}]$. We introduce the transformation:

$$n(x, t) = n_\infty + \rho(x, t). \tag{2.1}$$

Then the Cauchy problem (1.1) is reformulated as

$$\begin{cases} \partial_t \rho + n_\infty \nabla \cdot u + n_\infty \rho = -\nabla \cdot (\rho u) - \rho^2 \\ \partial_t u + u \cdot \nabla u - \delta \Delta u + \frac{p'(n_\infty)}{n_\infty} \nabla \rho = \nabla(c_1)^2 - \nabla(c_2)^2 - \left(\frac{p'(\rho + n_\infty)}{\rho + n_\infty} - \frac{p'(n_\infty)}{n_\infty} \right) \nabla \rho \\ \partial_t c_1 = \Delta c_1 - (a_{12} - a_{11} n_\infty) c_1 + a_{11} \rho c_1 \\ \partial_t c_2 = \Delta c_2 - (a_{22} - a_{21} n_\infty) c_2 + a_{21} \rho c_2, \end{cases} \tag{2.2}$$

with initial data

$$(\rho, u, c_1, c_2)|_{t=0} = (\rho_0, u_0, c_{1,0}, c_{2,0}) \to (0, 0, 0, 0), \tag{2.3}$$

as $|x| \to \infty$, where $\rho_0 = n_0 - n_\infty$. We assume that $a_{12} - a_{11} n_\infty > 0$ and $a_{22} - a_{21} n_\infty > 0$. In what follows, the integer $N \geq 4$ is always assumed.

Proposition 2.1. *There exists a positive number ϵ_0 which is small enough such that if*

$$\|[\rho_0, u_0, c_{1,0}, c_{2,0}]\|_{H^N} \leq \epsilon_0,$$

then the Cauchy problem (2.2)–(2.3) has a unique solution $(\rho, u, c_1, c_2)(t)$ globally in time which satisfies $(\rho, u, c_1, c_2)(t) \in X(0, \infty)$ and there are constants $C_0 > 0$, $\lambda_1 > 0$ and $\lambda_1 > 0$ such that

$$\|[\rho, u, c_1, c_2]\|_{H^N}^2 + \lambda_1 \int_0^t \|\nabla[u, c_1, c_2]\|_{H^N}^2 + \lambda_2 \int_0^t \|[\rho, c_1, c_2]\|_{H^N}^2 \leq C_0 \|[\rho_0, u_0, c_{1,0}, c_{2,0}]\|_{H^N}^2. \tag{2.4}$$

Proposition 2.2. *Let $U(t) = [\rho, u, c_1, c_2]$ be the solution to the Cauchy problem (2.2)–(2.3) obtained in Proposition 2.1, which satisfies the following L^q-time decay estimates for any $t \geq 0$:*

$$\|\rho\|_{L^q} \leq C(1 + t)^{-2 + \frac{3}{2q}}, \tag{2.5}$$

$$\|u\|_{L^q} \leq C(1 + t)^{\frac{-3}{2} + \frac{3}{2q}}, \tag{2.6}$$

$$\|c_1, c_2\|_{L^q} \leq C(1 + t)^{\frac{-3}{2}}, \tag{2.7}$$

with $2 \leq q < \infty$ and $C > 0$.

The proof of Theorem 1.1 obtained directly from the global existence proof in Proposition 2.1 and the derivation of rates in Theorem 1.1 is based on Proposition 2.2.

3 Global solution of the nonlinear system (2.2)

The goal of this section is to prove the global existence of solutions to the Cauchy problem (2.2) when initial data is a small, smooth perturbation near the steady state $(n_\infty, 0, 0, 0)$. The proof is based on some uniform a priori estimates combined with the local existence, which will be shown in Subsections 3.1 and 3.2.

3.1 Existence of local solutions

In this subsection, we show the proof of the existence of local solutions $[\rho, u, c_1, c_2]$ by constructing a sequence of functions that converges to a function satisfying the Cauchy problem. We construct a solution sequence $(\rho^j, u^j, c_1^j, c_2^j)_{j \geq 0}$ by iteratively solving the Cauchy problem on the following

$$\begin{cases} \partial_t \rho^{j+1} + n_\infty \nabla \cdot u^{j+1} + n_\infty \rho^{j+1} = -\rho^j \nabla \cdot u^{j+1} - \nabla \rho^{j+1} u^j - \rho^{j^2} \\ \partial_t u^{j+1} - \delta \Delta u^{j+1} = -u^j \cdot \nabla u^j + \nabla (c_1^j)^2 - \nabla (c_2^j)^2 - \frac{p'(\rho^j + n_\infty)}{\rho^j + n_\infty} \nabla \rho^j \\ \partial_t c_1^{j+1} - \Delta c_1^{j+1} + (a_{12} - a_{11} n_\infty) c_1^{j+1} = a_{11} \rho^j c_1^{j+1} \\ \partial_t c_2^{j+1} - \Delta c_2^{j+1} + (a_{22} - a_{21} n_\infty) c_2^{j+1} = a_{21} \rho^j c_2^{j+1}, \end{cases} \tag{3.1}$$

with initial data

$$(\rho^{j+1}, u^{j+1}, c_1^{j+1}, c_2^{j+1})|_{t=0} = U_0 = (\rho_0, u_0, c_{1,0}, c_{2,0}) \to (0, 0, 0, 0) \tag{3.2}$$

as $|x| \to \infty$, for $j \geq 0$. For simplicity, in what follows, we write $U^j = (\rho^j, u^j, c_1^j, c_2^j)$ and $U_0 = (\rho_0, u_0, c_{1,0}, c_{2,0})$, where $U^0 = (0, 0, 0, 0)$.
Now, we can start the following Lemma.

Lemma 3.1. *There are constants T_1 and $\epsilon_0 > 0$ such that if the initial data $U_0 \in H^N(\mathbb{R}^3)$ and $\|U_0\|_{H^N} \leq \epsilon_0$, then there exists a unique solution $U = (\rho, u, c_1, c_2)$ of the Cauchy problem (2.2)–(2.3) on $[0, T_1]$ with $U \in X(0, T_1)$.*

Proof. We first set $U^0 = (0, 0, 0, 0)$. Then, we use U^0 to solve the equations for U^1. The first equation is the first order partial differential equation and the second, third, and fourth equations are the second order parabolic equations. We obtain $u^1(x, t), c_1^1(x, t), c_2^1(x, t)$, and $\rho^1(x, t)$ in this order. Similarly, we define $(u^j, c_1^j, c_2^j, \rho^j)$ iteratively. Now, we prove the existence and uniqueness of solutions in space $C([0, T_1]; H^N(\mathbb{R}^3))$, where $T_1 > 0$ is suitably small. The proof is divided into four steps as follows.

In the first step, we show the uniform boundedness of the sequence of functions under our construction via energy estimates. We show that there exists a constant $M > 0$ such that $U^j \in C([0, T_1]; H^N(\mathbb{R}^3))$ is well defined and

$$\sup_{0 \leq t \leq T_1} \|U^j(t)\|_{H^N} \leq M, \tag{3.3}$$

for all $j \geq 0$. We use the induction to prove (3.3). It is trivial when $j = 0$. Suppose that it is true for $j \geq 0$ where M is small enough. To prove for $j + 1$, we need some energy estimate for U^{j+1}. Applying ∂^α to the first equation of (3.1), multiplying it by $\partial^\alpha \rho^{j+1}$ and integrating in x,

we obtain

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}(\partial^\alpha \rho^{j+1})^2 dx + n_\infty \int_{\mathbb{R}^3}|\partial^\alpha \rho^{j+1}|^2 dx$$

$$= -n_\infty \int_{\mathbb{R}^3}\partial^\alpha \rho^{j+1}\partial^\alpha \nabla \cdot u^{j+1}dx - \int_{\mathbb{R}^3}\partial^\alpha \rho^{j+1}\partial^\alpha(\nabla \rho^{j+1}\cdot u^j)dx$$

$$+\int_{\mathbb{R}^3}\partial^\alpha \rho^{j+1}\partial^\alpha(\rho^j \nabla \cdot u^{j+1})dx - \int_{\mathbb{R}^3}\partial^\alpha \rho^{j+1}\partial^\alpha \rho^{j2}dx.$$

The terms on the right hand side are further bounded by

$$C\|\nabla \cdot u^{j+1}\|_{H^N}\|\rho^{j+1}\|_{H^N} + C\|\nabla \cdot u^j\|_{L^\infty}\|\rho^{j+1}\|^2_{H^N}$$

$$+\|u^j\|_{H^N}\|\rho^{j+1}\|_{H^N}\|\nabla \rho^{j+1}\|_{H^{N-2}} + \|\rho^j\|_{H^N}\|\rho^{j+1}\|_{H^N}\|\nabla \cdot u^{j+1}\|_{H^N}$$

$$+C\|\rho^j\|_{H^{N-2}}\|\rho^{j+1}\|_{H^N}\|\rho^j\|_{H^N}.$$

Then, after taking the summation over $|\alpha| \le N$ and using the Cauchy inequality, one has

$$\frac{1}{2}\frac{d}{dt}\|\rho^{j+1}\|^2_{H^N}+\lambda_2\|\rho^{j+1}\|^2_{H^N}$$

$$\le C\|\nabla \cdot u^{j+1}\|^2_{H^N} + C\|u^j\|^2_{H^N}\|\rho^{j+1}\|^2_{H^N} + C\|\rho^j\|^2_{H^N}\|\rho^{j+1}\|^2_{H^N} + C\|\rho^j\|^2_{H^N}. \qquad (3.4)$$

Similarly, applying ∂^α to the second equation of (3.1), multiplying it by $\partial^\alpha u^{j+1}$, taking integrations in x, and then using integration by parts, we have

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}(\partial^\alpha u^{j+1})^2 dx + \delta\int_{\mathbb{R}^3}|\partial^\alpha \nabla \cdot u^{j+1}|^2 dx = \frac{p'(n_\infty)}{n_\infty}\int_{\mathbb{R}^3}\nabla \cdot \partial^\alpha u^{j+1}\partial^\alpha \rho^{j+1}dx$$

$$-\int_{\mathbb{R}^3}\nabla \cdot \partial^\alpha u^{j+1}\partial^\alpha c_1^{j2}dx + \int_{\mathbb{R}^3}\nabla \cdot \partial^\alpha u^{j+1}\partial^\alpha c_2^{j2}dx$$

$$-\int_{\mathbb{R}^3}\partial^\alpha u^{j+1}\cdot \partial^\alpha(u^j \cdot \nabla u^j)dx - \int_{\mathbb{R}^3}\partial^\alpha u^{j+1}\cdot \partial^\alpha\left(\frac{\nabla p(\rho^j + n_\infty)}{\rho^j + n_\infty}\right)dx.$$

Then, after taking the summation over $|\alpha| \le N$, the terms on the right side of the previous equation are bounded by

$$C\|\nabla \cdot u^{j+1}\|_{H^N}\|\rho^{j+1}\|_{H^N} + C\|c_1^j\|_{H^{N-3}}\|\nabla \cdot u^{j+1}\|_{H^N}\|c_1^j\|_{H^N}$$

$$+C\|c_2^j\|_{H^{N-3}}\|\nabla \cdot u^{j+1}\|_{H^N}\|c_2^j\|_{H^N} + \|u^j\|^2_{H^N}\|\nabla \cdot u^{j+1}\|_{H^N} + C\|\rho^j\|_{H^N}\|\nabla \cdot u^{j+1}\|_{H^N}.$$

By using the Cauchy inequality, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u^{j+1}\|^2_{H^N}+\lambda_1\|\nabla \cdot u^{j+1}\|^2_{H^N}\le C\|\rho^{j+1}\|^2_{H^N}+C\|c_1^j\|^2_{H^N}+C\|c_1^j\|^2_{H^N}\|\nabla \cdot u^{j+1}\|^2_{H^N}+C\|c_2^j\|^2_{H^N}$$

$$+C\|c_2^j\|^2_{H^N}\|\nabla \cdot u^{j+1}\|^2_{H^N}+C\|u^j\|^2_{H^N}\|\nabla \cdot u^{j+1}\|^2_{H^N}+\|\rho^j\|^2_{H^N}. \qquad (3.5)$$

In a similar way as above, we can estimate c_1 and c_2 as

$$\frac{1}{2}\frac{d}{dt}\|c_1^{j+1}\|^2_{H^N} + \|\nabla c_1^{j+1}\|^2_{H^N} + \lambda_2\|c_1^{j+1}\|^2_{H^N} \le C\|\rho^j\|^2_{H^N}\|c_1^{j+1}\|^2_{H^N} \qquad (3.6)$$

$$\frac{1}{2}\frac{d}{dt}\|c_2^{j+1}\|^2_{H^N} + \|\nabla c_2^{j+1}\|^2_{H^N} + \lambda_2\|c_2^{j+1}\|^2_{H^N} \le C\|\rho^j\|^2_{H^N}\|c_2^{j+1}\|^2_{H^N}. \qquad (3.7)$$

Taking the linear combination of inequalities (3.4)–(3.7), we have

$$\frac{1}{2}\frac{d}{dt}(\|\rho^{j+1}\|_{H^N}^2+\|u^{j+1}\|_{H^N}^2+\|c_1^{j+1}\|_{H^N}^2+\|c_2^{j+1}\|_{H^N}^2)+\lambda_1\|\nabla[u^{j+1},c_1^{j+1},c_2^{j+1}]\|_{H^N}^2$$
$$+\lambda_2\|[\rho^{j+1},c_1^{j+1},c_2^{j+1}]\|_{H^N}^2 \leq C\|[\rho^j,u^j,c_1^j,c_2^j]\|_{H^N}^2 + C\|[\rho^j,u^j]\|_{H^N}^2\|\rho^{j+1}\|_{H^N}^2$$
$$+C\|[u^j,c_1^j,c_2^j]\|_{H^N}^2\|\nabla\cdot u^{j+1}\|_{H^N}^2 + C\|\rho^j\|_{H^N}^2\|[c_1^{j+1},c_2^{j+1}]\|_{H^N}^2.$$

Thus, after integrating with respect to t, we have

$$\|U^{j+1}(t)\|_{H^N}^2+\lambda_1\int_0^t\|\nabla[u^{j+1},c_1^{j+1},c_2^{j+1}]\|_{H^N}^2ds + \lambda_2\int_0^t\|[\rho^{j+1},c_1^{j+1},c_2^{j+1}]\|_{H^N}^2ds$$

$$\leq C\|U^{j+1}(0)\|_{H^N}^2+C\int_0^t\|U^j(s)\|_{H^N}^2ds + C\int_0^t\|U^j(s)\|_{H^N}^2\|[\rho^{j+1},\nabla\cdot u^{j+1},c_1^{j+1},c_2^{j+1}]\|_{H^N}^2ds. \quad (3.8)$$

In the last inequality, we use the induction hypothesis. We obtain

$$\|U^{j+1}(t)\|_{H^N}^2+\lambda_1\int_0^t\|\nabla[u^{j+1},c_1^{j+1},c_2^{j+1}]\|_{H^N}^2ds + \lambda_2\int_0^t\|[\rho^{j+1},c_1^{j+1},c_2^{j+1}]\|_{H^N}^2ds$$

$$\leq C\epsilon_0^2 + CM^2T_1 + CM^2\int_0^t\|[\rho^{j+1},\nabla\cdot u^{j+1},c_1^{j+1},c_2^{j+1}]\|_{H^N}^2ds,$$

for $0 \leq t \leq T_1$. Now, we take the small constants $\epsilon_0 > 0$, $T_1 > 0$ and $M > 0$. Then we have

$$\|U^{j+1}(t)\|_{H^N}^2+\lambda_1\int_0^t\|\nabla[u^{j+1},c_1^{j+1},c_2^{j+1}]\|_{H^N}^2ds + \lambda_2\int_0^t\|[\rho^{j+1},c_1^{j+1},c_2^{j+1}]\|_{H^N}^2ds \leq M^2, \quad (3.9)$$

for $0 \leq t \leq T_1$. This implies that (3.3) holds true for $j + 1$. Hence (3.3) is proved for all $j \geq 0$.

