Electronic Journal of Qualitative Theory of Differential Equations
2021, No. **81**, 1–18; https://doi.org/10.14232/ejqtde.2021.1.81 www.math.u-szeged.hu/ejqtde/

Well-posedness for a fourth-order equation of Moore–Gibson–Thompson type

Carlos Lizama ✉ [1] and **Marina Murillo** * [2]

[1] Universidad de Santiago de Chile, Facultad de Ciencias, Departamento de Matemática y Ciencia de la Computación, Las Sophoras 173, Estación Central, Santiago, Chile
[2] Universitat Politècnica de València, Instituto Universitario de Matemática Pura y Aplicada, 46022 València, Spain

Received 21 March 2021, appeared 14 October 2021
Communicated by Roberto Livrea

Abstract. In this paper, we completely characterize, only in terms of the data, the well-posedness of a fourth order abstract evolution equation arising from the Moore–Gibson–Thomson equation with memory. This characterization is obtained in the scales of vector-valued Lebesgue, Besov and Triebel–Lizorkin function spaces. Our characterization is flexible enough to admit as examples the Laplacian and the fractional Laplacian operators, among others. We also provide a practical and general criteria that allows L^p–L^q-well-posedness.

Keywords: well-posedness, Moore–Gibson–Thompson equation, operator-valued multipliers, R-boundedness.

2020 Mathematics Subject Classification: Primary: 45Q05, 49K40. Secondary: 35K65, 35L80.

1 Introduction

In recent years, there has been considerable interest in mathematical models that are close to practical situations of the real life. In the context of acoustics, and in order to gain a better understanding of the nonlinear model, a typical and standard reference is the linearized part of the Westervelt equation [25] i.e.

$$\frac{\delta}{c_0^4}u'''(t) + \Delta u(t) - \frac{1}{c_0^2}u''(t) = 0, \qquad t \geq 0,$$

where u denotes the sound pressure, c_0 is the small signal sound speed, δ is the sound diffusivity and Δ denotes the Laplacian operator. An extension of the Westervelt equation that takes into account second sound effects and the associated thermal relaxation in viscous fluids is the Moore–Gibson–Thomson (MGT) equation

$$\tau u'''(t) + u''(t) - c^2\Delta u'(t) - b_0\Delta u(t) = 0, \qquad t \geq 0, \tag{1.1}$$

✉ Corresponding author. Email: carlos.lizama@usach.cl
*Email: mamuar1@upv.es

where $b_0 = \delta + \tau c^2$, see [16,27–29,35]. The MGT equation with memory

$$u'''(t) + au''(t) - b\Delta u'(t) - c\Delta u(t) + \int_0^t g(t-s)\Delta u(s)ds = 0, \qquad t \geq 0, \tag{1.2}$$

has been treated in [13, 20, 32, 33]. When $g \neq 0$, the memory term introduces further dissipation. From the physical point of view, the most relevant case in connection with (1.2) is

$$g(s) = de^{-\ell s}, \qquad d, \ell > 0.$$

Motivated by the above kernel, the following model

$$u''''(t) + \alpha u'''(t) + \beta u''(t) - \gamma\Delta u''(t) - \delta\Delta u'(t) - \rho\Delta u(t) = 0, \qquad t \geq 0, \tag{1.3}$$

has been recently proposed [21,34]. It can be obtained from (1.2) summing ∂_t(1.2) + ℓ(1.2).

It should be pointed out that third and fourth order derivatives in time are observed in various areas of research. In physics and engineering third and fourth order derivatives should always be considered when vibration occurs and particularly when this excitation induces multi-resonant modes of vibration [6]. They should also be considered at all times when a transition occurs such as: start up and shutdown; take-off and landing; and accelerating and decelerating [23]. Fourth order derivatives in time appear, for instance, in the study of chaotic hyperjerk systems [17], in the Taylor series expansion of the Hubble law [37] and in the kinematic performance of long-dwell mechanisms of linkage type, which are used in automatic machines to generate intermittent motions [24].

The model (1.3) was introduced and first studied by Dell'Oro and Pata [21] in their abstract version

$$u''''(t) + \alpha u'''(t) + \beta u''(t) + \gamma Au''(t) + \delta Au'(t) + \rho Au(t) = 0, \tag{1.4}$$

where A is a strictly positive unbounded linear operator with domain $D(A)$ densely embedded in a separable real Hilbert space H and $\alpha, \beta, \gamma, \delta, \rho \in \mathbb{R}$. In such abstract model, the equation (1.3) corresponds to the choice $H = L^2(\Omega)$ and $A = -\Delta$ with $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$. In [21] it was established the well-posedness for (1.4) by means of the existence of the solution semigroup, providing a detailed description of the spectrum of its infinitesimal generator and its relation with the growth bound. The stability properties of the related solution semigroup were then investigated and, in particular, a necessary and sufficient condition for exponential stability was established, in terms of the values of the stability numbers

$$\chi = \gamma - \frac{\delta}{\alpha}, \qquad \varpi = \beta - \frac{\rho\alpha}{\delta},$$

where $\alpha, \beta, \gamma, \delta$ and ρ are strictly positive. Later, Liu et al. [34] discussed the well-posedness of the solution for (1.4) with an additional memory term like in (1.2) by using the Faedo–Galerkin method. Then, the authors in [34] proved general decay results for the case $\chi > 0$ and $\varpi > 0$ based on the perturbed energy method and on some properties of convex functions.

However, we note that all above mentioned references studied (1.4) in the context of Hilbert spaces, and they do not include the important cases of the Lebesgue spaces $L^q(\Omega)$ except, of course, the case $q = 2$. Furthermore, the class of operators A studied so far does not allow the admissibility of more general types of differential operators like the Stokes operator, the fractional Laplacian operator or the biharmonic Δ^2, equipped with suitable boundary conditions.

On the other hand, using the method of operator-valued Fourier multipliers due to Arendt and Bu [4,5], well-posedness of the solutions for the *nonhomogeneous* MGT equation (1.2) in the class of $\mathcal{HT}$ (or UMD) spaces, that includes the scale of Lebesgue spaces $L^q(\Omega)$ among others, has been studied by Poblete and Pozo [36], Bu and Cai [7] and Conejero et al. [19]. This method allows the admissibility of very general linear operators A but, depending on the regularity on the time variable, sometimes needs a more restrictive condition on the associated operator-valued symbols, namely: R-boundedness [4,10]. This restrictive condition can be replaced by uniform boundedness if we assume, for instance, that time-regularity is needed in the scales of Besov spaces (that includes the class of Hölder continuous functions) [5,11,12] or the scale of Triebel–Lizorkin spaces [8,9,14].

In this paper we will take this last approach as method. We succeed in obtaining a completely new characterization of strongly well-posedness for the nonhomogeneous equation (1.4) in the the scales of Lebesgue, Besov and Triebel–Lizorkin spaces. For that purpose, we take advantage of a recent result proved in [19, Theorem 1.1] in order to simplify complex computations on the operator-valued symbols associated to the corresponding nonhomogeneous model (1.4). In the case of the scale of Lebesgue spaces, our result reads as follows: Assume that A is a closed linear operator with (not necessarily dense) domain $D(A)$ defined on a UMD space X. The following assertions are equivalent:

(i) The equation

$$u''''(t) + \alpha u'''(t) + \beta u''(t) + \gamma A u''(t) + \delta A u'(t) + \rho A u(t) = f(t), \quad t \in \mathbb{T} := [0, 2\pi], \tag{1.5}$$

is strongly L^p-well-posed, i.e. for each $f \in L^p(\mathbb{T}, X)$, there exists a unique solution

$$u \in W^{4,p}_{per}(\mathbb{T}, X) \cap W^{2,p}_{per}(\mathbb{T}, [D(A)]).$$

(ii) $\mathbb{Z} \subset \rho_s(A)$ and the set $\{k^4[k^4 - \alpha i k^3 - \beta k^2 - \gamma k^2 A + \delta i k A + \rho A]^{-1} : k \in \mathbb{Z}\}$ is R-bounded.

Moreover, if (i) (or (ii)) holds, then the following maximal regularity estimate

$$\|u\|_{L^p(\mathbb{T},X)} + \|u''\|_{W^{2,p}_{per}(\mathbb{T},X)} + \|u'''\|_{W^{3,p}_{per}(\mathbb{T},X)} + \|u''''\|_{W^{4,p}_{per}(\mathbb{T},X)} + \|Au\|_{L^p(\mathbb{T},[D(A)])} + \|Au'\|_{W^{1,p}_{per}(\mathbb{T},[D(A)])} + \|Au''\|_{W^{2,p}_{per}(\mathbb{T},[D(A)])} \le C\|f\|_{L^p(\mathbb{T},X)},$$

holds. The last estimate has many important applications. It is the central tool in the study of the following problems: existence and uniqueness of solutions of nonautonomous evolution equations; existence and uniqueness of solutions of quasilinear and nonlinear partial differential equations; stability theory for evolution equations; maximal regularity of solutions of elliptic differential equations; existence and uniqueness of solutions of Volterra integral equations; and uniqueness of mild solutions of the NavierâĂŞStokes equations. In these applications, a maximal regularity estimate is frequently used to reduce, via a fixed-point argument, a nonautonomous (resp. nonlinear) problem to an autonomous (resp. linear) problem. In some cases, maximal regularity is needed to apply an implicit function theorem. According to the literature, there has been a substantial amount of work, as one can see, for example, in Amann [2], Denk, Hieber and Prüss [22], Clément, Londen and Simonett [18], the survey by Arendt [3], and the bibliography therein.