For the second step, we prove that the sequence $(U^j)_{j\geq 0}$ is a Cauchy sequence in the Banach space $C([0,T_1];H^{N-1}(\mathbb{R}^3))$, which converges to the solution $U = (\rho,u,c_1,c_2)$ of the Cauchy problem (2.2)–(2.3), and satisfies $\sup_{0\leq t\leq T_1}\|[U^j(t)]\|_{H^{N-1}} \leq M$. See for example [16].

For simplicity, we denote $\delta f^{j+1} := f^{j+1} - f^j$. Subtracting the j-th equations from the $(j + 1)$-th equations, we have the following equations for $\delta\rho^{j+1}, \delta u^{j+1}, \delta c_1^{j+1}$ and δc_1^{j+1}:

$$\begin{cases}\partial_t\delta\rho^{j+1} + n_\infty\nabla\cdot(\delta u^{j+1}) + n_\infty\delta\rho^{j+1} = -\rho^j\nabla\cdot\delta u^{j+1} - \delta\rho^j\nabla\cdot u^j \\ -u^j\nabla\delta\rho^{j+1} - \delta u^j\nabla\rho^j + (\rho^j + \rho^{j-1})\delta\rho^j \\ \partial_t\delta u^{j+1} - \delta\Delta\delta u^{j+1} = -u^j\cdot\nabla\delta u^j - \delta u^j\cdot\nabla u^{j-1} + \nabla((c_1^j + c_1^{j-1})\delta c_1^j) \\ -\nabla((c_2^j + c_2^{j-1})\delta c_2^j) - (\frac{\nabla p(\rho^j+n_\infty)}{\rho^j+n_\infty} - \frac{\nabla p(\rho^{j-1}+n_\infty)}{\rho^{j-1}+n_\infty}) \\ \partial_t\delta c_1^{j+1} + \Delta\delta c_1^{j+1} + (a_{12} - a_{11}n_\infty)\delta c_1^{j+1} = a_{11}\rho^j\delta c_1^{j+1} + a_{11}\delta\rho^j c_1^j \\ \partial_t\delta c_2^{j+1} + \Delta\delta c_2^{j+1} + (a_{22} - a_{21}n_\infty)\delta c_2^{j+1} = a_{21}\rho^j\delta c_2^{j+1} + a_{21}\delta\rho^j c_2^j.\end{cases}$$

The estimate of $\delta\rho^{j+1}$ is as follows:

$$\frac{1}{2}\frac{d}{dt}\|\delta\rho^{j+1}\|_{H^{N-1}}^2+n_\infty\|\delta\rho^{j+1}\|_{H^{N-1}}^2\leq C\|\nabla\cdot\delta u^{j+1}\|_{H^{N-1}}\|\delta\rho^{j+1}\|_{H^{N-1}}$$
$$+C\|\rho^j\|_{H^{N-1}}\|\delta\rho^{j+1}\|_{H^{N-1}}\|\nabla\cdot\delta u^{j+1}\|_{H^{N-1}} + C\|\delta\rho^j\|_{H^{N-1}}\|\nabla\cdot u^j\|_{H^{N-1}}\|\delta\rho^{j+1}\|_{H^{N-1}}$$
$$+C\|\nabla\cdot u^j\|_{L^\infty}\|\delta\rho^{j+1}\|_{H^{N-1}}^2+C\|\delta\rho^{j+1}\|_{H^{N-2}}\|u^j\|_{H^{N-1}}\|\delta\rho^{j+1}\|_{H^{N-1}}$$
$$+C\|\delta\rho^{j+1}\|_{H^{N-1}}\|\delta u^j\|_{H^{N-1}}\|\nabla\rho^j\|_{H^{N-1}}+C\|\delta\rho^{j+1}\|_{H^{N-1}}\|\delta\rho^j\|_{H^{N-1}}.$$

Then

$$\frac{1}{2}\frac{d}{dt}\|\delta\rho^{j+1}\|^2_{H^{N-1}}+\lambda_2\|\delta\rho^{j+1}\|^2_{H^{N-1}}\le C\|\nabla\cdot\delta u^{j+1}\|^2_{H^{N-1}}+C\|\rho^j\|^2_{H^{N-1}}\|\delta\rho^{j+1}\|^2_{H^{N-1}}$$
$$+C\|\nabla\cdot u^j\|^2_{H^{N-1}}\|\delta\rho^j\|^2_{H^{N-1}}+C\|u^j\|^2_{H^{N-1}}\|\delta\rho^{j+1}\|^2_{H^{N-1}}$$
$$+C\|\nabla\rho^j\|^2_{H^{N-1}}\|\delta u^j\|^2_{H^{N-1}}+C\|\delta\rho^j\|^2_{H^{N-1}}. \tag{3.10}$$

The estimate of δu^{j+1} is

$$\frac{1}{2}\frac{d}{dt}\|\delta u^{j+1}\|^2_{H^{N-1}}+\delta\|\nabla\cdot\delta u^{j+1}\|^2_{H^{N-1}}\le C\|\nabla\cdot\delta u^{j+1}\|_{H^{N-1}}\|u^j\|_{H^{N-1}}\|\delta u^j\|_{H^{N-1}}$$
$$+\|\delta u^{j+1}\|_{H^{N-1}}\|\nabla\cdot u^j\|_{H^{N-1}}\|\delta u^j\|_{H^{N-1}}+C\|\delta u^{j+1}\|_{H^{N-1}}\|\delta u^j\|_{H^{N-1}}\|\nabla\cdot u^{j-1}\|_{H^{N-1}}$$
$$+C\|\delta c_1^j\|^2_{H^{N-1}}\|\nabla\cdot\delta u^{j+1}\|^2_{H^{N-1}}+C\|\delta c_2^{j+1}\|^2_{H^{N-1}}\|\nabla\cdot\delta u^{j+1}\|^2_{H^{N-1}}+C\|\delta\rho^{j+1}\|^2_{H^N}$$
$$+C\|\delta\rho^j\|_{H^{N-1}}\|\nabla\cdot\delta u^{j+1}\|_{H^{N-1}}.$$

Then

$$\frac{1}{2}\frac{d}{dt}\|\delta u^{j+1}\|^2_{H^{N-1}}+\lambda_1\|\nabla\cdot\delta u^{j+1}\|^2_{H^{N-1}}\le C\|u^j\|^2_{H^{N-1}}\|\delta u^j\|^2_{H^{N-1}}+\|\delta u^{j+1}\|^2_{H^{N-1}}\|\nabla\cdot u^j\|^2_{H^{N-1}}$$
$$+C\|\delta u^{j+1}\|^2_{H^{N-1}}\|\nabla\cdot u^{j-1}\|^2_{H^{N-1}}+C\|\delta u^j\|^2_{H^{N-1}}$$
$$+C\|\delta c_1^j\|^2_{H^{N-1}}+C\|\delta c_2^j\|^2_{H^{N-1}}+\|\delta\rho^j\|^2_{H^{N-1}}. \tag{3.11}$$

We have a similar way to estimate δc_1^{j+1} and δc_2^{j+1} as follows:

$$\frac{1}{2}\frac{d}{dt}\|\delta c_1^{j+1}\|^2_{H^{N-1}}+\|\nabla\delta c_1^{j+1}\|^2_{H^{N-1}}+\lambda_2\|\delta c_1^{j+1}\|^2_{H^{N-1}}$$
$$\le C\|\rho^j\|^2_{H^{N-1}}\|\delta c_1^{j+1}\|^2_{H^{N-1}}+C\|\delta c_1^j\|^2_{H^{N-1}}\|\rho^j\|^2_{H^{N-1}} \tag{3.12}$$

and

$$\frac{1}{2}\frac{d}{dt}\|\delta c_2^{j+1}\|^2_{H^{N-1}}+\|\nabla\delta c_2^{j+1}\|^2_{H^{N-1}}+\lambda_2\|\delta c_2^{j+1}\|^2_{H^{N-1}}$$
$$\le C\|\rho^j\|^2_{H^{N-1}}\|\delta c_2^{j+1}\|^2_{H^{N-1}}+C\|\delta c_2^j\|^2_{H^{N-1}}\|\rho^j\|^2_{H^{N-1}}. \tag{3.13}$$

We combine the equations (3.10)–(3.13) to obtain

$$\frac{1}{2}\frac{d}{dt}(\|\delta\rho^{j+1}\|^2_{H^{N-1}}+\|\delta u^{j+1}\|^2_{H^{N-1}}+\|\delta c_1^{j+1}\|^2_{H^{N-1}}+\|\delta c_2^{j+1}\|^2_{H^{N-1}})$$
$$+\lambda_1(\|\nabla\cdot\delta u^{j+1}\|^2_{H^{N-1}}+\|\nabla\delta c_1^{j+1}\|^2_{H^{N-1}}+\|\nabla\delta c_2^{j+1}\|^2_{H^{N-1}})$$
$$+\lambda_2(\|\delta\rho^{j+1}\|^2_{H^{N-1}}+\|\delta c_1^{j+1}\|^2_{H^{N-1}}+C\|\delta c_2^{j+1}\|^2_{H^{N-1}})$$
$$\le C(\|\delta u^{j+1}\|^2_{H^{N-1}}+\|\delta\rho^{j+1}\|^2_{H^{N-1}}+\|\delta c_1^{j+1}\|^2_{H^{N-1}}+\|\delta c_2^{j+1}\|^2_{H^{N-1}})$$
$$+C(\|\delta u^j\|^2_{H^{N-1}}+C\|\delta\rho^j\|^2_{H^{N-1}}+C\|\delta c_1^j\|^2_{H^{N-1}}+C\|\delta c_2^j\|^2_{H^{N-1}}).$$

By using Gronwall's inequality, we obtain

$$\sup_{0\le t\le T_1}(\|\delta\rho^{j+1}\|^2_{H^{N-1}}+\|\delta u^{j+1}\|^2_{H^{N-1}}+\|\delta c_1^{j+1}\|^2_{H^{N-1}}+\|\delta c_2^{j+1}\|^2_{H^{N-1}})$$
$$\le e^{\int_0^t c\,ds}\int_0^t\|\delta U^j(s)\|^2_{H^{N-1}}ds+e^{\int_0^t c\,ds}\|\delta U^{j+1}(0)\|^2_{H^{N-1}}ds$$
$$\le CT_1(e^{CT_1})\sup_{0\le t\le T_1}\|\delta U^j\|^2_{H^{N-1}}.$$

By taking $T_1 > 0$ sufficiently small we find that $(U^j)_{j \geq 0}$ is a Cauchy sequence in the Banach space $C([0, T_1]; H^{N-1}(\mathbb{R}^3))$. Thus, we have the limit function

$$U = U^0 + \lim_{m \to \infty} \sum_{j=0}^{m} (U^{j+1} - U^j)$$

in the same space $C([0, T_1]; H^{N-1}(\mathbb{R}^3))$, and satisfies

$$\sup_{0 \leq t \leq T_1} \|U\|_{H^{N-1}} \leq \sup_{0 \leq t \leq T_1} \liminf_{j \to \infty} \|U^j\|_{H^{N-1}} \leq M. \tag{3.14}$$

Thus, as $j \to \infty$ the limit exists such that

$$(U)_{j \geq 0} \to U(t)$$

strongly in $C([0, T_1]; H^{N-1})$ and as $j' \to \infty$, where $\{j'\}$ is a subsequence of $\{j\}$, we have

$$D(u, c_1, c_2)_{j'} \to D(u, c_1, c_2)$$

weakly in $L_2([0, T_1]; H^N)$ by step one. Also by step one, we know

$$(U)_{j''}(t) \to U(t)$$

weakly in H^N for every fixed $t \in [0, T_1]$, where $j'' = j''(t)$ is a subsequence of $\{j'\}$, depending on t. Thus, we have a solution $U(t) \in L_\infty([0, T_1]; H^N)$ for the problem (2.2)–(2.3).

For the third step, we show that $\|U^{j+1}(t)\|_{H^N}^2$ is continuous in time for each $j \geq 0$.

For simplicity, let us define the equivalent energy functional

$$\mathfrak{E}(U^{j+1}(t)) = \|\rho^{j+1}\|_{H^N}^2 + \|u^{j+1}\|_{H^N}^2 + \|c_1^{j+1}\|_{H^N}^2 + \|c_2^{j+1}\|_{H^N}^2.$$

Similarly to how we proved (3.8), we have

$$|\mathfrak{E}U^{j+1}(t) - \mathfrak{E}U^{j+1}(s)| = \left| \int_s^t \mathfrak{E}U^{j+1}(\theta) d\theta \right| \leq \int_s^t \|U^j(s)\|_{H^N}^2 d\theta$$

$$+ C \int_0^t (1 + \|U^j(s)\|_{H^N}^2) \|[\rho^{j+1}, \nabla \cdot u^{j+1}, c_1^{j+1}, c_2^{j+1}]\|_{H^N}^2 ds + C \int_s^t \|\nabla[c_1^{j+1}, c_2^{j+1}]\|_{H^N}^2 ds$$

$$\leq CM^2(t - s) + C(M^2 + 1) \int_s^t \|[\rho^{j+1}, \nabla \cdot u^{j+1}, c_1^{j+1}, c_2^{j+1}]\|_{H^N}^2 ds$$

$$+ C \int_s^t \|\nabla[c_1^{j+1}, c_2^{j+1}]\|_{H^N}^2 ds,$$

for any $0 \leq s \leq t \leq T_1$. The time integral on the right-hand side from the above inequality is bounded by (3.9), and hence $\mathfrak{E}U^{j+1}(t)$ is continuous in t for each $j \geq 0$. Therefore, $\|U^j(t)\|_{H^N}^2$ is continuous in time for each $j \geq 1$. Furthermore, $U = (\rho, u, c_1, c_2)$ is a local solution to the Cauchy problem (2.2)–(2.3).

For the fourth step, we show that the Cauchy problem (2.2)–(2.3) admits at most one solution in $C([0, T_1]; H^N(\mathbb{R}^3))$. We assume that there exist two local solutions $U, \tilde{U}$ in $C([0, T_1]; H^N)$ which satisfy (3.2). Let $\tilde{\rho} = \rho_1(x, t) - \rho_2(x, t)$, $\tilde{u}(x, t) = u_1(x, t) - u_2(x, t)$, $\tilde{c}_1(x, t) = c_{1,1}(x, t) - c_{1,2}(x, t)$ and $\tilde{c}_2(x, t) = c_{2,1}(x, t) - c_{2,2}(x, t)$ solve

$$\begin{cases} \partial_t \tilde{\rho} + n_\infty \nabla \cdot \tilde{u} + n_\infty \tilde{\rho} = -\nabla \cdot (\tilde{\rho} u_1) - \nabla \cdot (\rho_2 \tilde{u}) - (\rho_1 + \rho_2)\tilde{\rho} \\ \partial_t \tilde{u} + u_1 \cdot \nabla \tilde{u} - \delta \Delta \tilde{u} = -\tilde{u} \cdot \nabla u_2 - \dfrac{p'(\rho_1 + n_\infty)}{\rho_1 + n_\infty} \nabla \tilde{\rho} + \nabla((c_{1,1} + c_{1,2})\tilde{c}_1) \\ -\nabla((c_{2,1} + c_{2,2})\tilde{c}_2) - \left(\dfrac{p'(\rho_1 + n_\infty)}{\rho_1 + n_\infty} - \dfrac{p'(\rho_2 + n_\infty)}{\rho_2 + n_\infty} \right) \nabla \rho_2 \\ \partial_t \tilde{c}_1 = \Delta \tilde{c}_1 - a_{12}\tilde{c}_1 + a_{11}\rho_1 \tilde{c}_1 + a_{11}\tilde{\rho} c_{1,2} \\ \partial_t \tilde{c}_2 = \Delta \tilde{c}_2 - a_{22}\tilde{c}_2 + a_{21}\rho_1 \tilde{c}_2 + a_{21}\tilde{\rho} c_{2,2}. \end{cases} \tag{3.15}$$

Multiplying $\tilde{\rho}$ to both sides of the first equation of (3.15) and integrating over $\mathbb{R}^3$, we have

$$\int_{\mathbb{R}^3} \tilde{\rho} \partial_t \tilde{\rho} dx + n_\infty \int_{\mathbb{R}^3} \tilde{\rho} \nabla \cdot \tilde{u} dx + n_\infty \int_{\mathbb{R}^3} |\tilde{\rho}|^2 dx$$
$$= -\int_{\mathbb{R}^3} \tilde{\rho} \nabla \cdot (\tilde{\rho} u_1) dx + \int_{\mathbb{R}^3} \tilde{\rho} \nabla \cdot (\rho_2 \tilde{u}) dx + \int_{\mathbb{R}^3} (\rho_1 + \rho_2) \tilde{\rho}^2.$$

Using integration by parts and the Cauchy–Schwarz inequality, we have

$$\frac{1}{2} \frac{d}{dt} \|\tilde{\rho}\|_{L^2}^2 + n_\infty \|\tilde{\rho}\|_{L^2}^2 \leq \frac{n_\infty}{2} \|\tilde{\rho}\|_{L^2}^2 + \frac{n_\infty}{2} \|\nabla \cdot \tilde{u}\|_{L^2}^2 + \frac{1}{2} \|\nabla \cdot u_1\|_{L^\infty} \int_{\mathbb{R}^3} |\tilde{\rho}|^2 dx$$
$$+ \|\rho_2\|_{L^\infty} \int_{\mathbb{R}^3} (|\nabla \cdot \tilde{u}|^2 + |\tilde{\rho}|^2) dx + \|\nabla \rho_2\|_{L^\infty} \int_{\mathbb{R}^3} (|\tilde{u}|^2 + |\tilde{\rho}|^2) dx$$
$$+ \|[\rho_1 + \rho_2]\|_{L^\infty} \int_{\mathbb{R}^3} |\tilde{\rho}|^2 dx. \tag{3.16}$$

Next, we establish the energy estimates for $\tilde{u}$. By multiplying $\tilde{u}$ to both sides of the second equation of (3.15) and integrating in x, we have

$$\int_{\mathbb{R}^3} \tilde{u} \cdot \partial_t \tilde{u} dx + \int_{\mathbb{R}^3} \tilde{u} \cdot (u_1 \cdot \nabla \tilde{u}) dx - \delta \int_{\mathbb{R}^3} \tilde{u} \cdot \Delta \tilde{u} dx$$
$$= -\frac{p'(n_\infty)}{n_\infty} \int_{\mathbb{R}^3} \tilde{u} \cdot \nabla u_2 dx + \int_{\mathbb{R}^3} \tilde{u} \cdot \nabla \tilde{\rho} dx$$
$$+ \int_{\mathbb{R}^3} \tilde{u} \cdot \left(\frac{p'(\rho_1 + n_\infty)}{\rho_1 + n_\infty} - \frac{p'(n_\infty)}{n_\infty} \right) \nabla \tilde{\rho} + \int_{\mathbb{R}^3} \tilde{u} \cdot \nabla ((c_{1,1} + c_{1,2}) \tilde{c}_1) dx$$
$$- \int_{\mathbb{R}^3} \tilde{u} \cdot \nabla ((c_{2,1} + c_{2,2}) \tilde{c}_2) dx - \int_{\mathbb{R}^3} \tilde{u} \cdot \left(\frac{p'(\rho_1 + n_\infty)}{\rho_1 + n_\infty} dx - \frac{p'(\rho_2 + n_\infty)}{\rho_2 + n_\infty} \right) \nabla \rho_2 dx.$$

By using integration by parts and the Cauchy–Schwarz inequality, we have

$$\frac{1}{2} \frac{d}{dt} \|\tilde{u}\|_{L^2}^2 + \delta \|\nabla \cdot \tilde{u}\|_{L^2}^2 \leq \|\nabla \cdot u_1\|_{L^\infty} \|\tilde{u}\|_{L^2}^2 + \|\nabla \cdot u_2\|_{L^\infty} \|\tilde{u}\|_{L^2}^2 + \frac{p'(n_\infty)}{2n_\infty} \|\nabla \cdot \tilde{u}\|_{L^2}^2 + \frac{p'(n_\infty)}{2n_\infty} \|\tilde{\rho}\|_{L^2}^2$$
$$+ \|\rho_1\|_{L^\infty} (\|\nabla \cdot \tilde{u}\|_{L^2}^2 + \|\tilde{\rho}\|_{L^2}^2) + \|\nabla \rho_1\|_{L^\infty} (\|\tilde{u}\|_{L^2}^2 + \|\tilde{\rho}\|_{L^2}^2)$$
$$+ \|c_{1,1} + c_{1,2}\|_{L^\infty} (\|\nabla \cdot \tilde{u}\|_{L^2}^2 + \|\tilde{c}_1\|_{L^2}^2)$$
$$+ \|c_{2,1} + c_{2,2}\|_{L^\infty} (\|\nabla \cdot \tilde{u}\|_{L^2}^2 + \|\tilde{c}_2\|_{L^2}^2) + \|\nabla \rho_2\|_{L^\infty} (\|\tilde{u}\|_{L^2}^2 + \|\tilde{\rho}\|_{L^2}^2).$$

Since L^∞ norms of $\rho_i, u_i, c_{1,i}, c_{2,i}$ where $i = 1, 2$ are bounded, we have

$$\frac{1}{2} \frac{d}{dt} \|\tilde{u}\|_{L^2}^2 + \frac{\delta}{2} \|\nabla \cdot \tilde{u}\|_{L^2}^2 \leq C \|\tilde{u}\|_{L^2}^2 + C \|\tilde{\rho}\|_{L^2}^2 + C \|\tilde{c}_1\|_{L^2}^2 + C \|\tilde{c}_2\|_{L^2}^2. \tag{3.17}$$

We have a similar way to estimate $\tilde{c}_1$ and $\tilde{c}_2$ as follows:

$$\frac{1}{2} \frac{d}{dt} \|\tilde{c}_1\|_{L^2}^2 + \|\nabla \tilde{c}_1\|_{L^2}^2 + a_{12} \|\tilde{c}_1\|_{L^2}^2 \leq a_{11} \|\rho_1\|_{L^\infty} \|\tilde{c}_1\|_{L^2}^2 + \frac{a_{11}}{2} \|c_{1,2}\|_{L^\infty} (\|\tilde{\rho}\|_{L^2}^2 + \|\tilde{c}_1\|_{L^2}^2) \tag{3.18}$$

$$\frac{1}{2} \frac{d}{dt} \|\tilde{c}_2\|_{L^2}^2 + \|\nabla \tilde{c}_2\|_{L^2}^2 + a_{22} \|\tilde{c}_2\|_{L^2}^2 \leq a_{21} \|\rho_1\|_{L^\infty} \|\tilde{c}_2\|_{L^2}^2 + \frac{a_{21}}{2} \|c_{2,2}\|_{L^\infty} (\|\tilde{\rho}\|_{L^2}^2 + \|\tilde{c}_2\|_{L^2}^2). \tag{3.19}$$

By taking a linear combination of all estimates, we obtain

$$\frac{1}{2} \frac{d}{dt} (\|\tilde{\rho}\|_{L^2}^2 + \|\tilde{u}\|_{L^2}^2 + \|\tilde{c}_1\|_{L^2}^2 + \|\tilde{c}_2\|_{L^2}^2) + \lambda_1 (\|\nabla \cdot \tilde{u}\|_{L^2}^2 + \|\tilde{\nabla} c_1\|_{L^2}^2 + \|\tilde{\nabla} c_2\|_{L^2}^2)$$
$$+ \lambda_2 (\|\tilde{\rho}\|_{L^2}^2 + \|\tilde{c}_1\|_{L^2}^2 + \|\tilde{c}_2\|_{L^2}^2) \leq C (\|\tilde{\rho}\|_{L^2}^2 + \|\tilde{u}\|_{L^2}^2 + \|\tilde{c}_1\|_{L^2}^2 + \|\tilde{c}_2\|_{L^2}^2). \tag{3.20}$$

The Gronwall's inequality implies

$$\sup_{0 \le t \le T_1} (\|\tilde{\rho}\|_{L^2}^2 + \|\tilde{u}\|_{L^2}^2 + \|\tilde{c}_1\|_{L^2}^2 + \|\tilde{c}_2\|_{L^2}^2)$$

$$\le e^{cT_1} (\|\tilde{\rho}(0)\|_{L^2}^2 + \|\tilde{u}(0)\|_{L^2}^2 + \|\tilde{c}_1(0)\|_{L^2}^2 + \|\tilde{c}_2(0)\|_{L^2}^2). \quad (3.21)$$

Since the initial data of $(\tilde{\rho}, \tilde{u}, \tilde{c}_1, \tilde{c}_2)$ are all zero for $T > 0$, that implies the uniqueness of the local solution. $\quad\square$

3.2 A priori estimates

In this subsection, we provide some estimates for the solutions for any $t > 0$. We use the energy method to obtain uniform-in-time a priori estimates for smooth solutions to Cauchy problems (2.2)–(2.3).

Lemma 3.2 (A priori estimates). *Let $U(t) = (\rho, u, c_1, c_2) \in C([0, T]; H^N(\mathbb{R}^3))$ be the smooth solution to the Cauchy problem (2.2)–(2.3) for $T > 0$ with*

$$\sup_{0 \le t \le T} \|(\rho, u, c_1, c_2)(t)\|_N \le \epsilon \quad (3.22)$$

for $0 < \epsilon \le 1$. Then, there are $\epsilon_0 > 0$, $C_0 > 0$, $\lambda_1 > 0$ and $\lambda_2 > 0$ such that for any $\epsilon \le \epsilon_0$,

$$\|[\rho, u, c_1, c_2]\|_{H^N}^2 + \lambda_1 \int_0^t \|\nabla[u, c_1, c_2]\|_{H^N}^2 + \lambda_2 \int_0^t \|[\rho, c_1, c_2]\|_{H^N}^2 \le C_0 \|[\rho_0, u_0, c_{1,0}, c_{2,0}]\|_{H^N}^2 \quad (3.23)$$

holds for any $t \in [0, T]$.

Proof. First, we find the zero-order estimates. For the estimate of ρ, multiplying ρ to both sides of the first equation of (2.2) and taking integrations in $x \in \mathbb{R}^3$, we obtain

$$\int_{\mathbb{R}^3} \rho \rho_t \mathrm{d}x + n_\infty \int_{\mathbb{R}^3} \rho \nabla \cdot u \mathrm{d}x + n_\infty \int_{\mathbb{R}^3} |\rho|^2 \mathrm{d}x = -\int_{\mathbb{R}^3} \rho \nabla \cdot (\rho u) \mathrm{d}x - \int_{\mathbb{R}^3} \rho \rho^2 \mathrm{d}x.$$

Using integration by parts and the Cauchy–Schwarz inequality, we have

$$\frac{1}{2} \int_{\mathbb{R}^3} (\rho^2)_t \mathrm{d}x + n_\infty \int_{\mathbb{R}^3} |\rho|^2 \mathrm{d}x + n_\infty \int_{\mathbb{R}^3} \rho \nabla \cdot u \mathrm{d}x$$

$$\le \frac{1}{2} \sup_x |\nabla u| \int_{\mathbb{R}^3} |\rho|^2 \mathrm{d}x + \sup_x |\rho| \int_{\mathbb{R}^3} |\rho|^2 \mathrm{d}x$$

$$\le C \|\rho, u\|_{H^N} \int_{\mathbb{R}^3} |\rho|^2 \mathrm{d}x. \quad (3.24)$$

Now, we estimate u by multiplying the second equation of (2.2) by u and integrating over $\mathbb{R}^3$. Then, we have

$$\int_{\mathbb{R}^3} u \cdot u_t \mathrm{d}x + \int_{\mathbb{R}^3} u \cdot (u \cdot \nabla u) \mathrm{d}x - \delta \int_{\mathbb{R}^3} u \cdot \Delta u \mathrm{d}x + \frac{p'(n_\infty)}{n_\infty} \int_{\mathbb{R}^3} u \cdot \nabla \rho \mathrm{d}x$$

$$= \int_{\mathbb{R}^3} u \cdot \nabla c_1^2 \mathrm{d}x - \int_{\mathbb{R}^3} u \cdot \nabla c_2^2 \mathrm{d}x - \int_{\mathbb{R}^3} u \cdot \left(\frac{p'(\rho + n_\infty)}{\rho + n_\infty} - \frac{p'(n_\infty)}{n_\infty} \right) \nabla \rho \mathrm{d}x.$$

By using integration by parts and the Cauchy–Schwarz inequality, we have

$$\frac{1}{2} \int_{\mathbb{R}^3} (u^2)_t \mathrm{d}x + \delta \int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x - \frac{p'(n_\infty)}{n_\infty} \int_{\mathbb{R}^3} \rho \nabla \cdot u \mathrm{d}x$$

$$\le \|u\|_{H^1} \int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x + C \|u\|_{H^N} \int_{\mathbb{R}^3} (|c_1|^2 + |c_2|^2 + |\rho|^2) \mathrm{d}x. \quad (3.25)$$

For the estimates of c_1, we multiply c_1 to both sides of the equation of c_1 and integrate with respect to x, and we have

$$\int_{\mathbb{R}^3} c_1(c_1)_t \mathrm{d}x - \int_{\mathbb{R}^3} c_1 \Delta c_1 \mathrm{d}x + (a_{12} - n_\infty a_{11}) \int_{\mathbb{R}^3} |c_1|^2 \mathrm{d}x \le a_{11} \sup_x |\rho| \int_{\mathbb{R}^3} |c_1|^2 \mathrm{d}x.$$

By using integration by parts, we have

$$\frac{1}{2} \int_{\mathbb{R}^3} (c_1^2)_t \mathrm{d}x + \int_{\mathbb{R}^3} |\nabla c_1|^2 \mathrm{d}x + (a_{12} - n_\infty a_{11}) \int_{\mathbb{R}^3} |c_1|^2 \mathrm{d}x \le a_{11} \|\rho\|_{H^2} \int_{\mathbb{R}^3} |c_1|^2 \mathrm{d}x. \quad (3.26)$$

Similar to above, from the equation of c_2, we have

$$\frac{1}{2} \int_{\mathbb{R}^3} (c_2^2)_t \mathrm{d}x + \int_{\mathbb{R}^3} |\nabla c_2|^2 \mathrm{d}x + (a_{22} - n_\infty a_{21}) \int_{\mathbb{R}^3} |c_2|^2 \mathrm{d}x \le a_{21} \|\rho\|_{H^2} \int_{\mathbb{R}^3} |c_2|^2 \mathrm{d}x. \quad (3.27)$$

Consider the linear combination $d_1 \times (3.24) + (3.25) + (3.26) + (3.27)$, where $d_1 = \frac{p'(n_\infty)}{n_\infty^2}$. We see that as long as $\mathcal{E}_N^{\frac{1}{2}}(U) = \|U\|_{H^N}$ is small so that

$$(a_{12} - n_\infty a_{11}) > a_{11} \mathcal{E}_N^{\frac{1}{2}}(U),$$
$$(a_{22} - n_\infty a_{21}) > a_{21} \mathcal{E}_N^{\frac{1}{2}}(U)$$

are satisfied, the linear combination yields

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} \left(d_1 |\rho|^2 + |u|^2 + |c_1|^2 + |c_2|^2 \right) \mathrm{d}x + n_\infty \int_{\mathbb{R}^3} |\rho|^2 \mathrm{d}x + \delta \int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x$$
$$+ \int_{\mathbb{R}^3} |\nabla c_1|^2 \mathrm{d}x + \int_{\mathbb{R}^3} |\nabla c_2|^2 \mathrm{d}x + (a_{12} - n_\infty a_{11}) \int_{\mathbb{R}^3} |c_1|^2 \mathrm{d}x + (a_{22} - n_\infty a_{21}) \int_{\mathbb{R}^3} |c_2|^2 \mathrm{d}x$$
$$\le 0. \quad (3.28)$$

Now, we make estimates on the high-order derivatives of (ρ, u, c_1, c_2). Take α with $1 \le |\alpha| \le N$. Applying ∂^α to the first equation of (2.2), multiplying by $\partial^\alpha \rho$ and then integrating in x, we have

$$\int_{\mathbb{R}^3} \partial^\alpha \rho \partial^\alpha \rho_t \mathrm{d}x + n_\infty \int_{\mathbb{R}^3} \partial^\alpha \rho \partial^\alpha \nabla \cdot u \mathrm{d}x + n_\infty \int_{\mathbb{R}^3} \partial^\alpha \rho \partial^\alpha \rho \mathrm{d}x$$
$$= - \int_{\mathbb{R}^3} \partial^\alpha \rho \partial^\alpha \nabla \cdot (\rho u) \mathrm{d}x - \int_{\mathbb{R}^3} \partial^\alpha \rho \partial^\alpha \rho^2 \mathrm{d}x.$$

By using integration by parts and Cauchy-Schwarz inequality, we obtain

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} (\partial^\alpha \rho)^2 \mathrm{d}x + n_\infty \int_{\mathbb{R}^3} |\partial^\alpha \rho|^2 \mathrm{d}x + n_\infty \int_{\mathbb{R}^3} \partial^\alpha \rho \partial^\alpha \nabla \cdot u \mathrm{d}x$$
$$= \int_{\mathbb{R}^3} \partial^\alpha \rho \sum_{\beta=0}^\alpha C_\alpha^\beta \partial^\beta \nabla \cdot u \partial^{\alpha-\beta} \rho \mathrm{d}x + \int_{\mathbb{R}^3} \partial^\alpha \rho \sum_{\beta=0}^\alpha C_\alpha^\beta \partial^\beta u \cdot \partial^{\alpha-\beta} \nabla \rho \mathrm{d}x - \int_{\mathbb{R}^3} \partial^\alpha \rho \partial^\alpha \rho^2 \mathrm{d}x$$
$$\le C \|u\|_{H^N} \int_{\mathbb{R}^3} |\partial^\alpha \rho|^2 + C \|\rho\|_{H^N} \int_{\mathbb{R}^3} |\partial^\alpha \rho|^2 + |\partial^\alpha \nabla u|^2 \mathrm{d}x. \quad (3.29)$$

Similarly for $\partial^\alpha u$, what follows from (2.2)$_2$ is

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} (\partial^\alpha u)^2 \mathrm{d}x - \delta \int_{\mathbb{R}^3} \partial^\alpha u \cdot \partial^\alpha \Delta u \mathrm{d}x + \frac{p'(n_\infty)}{n_\infty} \int_{\mathbb{R}^3} \partial^\alpha u \cdot \partial^\alpha \nabla \rho \mathrm{d}x$$
$$= - \int_{\mathbb{R}^3} \partial^\alpha u \cdot \partial^\alpha (u \cdot \nabla u) \mathrm{d}x + \int_{\mathbb{R}^3} \partial^\alpha u \cdot \partial^\alpha \nabla c_1^2 \mathrm{d}x - \int_{\mathbb{R}^3} \partial^\alpha u \cdot \partial^\alpha \nabla c_2^2 \mathrm{d}x$$
$$- \int_{\mathbb{R}^3} \partial^\alpha u \cdot \partial^\alpha \left(\left(\frac{p'(\rho+n_\infty)}{\rho+n_\infty} - \frac{p'(n_\infty)}{n_\infty} \right) \nabla \rho \right) \mathrm{d}x.$$

By using integration by parts and the Cauchy–Schwarz inequality, we have

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}(\partial^\alpha u)^2\,dx + \delta\int_{\mathbb{R}^3}|\partial^\alpha\nabla u|^2dx - \frac{p'(n_\infty)}{n_\infty}\int_{\mathbb{R}^3}\partial^\alpha\nabla\cdot u\,\partial^\alpha\rho\,dx$$

$$\leq C\|u\|_{H^N}\int_{\mathbb{R}^3}|\partial^\alpha u|^2dx + C\|c_1\|_{H^N}\int_{\mathbb{R}^3}(|\partial^\alpha u|^2+|\partial^\alpha\nabla c_1|^2)dx$$

$$+ C\|c_2\|_{H^N}\int_{\mathbb{R}^3}(|\partial^\alpha u|^2+|\partial^\alpha\nabla c_2|^2)dx + C\|\rho\|_{H^N}\int_{\mathbb{R}^3}|\partial^\alpha u|^2dx + |\partial^\alpha\rho|^2dx. \qquad (3.30)$$

Similarly, we estimate c_1, c_2 as follows:

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}(\partial^\alpha c_1)^2 + \int_{\mathbb{R}^3}|\nabla\partial^\alpha c_1|^2ds + (a_{12}-n_\infty a_{11})\int_{\mathbb{R}^3}|\partial^\alpha c_1|^2ds$$

$$\leq C\|\rho\|_{H^N}\int_{\mathbb{R}^3}\|\partial^\alpha c_1|^2ds + C\|c_1\|_{H^N}\int_{\mathbb{R}^3}(|\partial^\alpha c_1|^2+|\partial^\alpha\rho|^2)ds, \qquad (3.31)$$

and

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}(\partial^\alpha c_2)^2 + \int_{\mathbb{R}^3}|\nabla\partial^\alpha c_2|^2ds + (a_{22}-n_\infty a_{21})\int_{\mathbb{R}^3}|\partial^\alpha c_2|^2ds$$

$$\leq C\|\rho\|_{H^N}\int_{\mathbb{R}^3}\|\partial^\alpha c_2|^2ds + C\|c_2\|_{H^N}\int_{\mathbb{R}^3}(|\partial^\alpha c_2|^2+|\partial^\alpha\rho|^2)ds. \qquad (3.32)$$

Then, after taking the summation over $1\leq|\alpha|\leqslant N$ and the combination $(3.29)\times d_1 + (3.30) + (3.31) + (3.32)$, we obtain

$$\frac{1}{2}\frac{d}{dt}\sum_{1\leq|\alpha|\leq N}C_\alpha\int_{\mathbb{R}^3}|\partial^\alpha(\rho,u,c_1,c_2)|^2 + \lambda_1\sum_{1\leq|\alpha|\leq N}\int_{\mathbb{R}^3}|\partial^\alpha\nabla(u,c_1,c_2)|^2dx$$

$$+ \lambda_2\sum_{1\leq|\alpha|\leq N}\int_{\mathbb{R}^3}|\partial^\alpha(\rho,c_1,c_2)|^2dx \leq 0, \quad (3.33)$$

for some positive constants C_α, λ_1 and λ_2. Therefore (3.23) follows from the further linear combination of (3.28) and (3.33) and the time integration over $[0, T]$. This completes the proof of Lemma 3.2. $\qquad\square$

Now, we are ready to present the proof of Proposition 2.1.

Proof of Proposition 2.1. Choose a positive constant $M = \min\{\epsilon_0, \epsilon_1\}$, where $\epsilon_0 > 0$ and $\epsilon_1 > 0$ are given in Lemma 3.1 and Lemma 3.2.

Let $U_0 \in H^N(\mathbb{R}^3)$ satisfy $\|U_0\|_{H^N} < \frac{M}{2\sqrt{C_0+1}}$. Now, let us define

$$T = \{t \geq 0 : \sup_{0\leq s\leq t}\|U(s)\|_{H^N}\leq M\}.$$

Since $\|U_0\|_{H^N}\leq \frac{M}{2\sqrt{C_0+1}} \leq \frac{M}{2} < M \leq \epsilon_0$, then $T > 0$ holds from the local existence result. If T is finite, from the definition of T, we have

$$\sup_{0\leq s\leq t}\|U\|_{H^N} = M. \qquad (3.34)$$

On the other hand, from a priori estimates, we have

$$\sup_{0\leq s\leq t}\|U(s)\|_{H^N}\leq \sqrt{C_0}\|U_0\|_{H^N}\leq \frac{M\sqrt{C_0}}{2\sqrt{C_0+1}} \leq \frac{M}{2},$$

which is a contradiction to (3.34). Therefore, $T = \infty$ holds. This implies that the local solution $U(t)$ obtained in Lemma 3.1 can be extended to infinity in time. Thus, we have a global solution $(\rho, u, c_1, c_2)(t) \in C([0,\infty); H^N)$. This completes the proof of Proposition 2.1. $\qquad\square$

4 Linearized homogeneous system

In this section, to study the time-decay property of solutions to the nonlinear system (2.2), we have to consider the following Cauchy problem arising from the system (2.2)–(2.3)

$$\begin{cases} \partial_t \rho + n_\infty \nabla \cdot u + n_\infty \rho = g_1 \\ \partial_t u - \delta \Delta u + \frac{p'(n_\infty)}{n_\infty} \nabla \rho = g_2 \\ \partial_t c_1 - \Delta c_1 + (a_{12} - a_{11}) c_1 = g_3 \\ \partial_t c_2 - \Delta c_2 + (a_{22} - a_{21}) c_2 = g_4, \end{cases} \tag{4.1}$$

with initial data

$$(\rho, u, c_1, c_2)|_{t=0} = U_0 = (\rho_0, u_0, c_{1,0}, c_{2,0}). \tag{4.2}$$

Here, the nonlinear source term takes the form

$$\begin{cases} g_1 = -\nabla \cdot (\rho u) - \rho^2 \\ g_2 = -u \cdot \nabla u + \nabla c_1^2 - \nabla c_2^2 - \left(\frac{p'(\rho + n_\infty)}{\rho + n_\infty} - \frac{p'(n_\infty)}{n_\infty} \right) \nabla \rho. \\ g_3 = a_{11} \rho c_1 \\ g_4 = a_{21} \rho c_2. \end{cases} \tag{4.3}$$

To obtain the time-decay rates of the solution to the system (4.1) in the next section, we are concerned with the following Cauchy problem for the linearized homogenous system corresponding to (4.1)

$$\begin{cases} \partial_t \rho + n_\infty \nabla \cdot u + n_\infty \rho = 0 \\ \partial_t u - \delta \Delta u + \frac{p'(n_\infty)}{n_\infty} \nabla \rho = 0 \\ \partial_t c_1 - \Delta c_1 + (a_{12} - a_{11}) c_1 = 0 \\ \partial_t c_2 - \Delta c_2 + (a_{22} - a_{21}) c_2 = 0. \end{cases} \tag{4.4}$$

In this section, we always denote $U_1 = [\rho, u]$ as the solution to the linearized homogeneous system

$$\begin{cases} \partial_t \rho + n_\infty \nabla \cdot u + n_\infty \rho = 0 \\ \partial_t u - \delta \Delta u + \frac{p'(n_\infty)}{n_\infty} \nabla \rho = 0, \end{cases} \tag{4.5}$$

with the initial data $U_1|_{t=0} = U_{1,0} = (\rho_0, u_0)$ in $\mathbb{R}^3$.

4.1 Representation of solutions

We first find the explicit representation of the Fourier transform of the solution $U_1 = [\rho, u]$ for the system

$$\begin{cases} \rho_t + n_\infty \nabla \cdot u + n_\infty \rho = 0 \\ u_t - \delta \Delta u + \frac{p'(n_\infty)}{n_\infty} \nabla \rho = 0, \end{cases} \tag{4.6}$$

with initial data $U_1|_{t=0} = U_{1,0} = (\rho_0, u_0)$.

After taking the Fourier transform in x for the first equation of (4.6), we have

$$\hat{\rho}_t + n_\infty i \xi \hat{u} + n_\infty \hat{\rho} = 0, \tag{4.7}$$

with initial data $\hat{\rho}|_{t=0} = \hat{\rho}_0.$

Similarly, by taking the Fourier transform for the second equation of (4.6), we get

$$\hat{u}_t + \delta|\xi|^2\hat{u} + \frac{p'(n_\infty)}{n_\infty}i\xi\hat{\rho} = 0, \tag{4.8}$$

with initial data $\hat{u}|_{t=0} = \hat{u}_0$.

Further, by taking the dot product of (4.8) with $\tilde{\xi}$, we have

$$\tilde{\xi} \cdot \hat{u}_t + \delta|\xi|^2\tilde{\xi} \cdot \hat{u} + i\frac{p'(n_\infty)}{n_\infty}\tilde{\xi} \cdot \xi\hat{\rho} = 0. \tag{4.9}$$

Here and in the sequel we set $\tilde{\xi} = \frac{\xi}{|\xi|}$ for $|\xi| \neq 0$.

Then, we have

$$\begin{cases} \hat{\rho}_t + in_\infty\xi \cdot \hat{u} + n_\infty\hat{\rho} = 0 \\ \tilde{\xi} \cdot \hat{u}_t + \delta|\xi|^2\tilde{\xi} \cdot \hat{u} + i\frac{p'(n_\infty)}{n_\infty}\tilde{\xi} \cdot \xi\hat{\rho} = 0. \end{cases} \tag{4.10}$$

We can rewrite (4.10) as

$$\partial_t\hat{U} = A(\xi)\hat{U}, \tag{4.11}$$

with $\hat{U}(\xi, t) = (\hat{\rho}(\xi, t), \tilde{\xi} \cdot \hat{u}(\xi, t))^T$ and

$$A(\xi) = \begin{bmatrix} -n_\infty & -in_\infty|\xi| \\ -i\frac{p'(n_\infty)}{n_\infty}|\xi| & -\delta|\xi|^2 \end{bmatrix},$$

where T denotes the transpose of a row vector. Then,

$$\det(A - \lambda I) = \lambda^2 + (\delta\xi^2 + n_\infty)\lambda + \delta n_\infty|\xi|^2 + p'(n_\infty)|\xi|^2 = 0.$$

The eigenvalues of the system are as follows

$$\lambda_1 = -\frac{1}{2}(\delta\xi^2 + n_\infty) + \frac{1}{2}\sqrt{(\delta\xi^2 + n_\infty)^2 - 4|\xi|^2(\delta n_\infty + p'(n_\infty))}$$

$$\lambda_2 = -\frac{1}{2}(\delta\xi^2 + n_\infty) - \frac{1}{2}\sqrt{(\delta\xi^2 + n_\infty)^2 - 4|\xi|^2(\delta n_\infty + p'(n_\infty))}.$$

Therefore, the eigenvectors corresponding to the eigenvalues λ of $A(\xi)$ that satisfy $(A - \lambda I)X = 0$ are

$$v_1 = \begin{bmatrix} in_\infty|\xi| \\ -(n_\infty + \lambda_1) \end{bmatrix}$$

and

$$v_2 = \begin{bmatrix} in_\infty|\xi| \\ -(n_\infty + \lambda_2) \end{bmatrix}.$$

From the work above, one can define the general solution of (4.10) as

$$\begin{bmatrix} \hat{\rho} \\ \tilde{\xi} \cdot \hat{u} \end{bmatrix} = \begin{bmatrix} in_\infty|\xi|e^{\lambda_1 t} & in_\infty|\xi|e^{\lambda_2 t} \\ -(n_\infty + \lambda_1)e^{\lambda_1 t} & -(n_\infty + \lambda_2)e^{\lambda_2 t} \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}, \tag{4.12}$$

where d_1, d_2 satisfy

$$\begin{bmatrix} \hat{\rho}|_{t=0} \\ \tilde{\xi} \cdot \hat{u}|_{t=0} \end{bmatrix} = \begin{bmatrix} in_\infty|\xi| & in_\infty|\xi| \\ -(n_\infty + \lambda_1) & -(n_\infty + \lambda_2) \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}.$$

From this, we deduce that

$$\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \frac{1}{in_\infty|\xi|(\lambda_1 - \lambda_2)} \begin{bmatrix} -(n_\infty + \lambda_2) & -in_\infty|\xi| \\ (n_\infty + \lambda_1) & in_\infty|\xi| \end{bmatrix} \begin{bmatrix} \hat{\rho}_0 \\ \tilde{\xi} \cdot \hat{u}_0 \end{bmatrix}. \tag{4.13}$$

Therefore, we have

$$\begin{bmatrix} \hat{\rho} \\ \tilde{\xi} \cdot \hat{u} \end{bmatrix} = \frac{1}{in_\infty |\xi|(\lambda_1 - \lambda_2)} \begin{bmatrix} in_\infty |\xi| e^{\lambda_1 t} & in_\infty |\xi| e^{\lambda_2 t} \\ -(n_\infty + \lambda_1) e^{\lambda_1 t} & -(n_\infty + \lambda_2) e^{\lambda_2 t} \end{bmatrix} \begin{bmatrix} -(n_\infty + \lambda_2) & -in_\infty |\xi| \\ (n_\infty + \lambda_1) & in_\infty |\xi| \end{bmatrix} \begin{bmatrix} \hat{\rho}_0 \\ \tilde{\xi} \cdot \hat{u}_0 \end{bmatrix}. \quad (4.14)$$

It is straightforward to obtain

$$\hat{\rho} = \frac{(\lambda_1 + n_\infty) e^{\lambda_2 t} - (\lambda_2 + n_\infty) e^{\lambda_1 t}}{(\lambda_1 - \lambda_2)} \hat{\rho}_0 - in_\infty \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{(\lambda_1 - \lambda_2)} \xi \cdot \hat{u}_0 \quad (4.15)$$

and

$$\tilde{\xi} \cdot \hat{u} = \frac{(n_\infty + \lambda_1)(n_\infty + \lambda_2)}{in_\infty |\xi|} \left(\frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2} \right) \hat{\rho}_0 + \frac{(\lambda_1 + n_\infty) e^{\lambda_1 t} - (\lambda_2 + n_\infty) e^{\lambda_2 t}}{\lambda_1 - \lambda_2} \tilde{\xi} \cdot \hat{u}_0. \quad (4.16)$$

Moreover, by taking the curl for the second equation of (4.6), we have

$$\nabla \times u_t - \delta \nabla \times \Delta u + \frac{p'(n_\infty)}{n_\infty} \nabla \times \nabla \rho = 0, \quad (4.17)$$

since $\nabla \times \nabla \rho = 0$ implies

$$\partial_t (\nabla \times u) - \delta \nabla \times \Delta u = 0.$$

Taking the Fourier transform in x for the above equation, we have

$$\partial_t (\tilde{\xi} \times \hat{u}) + \delta |\xi|^2 (\tilde{\xi} \times \hat{u}) = 0. \quad (4.18)$$

Initial data is given as

$$(\tilde{\xi} \times \hat{u})|_{t=0} = \tilde{\xi} \times \hat{u}_0. \quad (4.19)$$

By solving the initial value problem (4.18) and (4.19), we have

$$\tilde{\xi} \times \hat{u} = e^{-\delta |\xi|^2 t} \tilde{\xi} \times \hat{u}_0. \quad (4.20)$$

For $t \geq 0$ and $\xi \in \mathbb{R}^3$ with $|\xi| \neq 0$, one has the decomposition $\hat{u} = \tilde{\xi} \tilde{\xi} \cdot \hat{u} - \tilde{\xi} \times (\tilde{\xi} \times \hat{u})$. It is straightforward to get

$$\hat{u} = \frac{(n_\infty + \lambda_1)(n_\infty + \lambda_2)}{in_\infty |\xi|^2} \left(\frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2} \right) \xi \cdot \hat{\rho}_0$$
$$+ \left(\frac{(\lambda_1 + n_\infty) e^{\lambda_1 t} - (\lambda_2 + n_\infty) e^{\lambda_2 t}}{\lambda_1 - \lambda_2} \right) \tilde{\xi} \tilde{\xi} \cdot \hat{u}_0 - e^{-\delta |\xi|^2 t} \tilde{\xi} \times (\tilde{\xi} \times \hat{u}_0). \quad (4.21)$$

Then

$$\hat{u} = \frac{(n_\infty + \lambda_1)(n_\infty + \lambda_2)}{in_\infty |\xi|} \left(\frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2} \right) \frac{\xi}{|\xi|} \hat{\rho}_0$$
$$+ \left(\frac{(\lambda_1 + n_\infty) e^{\lambda_1 t} - (\lambda_2 + n_\infty) e^{\lambda_2 t}}{\lambda_1 - \lambda_2} \right) \frac{\xi \otimes \xi}{|\xi|^2} \hat{u}_0 + e^{-\delta |\xi|^2 t} (I_3 - \frac{\xi \otimes \xi}{|\xi|^2}) \hat{u}_0. \quad (4.22)$$

After summarizing the above computations on the explicit representation of the Fourier transform of the solution $U_1 = [\rho, u]$, we have

$$\begin{bmatrix} \hat{\rho}(\xi, t) \\ \hat{u}(\xi, t) \end{bmatrix} = \hat{G}(\xi, t) \begin{bmatrix} \hat{\rho}(\xi, 0) \\ \hat{u}(\xi, 0) \end{bmatrix}.$$

We can verify the exact expression of the Fourier transform $\hat{G}(\xi, t)$ of Green's function $G(\xi, t) = e^{tB}$ as

$$
\hat{G}(\xi, t) = \begin{bmatrix} \hat{G}_{11} & \hat{G}_{12} \\ \hat{G}_{21} & \hat{G}_{22} \end{bmatrix}
$$

$$
= \begin{bmatrix} \frac{(\lambda_1 + n_\infty)e^{\lambda_2 t} - (\lambda_2 + n_\infty)e^{\lambda_1 t}}{\lambda_1 - \lambda_2} & -in_\infty \xi \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{(\lambda_1 - \lambda_2)} \\ \frac{(n_\infty + \lambda_1)(n_\infty + \lambda_2)\xi}{in_\infty |\xi|^2} \left(\frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2} \right) & \frac{(\lambda_1 + n_\infty)e^{\lambda_1 t} - (\lambda_2 + n_\infty)e^{\lambda_2 t}}{\lambda_1 - \lambda_2} \frac{\xi \otimes \xi}{|\xi|^2} + e^{-\delta \xi^2 t} \left(I_3 - \frac{\xi \otimes \xi}{|\xi|^2} \right) \end{bmatrix}. \quad (4.23)
$$

4.2 L^2–L^q time-decay property

In this subsection, we use (4.23) to obtain the refined L^2–L^q time-decay property for

$$
U_1 = (\rho, u) = e^{tB}U_{1,0},
$$

where e^{tB} is the linear solution operator for $t \geq 0$. For this, we need to find the time-frequency pointwise estimate on $\hat{\rho}, \hat{u}$ in the following lemma.

Lemma 4.1. *Let $U_1 = [\rho, u]$ be the solution to the linear homogeneous system* (4.6) *with the initial data $U_1|_{t=0} = (\rho_0, u_0)$. Then there exist constants $\epsilon > 0, \lambda > 0, C > 0$ such that for all $t > 0, |\xi| \leq \epsilon$,*

$$
|\hat{\rho}(\xi, t)| \leq C(|\xi|^2 e^{-\lambda |\xi|^2 t} + e^{-n_\infty \lambda t})|\hat{\rho}_0(\xi)| + C(|\xi|e^{-\lambda |\xi|^2 t} + |\xi|e^{-n_\infty \lambda t})|\hat{u}_0(\xi)|, \quad (4.24)
$$

$$
|\hat{u}(\xi, t)| \leq C|\xi|(e^{-\lambda |\xi|^2 t} + e^{-n_\infty \lambda t})|\hat{\rho}_0(\xi)| + C(e^{-\lambda |\xi|^2 t} + |\xi|^2 e^{-n_\infty \lambda t})|\hat{u}_0(\xi)|, \quad (4.25)
$$

and for all $t > 0, |\xi| \geq \epsilon$,

$$
|\hat{\rho}(\xi, t)| \leq Ce^{-\lambda t}|\hat{\rho}_0(\xi), \hat{u}_0(\xi)|, \quad (4.26)
$$

$$
|\hat{u}(\xi, t)| \leq Ce^{-\lambda t}|\hat{\rho}_0(\xi), \hat{u}_0(\xi)|. \quad (4.27)
$$

Proof. In order to obtain the upper bound of $\hat{\rho}(\xi, t)$ and $\hat{u}(\xi, t)$, we have to estimate $\hat{G}_{11}$, $\hat{G}_{12}$, $\hat{G}_{21}$, and $\hat{G}_{22}$ in (4.23). To do so, we need to deal with the low frequency $|\xi| \leq \epsilon$ and high frequency $|\xi| > \epsilon$. By using the definition of the eigenvalue, we can analyze the eigenvalue for $|\xi| \to 0$ as

$$
\lambda_1 \sim -O(1)|\xi|^2,
$$
$$
\lambda_2 \sim -n_\infty + O(1)|\xi|^2.
$$

On the other hand, we have the leading orders of the eigenvalue for $|\xi| \to \infty$ as

$$
\lambda_1 \sim -O(1),
$$
$$
\lambda_2 \sim -\delta \xi^2 + O(1).
$$

Now, we can estimate $\hat{G}(\xi, t)$ as follows: For $|\xi| \leq \epsilon$,

$$
|\hat{G}_{11}| \leq C(|\xi|^2 e^{-\lambda |\xi|^2 t} + e^{-n_\infty \lambda t}),
$$
$$
|\hat{G}_{12}| \leq |\xi|(e^{-\lambda |\xi|^2 t} + e^{-n_\infty \lambda t}),
$$
$$
|\hat{G}_{21}| \leq C|\xi|(e^{-\lambda |\xi|^2 t} + e^{-n_\infty \lambda t}),
$$
$$
|\hat{G}_{22}| \leq C(e^{-\lambda |\xi|^2 t} + |\xi|^2 e^{-n_\infty \lambda t}) + Ce^{-\delta |\xi|^2 t},
$$
$$
\leq C(e^{-\lambda |\xi|^2 t} + |\xi|^2 e^{-n_\infty \lambda t}),
$$

and for $|\xi| > \epsilon$

$$|\hat{G}_{11}| \leq Ce^{-O(1)\lambda t} \leq Ce^{-\lambda t},$$
$$|\hat{G}_{12}| = |\hat{G}_{21}| \leq Ce^{-\lambda t},$$
$$|\hat{G}_{22}| \leq Ce^{-\delta|\xi|^2 t} + Ce^{-O(1)t} \leq Ce^{-\lambda t}.$$

Since the real parts of the eigenvalues are negative except when $\xi = 0$, $\hat{G}$ decays exponentially when the eigenvalues coalesce.

Therefore, after plugging the above computations into (4.15) and (4.22), it holds that

$$|\hat{\rho}(\xi, t)| \leq C(|\xi|^2 e^{-\lambda|\xi|^2 t} + e^{-n_\infty \lambda t})|\hat{\rho}_0(\xi)| + C(|\xi| e^{-\lambda|\xi|^2 t} + |\xi| e^{-n_\infty \lambda t})|\hat{u}_0(\xi)|$$

and

$$|\hat{u}(\xi, t)| \leq C|\xi|(e^{-\lambda|\xi|^2 t} + e^{-n_\infty \lambda t})|\hat{\rho}_0(\xi)| + C(e^{-\lambda|\xi|^2 t} + |\xi|^2 e^{-n_\infty \lambda t})|\hat{u}_0(\xi)|,$$

for $|\xi| \leq \epsilon$. This proves (4.24) and (4.25). Finally, (4.26) and (4.27) can be proven in the completely same way as for (4.24) and (4.25). This completes the proof of Lemma 4.1. $\qquad\square$

Theorem 4.2. *Let $2 \leq q \leq \infty$, and let $m \geq 0$ be an integer. Suppose that $U_1 = e^{Bt}U_{1,0}$ is the solution to the Cauchy problem (4.6) with the initial data $U_{1,0} = (\rho_0, u_0)$. Then $U_1 = [\rho, u]$ satisfies the following time-decay property:*

$$\|\nabla^m \rho(t)\|_{L^q} \leq C(1+t)^{-\frac{3}{2}(1-\frac{1}{q})-\frac{m+1}{2}}\|\rho_0, u_0\|_{L^1} + e^{-\lambda t}\|\nabla^{m+[3(\frac{1}{2}-\frac{1}{q})]_+}(\rho_0, u_0)\|_{L^2}, \quad (4.28)$$

$$\|\nabla^m u(t)\|_{L^q} \leq C(1+t)^{-\frac{3}{2}(1-\frac{1}{q})-\frac{m}{2}}\|\rho_0, u_0\|_{L^1} + e^{-\lambda t}\|\nabla^{m+[3(\frac{1}{2}-\frac{1}{q})]_+}(\rho_0, u_0)\|_{L^2}, \quad (4.29)$$

for any $t \geq 0$, where $C = C(m, q)$ and $[3(\frac{1}{2} - \frac{1}{q})]_+$ is defined as

$$\left[3\left(\frac{1}{2} - \frac{1}{q}\right)\right]_+ = \begin{cases} 0 & \text{if } q = 2 \\ [3(\frac{1}{2} - \frac{1}{q})]_- + 1 & \text{if } q \neq 2 \end{cases} \quad (4.30)$$

where $[\cdot]_-$ denotes the integer part of the argument.

Proof. Take $2 \leq q \leq \infty$ and an integer $m \geq 0$. Set $U_1 = e^{Bt}U_{1,0}$. From the Hausdorff–Young inequality,

$$\|\nabla^m \rho(t)\|_{L^q(\mathbb{R}^3_x)} \leq C\||\xi|^m \hat{\rho}(\xi, t)\|_{L^{q'}(\mathbb{R}^3_\xi)}$$
$$\leq C\||\xi|^m \hat{\rho}(\xi, t)\|_{L^{q'}(|\xi| \leq \epsilon)} + C\||\xi|^m \hat{\rho}(\xi, t)\|_{L^{q'}(|\xi| \geq \epsilon)}, \quad (4.31)$$

where $\frac{1}{q} + \frac{1}{q'} = 1$.

We estimate the first term of (4.31) by using (4.24), as follows:

$$\||\xi|^m \hat{\rho}(\xi, t)\|_{L^{q'}(|\xi| \leq \epsilon)}^{q'} \leq c \int_{|\xi| \leq \epsilon} [(|\xi|^{(m+2)q'} e^{-\lambda q'|\xi|^2 t} + |\xi|^{mq'} e^{-n_\infty \lambda q' t})|\hat{\rho}_0(\xi)|^{q'}$$
$$+ c(|\xi|^{mq'+q'} e^{-\lambda q'|\xi|^2 t} + |\xi|^{mq'+q'} e^{-n_\infty \lambda q' t})|\hat{u}_0(\xi)|^{q'}]d\xi$$
$$\leq C \sup_\xi |\hat{\rho}_0|^{q'} \int_{|\xi| \leq \epsilon} (|\xi|^{(m+2)q'} e^{-q'\lambda|\xi|^2(1+t)+q'\lambda|\xi|^2} + |\xi|^{mq'} e^{-n_\infty \lambda q' t})d\xi$$
$$+ C \sup_{\hat{\xi}} |\hat{u}_0|^{q'} \int_{|\xi| \leq \epsilon} (|\xi|^{(m+1)q'} e^{-\lambda q'|\xi|^2(1+t)+\lambda q'|\xi|^2} + |\xi|^{(m+1)q'} e^{-n_\infty \lambda q' t})d\xi$$
$$\leq C(1+t)^{-\frac{mq'+2q'+3}{2}}\|\rho_0\|_{L^1}^{q'} + C(1+t)^{-\frac{mq'+q'+3}{2}}\|u_0\|_{L^1}^{q'}$$
$$+ Ce^{-n_\infty \lambda q' t}\|[\rho_0, u_0]\|_{L^1}^{q'}.$$

Thus,

$$
\begin{aligned}
\||\xi|^m \hat{\rho}(\xi,t)\|_{L^{q'}(|\xi|\le\epsilon)} &\le C(1+t)^{-\frac{3}{2q'}-\frac{m+2}{2}}\|\rho_0\|_{L^1} + C(1+t)^{-\frac{3}{2q'}-(\frac{m+1}{2})}\|u_0\|_{L^1} \\
&\quad + Ce^{-n_\infty \lambda t}\|[\rho_0,u_0]\|_{L^1} \\
&\le C(1+t)^{-\frac{3}{2}[1-\frac{1}{q}]-\frac{m+1}{2}}\|[\rho_0,u_0]\|_{L^1}.
\end{aligned}
\tag{4.32}
$$

Now, we estimate the second term of (4.31) from (4.26) as

$$
\||\xi|^m \hat{\rho}(\xi,t)\|_{L^{q'}(|\xi|\ge\epsilon)} \le C\left[\int_{|\xi|\ge\epsilon} |\xi|^{mq'} e^{-q'\lambda t}|\hat{\rho}_0(\xi),\hat{u}_0(\xi)|^{q'} d\xi\right]^{\frac{1}{q'}}
$$

Now, take $\epsilon_1 > 0$ which is small enough. By the Hölder inequality $\frac{1}{q'} = \frac{1}{2} + \frac{2-q'}{2q'}$, we have

$$
\begin{aligned}
\||\xi|^m \hat{\rho}(\xi,t)\|_{L^{q'}(|\xi|\ge\epsilon)} &\le C\left[\int_{|\xi|\ge\epsilon} |\xi|^{-(3+\epsilon)(\frac{2-q'}{2})}|\xi|^{(3+\epsilon)(\frac{2-q'}{2})+mq'} e^{-q'\lambda t}|\hat{\rho}_0(\xi),\hat{u}_0(\xi)|^{q'} d\xi\right]^{\frac{1}{q'}} \\
&\le Ce^{-\lambda t}\left[\int_{|\xi|\ge\epsilon} |\xi|^{-(3+\epsilon)} d\xi\right]^{\frac{2-q'}{2q'}}\left[\int_{|\xi|\ge\epsilon} |\xi|^{((3+\epsilon)(\frac{2-q'}{2})+mq')\frac{2}{q'}}|\hat{\rho}_0(\xi),\hat{u}_0(\xi)|^{q'(\frac{2}{q'})} d\xi\right]^{(\frac{1}{q'})(\frac{q'}{2})} \\
&\le Ce^{-\lambda t}\||\xi|^{-(3+\epsilon)}\|^{\frac{2-q'}{2q'}}\||\xi|^{(3+\epsilon)\frac{2-q'}{2q'}+m}[\hat{\rho}_0(\xi),\hat{u}_0(\xi)]\|_{L^2} \\
&\le Ce^{-\lambda t}\|\nabla^{m+(3+\epsilon)\frac{2-q'}{2q'}}[\rho_0,u_0]\|_{L^2} \\
&\le Ce^{-\lambda t}\|\nabla^{m+3[\frac{1}{q'}-\frac{1}{2}]_+}[\rho_0,u_0]\|_{L^2} \\
&\le Ce^{-\lambda t}\|\nabla^{m+3[\frac{1}{2}-\frac{1}{q}]_+}[\rho_0,u_0]\|_{L^2},
\end{aligned}
\tag{4.33}
$$

after plugging (4.33) and (4.32) into (4.31) implies (4.28).

To prove (4.29), it similarly holds that

$$
\begin{aligned}
\|\nabla^m u(t)\|_{L^q(\mathbb{R}_x^3)} &\le C\||\xi|^m \hat{u}(\xi,t)\|_{L^{q'}(\mathbb{R}_\xi^3)} \\
&\le C\||\xi|^m \hat{u}(\xi,t)\|_{L^{q'}(|\xi|\le\epsilon)} + C\||\xi|^m \hat{u}(\xi,t)\|_{L^{q'}(|\xi|\ge\epsilon)},
\end{aligned}
\tag{4.34}
$$

where from (4.25), the first term is

$$
\begin{aligned}
\||\xi|^m \hat{u}(\xi,t)\|_{L^{q'}(|\xi|\le\epsilon)}^{q'} &\le C\int_{|\xi|\le\epsilon} (|\xi|^{mq'+q'}(e^{-q'\lambda|\xi|^2(t+1)}+e^{-n_\infty\lambda q't})|\hat{\rho}_0(\xi)|^{q'})d\xi \\
&\quad + C\int_{\xi\le\epsilon} (|\xi|^{mq'}e^{-\lambda q'|\xi|^2(t+1)}+|\xi|^{(m+2)q'}e^{-n_\infty\lambda q't})|\hat{u}_0(\xi)|^{q'}d\xi \\
&\le C(1+t)^{-\frac{mq'+q'+3}{2}}\|[\rho_0]\|_{L^1}^{q'} + (1+t)^{-\frac{mq'+3}{2}}\|u_0\|_{L^1}^{q'} \\
&\quad + Ce^{-n_\infty\lambda q't}\|[\rho_0,u_0]\|^{q'}{}_{L^1}.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\||\xi|^m \hat{u}(\xi,t)\|_{L^{q'}(|\xi|\le\epsilon)} &\le C(1+t)^{-\frac{3}{2q'}-\frac{m+1}{2}}\|[\rho_0\|_{L^1} \\
&\quad + (1+t)^{-\frac{3}{2q'}-\frac{m}{2}}\|u_0\|_{L^1} + Ce^{-n_\infty\lambda t}\|[\rho_0,u_0]\|_{L^1} \\
&\le C(1+t)^{-\frac{3}{2}[1-\frac{1}{q}]-\frac{m+1}{2}}\|\rho_0\|_{L^1} + (1+t)^{-\frac{3}{2}[1-\frac{1}{q}]-\frac{m}{2}}\|u_0\|_{L^1} \\
&\le C(1+t)^{-\frac{3}{2}[1-\frac{1}{q}]-\frac{m}{2}}\|[\rho_0,u_0]\|_{L^1}.
\end{aligned}
\tag{4.35}
$$

Similarly to obtaining (4.33), one has

$$\||\xi|^m \hat{u}(\xi,t)\|_{L^{q'}(|\xi|\geq\epsilon)} \leq Ce^{-\lambda t}\|\nabla^{m+3[\frac{1}{2}-\frac{1}{q}]+}[\rho_0,u_0]\|_{L^2}. \tag{4.36}$$

Thus, plugging (4.35) and (4.36) into (4.34) implies (4.29). This completes the proof of Theorem 4.2. □

Corollary 4.3. *Assume that $U_1 = e^{Bt}U_{1,0}$ is the solution to the Cauchy problem (4.6) with initial data $U_{1,0} = [\rho_0, u_0]$. Then $U_1 = [\rho, u]$ satisfies the following:*

$$\|\rho(t)\|_{L^2} \leq C(1+t)^{-\frac{5}{4}}\|[\rho_0,u_0]\|_{L^1} + e^{-\lambda t}\|[\rho_0,u_0]\|_{L^2}, \tag{4.37}$$

$$\|u(t)\|_{L^2} \leq C(1+t)^{-\frac{3}{4}}\|[\rho_0,u_0]\|_{L^1} + e^{-\lambda t}\|[\rho_0,u_0]\|_{L^2}, \tag{4.38}$$

$$\|\rho(t)\|_{L^\infty} \leq C(1+t)^{-2}\|[\rho_0,u_0]\|_{L^1} + e^{-\lambda t}\|[\rho_0,u_0]\|_{\dot{H}^2}, \tag{4.39}$$

$$\|u(t)\|_{L^\infty} \leq C(1+t)^{-\frac{3}{2}}\|[\rho_0,u_0]\|_{L^1} + e^{-\lambda t}\|[\rho_0,u_0]\|_{\dot{H}^2}. \tag{4.40}$$

5   Time-decay rates for the nonlinear system

In this section, we will prove (2.5)–(2.7) in Proposition 2.2. The main idea is to introduce a general approach to combine the energy estimates and spectral analysis. We will apply the linear $L^2 - L^q$ time-decay property of the linearized homogeneous system (4.4), studied in the previous section, to the nonlinear case. We need the mild form of the original nonlinear Cauchy problem (2.2). Throughout this section, we suppose that $U = [\rho, u, c_1, c_2]$ is the solution to the Cauchy problem (2.3) with initial data $U_0 = (\rho_0, u_0, c_{1,0}, c_{2,0})$.

Then, by Duhamel's principle, the solution $U = [\rho, u, c_1, c_2]$ can be formally written as

$$U(t) = e^{Bt}U_0 + \int_0^t e^{(t-s)B}[g_1, g_2, g_3, g_4]ds, \tag{5.1}$$

where $e^{Bt}U_0$ is the solution to the Cauchy problem (4.1) with initial data $U_0 = (\rho_0, u_0, c_{1,0}, c_{2,0})$. Here, the nonlinear source term takes the form (4.3).

5.1   Time rate for the energy functional and high-order energy functional

In this subsection, we will prove the time-decay rate for the energy functional $\|U(t)\|_{H^N}^2$ and the time-decay rate for the high-order energy functional $\|\nabla U(t)\|_{H^N}^2$. For that, we investigate the time-decay rates of solutions in Proposition 2.1 under extra conditions on the given initial data $U_0 = [\rho_0, u_0, c_{1,0}, c_{2,0}]$. We define

$$\epsilon_{H^N}(U_0) = \|U_0\|_{H^N} + \|[\rho_0, u_0]\|_{L^1}, \tag{5.2}$$

for an integer $N \geq 4$. We also define $\mathcal{E}_N U(t) \sim \|[\rho, u, c_1, c_2]\|_{H^N}^2$ as the energy functional and $\mathcal{D}_N U(t) \sim \|[\nabla(u, c_1, c_2)]\|_{H^N}^2$, $\mathcal{D}_N^h U(t) \sim \|[\rho, c_1, c_2]\|_{H^N}^2$ as the dissipation rates.

First, we start with this proposition for the energy functional and the high-order energy functional.

Proposition 5.1. *Let $U = [\rho, u, c_1, c_2]$ be the solution to the Cauchy problem* (2.2) *with initial data $U_0 = (\rho_0, u_0, c_{1,0}, c_{2,0})$. If $\epsilon_{N+1}(U_0) > 0$ is small enough, then the solution $U = [\rho, u, c_1, c_2]$ satisfies*

$$\|U(t)\|_{H^N} \leq \epsilon_{N+1}(U_0)(1+t)^{\frac{-3}{4}}, \tag{5.3}$$

and

$$\|\nabla U(t)\|_{H^N} \leq \epsilon_{N+1}(U_0)(1+t)^{\frac{-5}{4}}, \tag{5.4}$$

for any $t \geq 0$.

Proof. Suppose $\epsilon_{N+1}(U_0)$ is sufficiently small. From Proposition 2.1 the solution $U = [\rho, u, c_1, c_2]$ satisfies:

$$\frac{d}{dt}\mathcal{E}_N(U(t)) + \lambda_1 \mathcal{D}_N(U(t)) + \lambda_2 \mathcal{D}_N^h(U(t)) \leq 0, \tag{5.5}$$

for $t \geq 0$.

Now, we proceed by making the time-weighted estimate and iteration for the inequality (5.5). Let $l \geq 0$. Multiplying (5.5) by $(1+t)^l$ and integrating over $[0, t]$ gives

$$(1+t)^l \mathcal{E}_N U(t) + \lambda_1 \int_0^t (1+s)^l \mathcal{D}_N(U(s))ds + \lambda_2 \int_0^t (1+s)^l \mathcal{D}_N^h(U(s))ds$$

$$\leq \mathcal{E}_N(U_0) + l \int_0^t (1+s)^{l-1}\mathcal{E}_N U(s)ds$$

$$\leq \mathcal{E}_N(U_0) + Cl \int_0^t (1+s)^{l-1}(\mathcal{D}_{N-1}U(s) + \mathcal{D}_N^h(U(s)) + \|u(s)\|_{L^2}^2)ds,$$

where we have used

$$\mathcal{E}_N U(t) \leq C\mathcal{D}_{N-1}U(t) + C\mathcal{D}_N^h(U(t)) + \|u(t)\|_{L^2}^2.$$

Using (5.5) again, we have

$$\mathcal{E}_{N+1}(U(t)) + \lambda_1 \int_0^t \mathcal{D}_{N+1}(U(t)) + \lambda_2 \int_0^t \mathcal{D}_{N+1}^h(U(t)) \leq \mathcal{E}_{N+1}(U_0),$$

and

$$(1+t)^{l-1}\mathcal{E}_{N+1}U(t) + \lambda_1 \int_0^t (1+s)^{l-1}\mathcal{D}_{N+1}(U(s))ds + \lambda_2 \int_0^t (1+s)^{l-1}\mathcal{D}_{N+1}^h(U(s))ds$$

$$\leq \mathcal{E}_{N+1}(U_0) + C(l-1) \int_0^t (1+s)^{l-2}\mathcal{E}_{N+1}U(s)ds$$

$$\leq \mathcal{E}_{N+1}(U_0) + C(l-1) \int_0^t (1+s)^{l-2}(\mathcal{D}_N U(s) + C\mathcal{D}_{N+1}^h(U(s)) + \|u(s)\|_{L^2}^2)ds.$$

By iterating the above estimates for $1 < l < 2$, we have

$$(1+t)^l \mathcal{E}_N U(t) + \lambda_1 \int_0^t (1+s)^l \mathcal{D}_N(U(s))ds + \lambda_2 \int_0^t (1+s)^l \mathcal{D}_N^h(U(s))ds$$

$$\leq \mathcal{E}_{N+1}(U_0) + C \int_0^t (1+s)^{l-1}\|u(s)\|_{L^2}^2 ds. \tag{5.6}$$

To estimate the integral term on the right-hand side of (5.6), let us define

$$\mathcal{E}_{N,\infty}(U(t)) = \sup_{0 \leq s \leq T}(1+t)^{\frac{3}{2}}\mathcal{E}_N U(t).$$

Now, we estimate the integral term on the right-hand side of (5.6) by applying the linear estimate on u in (4.38) to the mild form (5.1), giving us

$$\|u(t)\|_{L^2} \leq C(1+t)^{\frac{-3}{4}}\|\rho_0, u_0\|_{L^1} + Ce^{-\lambda t}\|\rho_0, u_0\|_{L^2}$$
$$+ C\int_0^t (1+t-s)^{\frac{-3}{4}}\|g_1, g_2\|_{L^1}ds + C\int_0^t e^{-\lambda(t-s)})\|g_1, g_2\|_{L^2}ds. \qquad (5.7)$$

Recall the definitions (4.3) of g_1 and g_2. It is direct to check that for any $0 \leq s \leq t$,

$$\|g_1(s), g_2(s)\|_{L^1 \cap L^2} \leq C\mathcal{E}_N U(t) \leq C(1+s)^{\frac{-3}{2}}\mathcal{E}_{N,\infty} U(t),$$

where

$$\mathcal{E}_{N,\infty}(U(t)) = \sup_{0 \leq s \leq T}(1+t)^{\frac{3}{2}}\mathcal{E}_N U(t).$$

Putting the above inequalities into (5.7), gives

$$\|u(t)\|_{L^2} \leq C(1+t)^{\frac{-3}{4}}(\|\rho_0, u_0\|_{L^1 \cap L^2} + \mathcal{E}_{N,\infty} U(t)). \qquad (5.8)$$

Next, we prove the uniform-in-time boundedness of $\mathcal{E}_{N,\infty} U(t)$ which yields the time-decay rates of the energy functional $\mathcal{E}_N U(t)$. In fact, by taking $l = \frac{3}{2} + \epsilon$ in (5.6) where $\epsilon > 0$ is sufficiently small, it follows that

$$(1+t)^{\frac{3}{2}+\epsilon}\mathcal{E}_N U(t) + \lambda_1 \int_0^t (1+s)^{\frac{3}{2}+\epsilon}\mathcal{D}_N(U(s))ds + \lambda_2 \int_0^t (1+s)^{\frac{3}{2}+\epsilon}\mathcal{D}_N^h(U(s))ds$$
$$\leq \mathcal{E}_{N+1}(U_0) + C\int_0^t (1+s)^{\frac{1}{2}+\epsilon}\|u(s)\|_{L^2}^2 ds.$$

Here, using (5.10) and the fact that $\mathcal{E}_{N,\infty}(U(t))$ is non-decreasing in t, it further holds that

$$\int_0^t (1+s)^{\frac{1}{2}+\epsilon}\|u(t)\|_{L^2}^2 ds \leq C(1+t)^{\epsilon}(\mathcal{E}_{N,\infty}^2 U(t)) + \|\rho_0, u_0\|_{L^1 \cap L^2}^2).$$

Therefore, it follows that

$$(1+t)^{\frac{3}{2}+\epsilon}\mathcal{E}_N U(t) + \lambda_1 \int_0^t (1+s)^{\frac{3}{2}+\epsilon}\mathcal{D}_N(U(s))ds + \lambda_2 \int_0^t (1+s)^{\frac{3}{2}+\epsilon}\mathcal{D}_N^h(U(s))ds$$
$$\leq \mathcal{E}_{N+1}(U_0) + C(1+t)^{\epsilon}(\mathcal{E}_{N,\infty}^2 U(t)) + \|\rho_0, u_0\|_{L^1 \cap L^2}^2),$$

which implies

$$(1+t)^{\frac{3}{2}}\mathcal{E}_N U(t) \leq C(\mathcal{E}_{N+1}(U_0) + \|\rho_0, u_0\|_{L^1}^2 + \mathcal{E}_{N,\infty}^2 U(t)),$$

and thus

$$\mathcal{E}_{N,\infty} U(t) \leq C(\epsilon_{N+1}^2(U_0) + \mathcal{E}_{N,\infty}^2 U(t)).$$

Since $\epsilon_{N+1}(U_0) > 0$ is sufficiently small, it holds that $\mathcal{E}_{N,\infty} U(t)) \leq C\epsilon_{N+1}^2(U_0)$ for any $t \geq 0$, which gives $\|U(s)\|_{H^N} \leq C(\mathcal{E}_N U(t))^{\frac{1}{2}} \leq C\epsilon_{N+1}(U_0)(1+t)^{-\frac{3}{4}}$. This proves (5.3).

Now, we estimate the high-order energy functional. By comparing the definitions of $\mathcal{E}_N U(t)$, $\mathcal{D}_N U(t)$ and $\mathcal{D}_N^h U(t)$, it follows from (5.5) that we have

$$\frac{d}{dt}\|\nabla U(t)\|_{H^N}^2 + \lambda\|\nabla U(t)\|_{H^N}^2 \leq C\|\nabla u(t)\|_{L^2}^2,$$

which implies

$$\|\nabla U(t)\|_{H^N}^2 \le e^{-\lambda t}\|\nabla U_0\|_{H^N}^2 + C\int_0^t e^{-\lambda(t-s)}\|\nabla u(s)\|_{L^2}^2 ds, \tag{5.9}$$

for any $t \ge 0$.

Similarly to obtaining (5.8), we estimate the time integral term on the (r.h.s.) of the above inequality. One can apply the linear estimate (4.29) to the mild form (5.1) so that

$$\|\nabla u(t)\|_{L^2} \le C(1+t)^{\frac{-5}{4}}\|\rho_0, u_0\|_{L^1} + Ce^{-\lambda t}\|[\rho_0, u_0]\|_{\dot{H}^1}$$
$$+ C\int_0^t (1+t-s)^{\frac{-5}{4}}\|[g_1(s), g_2(s)]\|_{L^1} ds + C\int_0^t e^{-\lambda(t-s)}\|[g_1(s), g_2(s)]\|_{\dot{H}^1} ds. \tag{5.10}$$

Recall the definition (4.3) of g_1 and g_2. It is straightforward to check that for any $0 \le s \le t$,

$$\|[g_1(s), g_2(s)]\|_{L^1 \cap \dot{H}^1} \le C\mathcal{E}_N U(s) \le C\epsilon_{N+1}^2(U_0)(1+s)^{\frac{-3}{2}}.$$

Putting this into (5.10) gives

$$\|\nabla u(t)\|_{L^2} \le C\epsilon_{N+1}(U_0)(1+t)^{\frac{-5}{4}}. \tag{5.11}$$

Then, by using (5.11) in (5.9), we have

$$\|\nabla U(t)\|_{H^N}^2 \le e^{-\lambda t}\|\nabla U_0\|_{H^N}^2 + C\epsilon_{N+1}^2(U_0)(1+t)^{\frac{-5}{2}},$$

which implies (5.4). The proof of Proposition 5.1 is complete. □

5.2 Time-decay rate in L^q

In this subsection, we will prove Proposition 2.2 for time-decay rates in L^q with $2 \le q \le \infty$ corresponding to (1.4)–(1.6) in Theorem 1.1. For $N \ge 4$, Proposition 5.1 shows that if $\epsilon_{N+1}(U_0)$ is small enough,

$$\|U(s)\|_{H^N} \le C\epsilon_{N+1}(U_0)(1+t)^{-\frac{3}{4}}, \tag{5.12}$$

and

$$\|\nabla U(t)\|_{H^N} \le C\epsilon_{N+1}(U_0)(1+t)^{\frac{-5}{4}}. \tag{5.13}$$

Now, let us establish the estimates on u, ρ as follows.

Estimate on $\|u(t)\|_{L^q}$. For the L^2 rate, it is easy to see from (5.8) and (5.12) that

$$\|u(t)\|_{L^2} \le C\epsilon_{N+1}(U_0)(1+t)^{\frac{-3}{4}} \le C(1+t)^{\frac{-3}{4}}.$$

For the L^∞ rate, by applying the L^∞ linear estimate on u in (4.40) to the mild form (5.1), we have

$$\|u(t)\|_{L^\infty} \le C(1+t)^{\frac{-3}{2}}\|\rho_0, u_0\|_{L^1} + Ce^{-\lambda t}\|\nabla^2[\rho_0, u_0]\|_{L^2}$$
$$+ C\int_0^t (1+t-s)^{\frac{-3}{2}}\|[g_1(s), g_2(s)]\|_{L^1} ds + C\int_0^t e^{-\lambda(t-s)}\|\nabla^2[g_1(s), g_2(s)]\|_{L^2} ds$$
$$\le C(1+t)^{\frac{-3}{2}}\|\rho_0, u_0\|_{L^1 \cap \dot{H}^2} + C\int_0^t (1+t-s)^{\frac{-3}{2}}\|[g_1(s), g_2(s)]\|_{L^1 \cap \dot{H}^2} ds. \tag{5.14}$$

Since by (5.12) and (5.13)

$$\|[g_1(s), g_2(s)]\|_{L^1 \cap \dot{H}^2} \le C\|\nabla U(t)\|_{H^N}\|U(s)\|_{H^N} \le C\epsilon_{N+1}^2(U_0)(1+s)^{-2},$$

it follows that

$$\|u(t)\|_{L^\infty} \le C\epsilon_{N+1}(U_0)(1+t)^{\frac{-3}{2}}.$$

Then, by $L^2 - L^\infty$ interpolation,

$$\|u\|_{L^q} \le C\epsilon_{N+1}(U_0)(1+t)^{\frac{-3}{2}+\frac{3}{2q}} \tag{5.15}$$

for $2 \le q \le \infty$.

Estimate on $\|\rho(t)\|_{L^q}$. For the L^2 rate, utilizing the L^2 estimate on ρ in (4.37) to (5.1), we have

$$\|\rho(t)\|_{L^2} \le C(1+t)^{\frac{-5}{4}}\|\rho_0, u_0\|_{L^1} + Ce^{-\lambda t}\|\rho_0, u_0\|_{L^2} + C\int_0^t (1+t-s)^{\frac{-5}{4}}\|g_1, g_2\|_{L^1}ds$$

$$+ C\int_0^t e^{-\lambda(t-s)}\|g_1(s), g_2(s)\|_{L^2}ds. \tag{5.16}$$

Due to (5.12),

$$\|g_1(s), g_2^*(s)\|_{L^1\cap L^2} \le C\|U(s)\|_{H^N}^2 \le C\epsilon_{N+1}^2(U_0)(1+t)^{\frac{-3}{2}}.$$

Then (5.16) implies the slower decay estimate

$$\|\rho(t)\|_{L^2} \le C\epsilon_{N+1}(U_0)(1+t)^{\frac{-5}{4}} \le C(1+t)^{\frac{-5}{4}}. \tag{5.17}$$

For the L^∞ rate, utilizing the L^∞ estimate on ρ in (4.39) to (5.1), we have

$$\|\rho(t)\|_{L^\infty} \le (1+t)^{-2}\|\rho_0, u_0\|_{L^1\cap \dot{H}^2} + C\int_0^t (1+t-s)^{-2}\|[g_1(s), g_2(s)]\|_{L^1\cap \dot{H}^2}ds. \tag{5.18}$$

Since by (5.12) and (5.13)

$$\|[g_1(s), g_2(s)]\|_{L^1\cap \dot{H}^2} \le C\|\nabla U(t)\|_{H^N}\|U(s)\|_{H^N} \le C\epsilon_{N+1}^2(U_0)(1+s)^{-2},$$

which yields from (5.18) that

$$\|\rho(t)\|_{L^\infty} \le C\epsilon_{N+1}(U_0)(1+s)^{-2}.$$

Therefore, by $L^2 - L^\infty$ interpolation,

$$\|\rho(t)\|_{L^q} \le C\epsilon_{N+1}(U_0)(1+s)^{-2+\frac{3}{2q}} \tag{5.19}$$

for $2 \le q \le \infty$.

Next, we estimate the time-decay rate of $[c_1, c_2]$. We start with the estimate on $\|c_1(t)\|_{L^q}$. For the L^2 rate,

$$\|c_1\|_{L^2} \le C\|\hat{c}_1\|_{L^2(\xi)} \tag{5.20}$$

$$\le C\left[\int_\xi e^{-2(|\xi|^2+(a_{12}-a_{11}n_\infty))t}|\hat{c}_0|^2d\xi\right]^{\frac{1}{2}} + a_{11}\int_0^t \left[\int_\xi [e^{-2(|\xi|^2+(a_{12}-a_{11}n_\infty))(t-s)}|\rho\hat{c}_1|^2d\xi\right]^{\frac{1}{2}}ds$$

$$\le e^{-(a_{12}-a_{11}n_\infty)t}\left[\int_\xi e^{-2|\xi|^2(t)}|\hat{c}_0|^2d\xi\right]^{\frac{1}{2}} + C\int_0^t e^{-(a_{12}-a_{11}n_\infty)(t-s)}\left[\int_\xi e^{-2|\xi|^2(t-s+1)}|\rho\hat{c}_1|^2d\xi\right]^{\frac{1}{2}}ds$$

$$\le Ce^{-(a_{12}-a_{11}n_\infty)t}\|\hat{c}_0\|_{L^2} + C\int_0^t e^{-(a_{12}-a_{11}n_\infty)(t-s)}\sup_\xi e^{-|\xi|^2(t-s+1)}\|\rho c_1(s)\|_{L^2}ds \tag{5.21}$$

Due to (5.12),

$$\|\rho c_1(s)\|_{L^2} \le C\|U(s)\|_N^2 \le C\epsilon_{N+1}^2(U_0)(1+t)^{\frac{-3}{2}}.$$

Then (5.20) implies the slower decay estimate

$$\|c_1\|_{L^2} \le C\epsilon_{N+1}(U_0)(1+t)^{\frac{-3}{2}}. \tag{5.22}$$

Similarly, we have

$$\|c_2\|_{L^2} \le C\epsilon_{N+1}(U_0)(1+t)^{\frac{-3}{2}}. \tag{5.23}$$

For L^∞ rate, from the Hausdorff–Young inequality and the Hölder inequality, we have

$$
\begin{aligned}
\|c_1\|_{L^\infty} \le C\|\hat{c}_1\|_{L^1} &\le C\int_{\xi \le \epsilon} e^{-(|\xi|^2+(a_{12}-a_{11}n_\infty))t}|\hat{c}_{1,0}|d\xi \\
&\quad + C\int_0^t \int_{\xi \le \epsilon} e^{-(|\xi|^2+(a_{12}-a_{11}n_\infty))(t-s)}|\rho\hat{c}_1|d\xi ds \\
&\quad + C\int_{|\xi| \ge \epsilon} e^{-(a_{12}-a_{11}n_\infty))t}|\hat{c}_{1,0}|d\xi + C\int_0^t \int_{|\xi| \ge \epsilon} e^{-(a_{12}-a_{11}n_\infty))(t-s)}|\rho\hat{c}_1|d\xi ds \\
&\le Ce^{-(a_{12}-a_{11}n_\infty)t}(1+t)^{\frac{-3}{2}}\|c_0\|_{L^1} + C\int_0^t e^{-(a_{12}-a_{11}n_\infty)(t-s)}\|\rho\hat{c}_1(s)\|_{L^1} \\
&\quad + Ce^{-(a_{12}-a_{11}n_\infty))t}\left[\int_{|\xi| \ge \epsilon}|\xi|^{-4}d\xi\right]^{\frac{1}{2}}\left[\int_{|\xi| \ge \epsilon}|\xi|^4|\hat{c}_{1,0}|^2 d\xi\right]^{\frac{1}{2}} \\
&\quad + C\int_0^t e^{-(a_{12}-a_{11}n_\infty))(t-s)}\left[\int_{|\xi| \ge \epsilon}|\xi|^{-4}d\xi\right]^{\frac{1}{2}}\left[\int_{|\xi| \ge \epsilon}|\xi|^4|\rho\hat{c}_1|^2 d\xi\right]^{\frac{1}{2}}ds \\
&\le Ce^{-(a_{12}-a_{11}n_\infty)t}(1+t)^{\frac{-3}{2}}\|c_0\|_{L^1} + C\int_0^t e^{-(a_{12}-a_{11}n_\infty)(t-s)}\|\rho c_1(s)\|_{L^1}ds \\
&\quad + Ce^{-(a_{12}-a_{11}n_\infty))t}\|\nabla^2 c_0\|_{L^2} + C\int_0^t e^{-(a_{12}-a_{11}n_\infty)(t-s)}\|\nabla^2(\rho c_1(s))\|_{L^2}ds \tag{5.24}
\end{aligned}
$$

Since by (5.12)

$$\|\rho c_1(s)\|_{L^1 \cap \dot{H}^2} \le C\|U(s)\|_N^2 \le C\epsilon_{N+1}^2(U_0)(1+t)^{\frac{-3}{2}}.$$

Then, (5.24) implies the slower decay estimate

$$\|c_1\|_{L^\infty} \le C\epsilon_{N+1}(U_0)(1+t)^{\frac{-3}{2}}. \tag{5.25}$$

Similarly, we have

$$\|c_2\|_{L^\infty} \le C\epsilon_{N+1}(U_0)(1+t)^{\frac{-3}{2}}. \tag{5.26}$$

So, by $L^2 - L^\infty$ interpolation,

$$\|c_1, c_2\|_{L^q} \le C\epsilon_{N+1}(U_0)(1+t)^{\frac{-3}{2}}, \tag{5.27}$$

for $2 \le q \le \infty$.

This completes the proof of Proposition 2.2 and hence Theorem 1.1.

6 Conclusion

We have studied a chemotaxis model where a compressible fluid model for cells and a diffusive Lotka–Volterra model for chemoattractants and repellents are used. The previous results for chemotaxis are mostly extensions of the Keller and Segel model or in the case of fluid dynamical models, the incompressible fluid models for the cells are used. We showed the

existence of global solutions and their asymptotic behavior in three dimensions with the initial data as a small perturbation of the constant state $(n_\infty, 0, 0, 0)$. Our method is based on the basic energy estimates used for the a priori estimates and the iterative method in solving the Cauchy problem (1.1). Moreover, we have also shown the decay estimates of solutions to the Cauchy problem (1.1) in $\mathbb{R}^3$, in which the detailed analysis of Green's functions of the linear system is combined with the refined energy estimates with the help of Duhamel's principle. We proved the decay property of solutions as time goes to infinity. Our results are complementary to Ambrosi, Bussolino and Preziosi [2], where the modeling aspects such as qualitative analysis and numerical simulations of the compressible fluid model for cells with chemoattractants are examined for vasculogenesis.

Acknowledgements

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