Our new characterization of strongly L^p-well-posedness shows to be flexible in certain combination of strictly positive parameters $\alpha, \beta, \gamma, \delta$ and ρ, and that is amenable enough to allow fractional powers of operators. In fact, as a consequence of our results we deduce that if A is an R-sectorial operator of angle $\pi/2$ on $L^q(\Omega)$, $\Omega \subset \mathbb{R}^N, 1 < q < \infty$ and

$$\rho + \beta\gamma < \alpha\delta$$

then for any given $f \in L^p(\mathbb{T}, L^q(\Omega))$, $1 < p < \infty$, the initial value problem (1.5) admits a unique solution $u \in W^{4,p}_{per}(\mathbb{T}, L^q(\Omega)) \cap W^{2,p}_{per}(\mathbb{T}, [D(A)])$. As a consequence, we obtain optimal results, that we illustrate with two examples: $A = \Delta$ the Laplacian, and $A = -(-\Delta)^s$ the fractional Laplacian of order $1/2 < s < 1$.

2 Preliminaries

We start this section introducing the notion of L^p-Fourier multiplier. We will denote the space of bounded linear operators from X into Y endowed with the uniform operator topology as $\mathcal{B}(X, Y)$. If $X = Y$ we simply abbreviate $\mathcal{B}(X)$.

Definition 2.1. Let X and Y be Banach spaces and $1 \le p < \infty$. We say that $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{B}(X, Y)$ is an L^p-Fourier multiplier if, for each $f \in L^p(\mathbb{T}, X)$, there exists $u \in L^p(\mathbb{T}, Y)$ such that $\hat{u}(k) = M_k \hat{f}(k)$ for all $k \in \mathbb{Z}$, where

$$\hat{f}(k) := \frac{1}{2\pi}\int_0^{2\pi} e^{-ikt} f(t)dt$$

denotes the k-th Fourier coefficient of f.

Our characterization will be provided in terms of the R-boundedness of certain sets of operators. For that purpose, we need to recall the notion of R-boundedness.

Definition 2.2. Let X and Y be Banach spaces. A set $\mathcal{T} \subset \mathcal{B}(X, Y)$ is called R-bounded if there is a constant $c \ge 0$ such that

$$\|(T_1x_1, \ldots, T_nx_n)\|_R \le c\|(x_1, \ldots, x_n)\|_R, \tag{2.1}$$

for all $T_1, \ldots, T_n \in \mathcal{T},\ x_1, \ldots, x_n \in X,\ n \in \mathbb{N}$ where

$$\|(x_1, \ldots, x_n)\|_R := \frac{1}{2^n}\sum_{\epsilon_j\in\{-1,1\}^n}\Big\|\sum_{j=1}^n \epsilon_j x_j\Big\|.$$

The least c such that (2.1) is satisfied is called the R-bound of $\mathcal{T}$ and is denoted $R(\mathcal{T})$.

The property of R-boundedness is preserved under sum or product by a constant. Moreover, if X and Y are Hilbert spaces, R- boundedness is equivalent to uniform boundedness. More information about these properties are summarized in [22].

The class of Banach spaces X such that the Hilbert transform defined by

$$(Hf)(t) = \lim_{\epsilon,R\to\infty}\frac{1}{\pi}\int_{\epsilon\le|s|\le R}\frac{f(t-s)}{s}ds, \qquad t \in \mathbb{R},$$

is bounded in $L^p(\mathbb{R}; X)$ for some $p \in (1, \infty)$ is denoted by $\mathcal{HT}$. The basic reference for the class $\mathcal{HT}$ is the survey article by Burkholder [15], where two other characterizations for the class $\mathcal{HT}$ are also given, a probabilistic one, and a geometrical one. To describe the latter, recall that a Banach space X is termed ξ-convex, if there is a function $\xi : X \times X \to \mathbb{R}$ which is convex in each of its variables and such that $\xi(0,0) > 0$ and

$$\xi(x,y) \leq |x+y| \quad \text{for all } x,y \in X \text{ with } |x| = |y| = 1.$$

A Banach space X belongs to the class $\mathcal{HT}$ if and only if X is ξ-convex if and only if X has the unconditional martingale difference property (UMD) [15]. The UMD spaces include Hilbert spaces, Sobolev spaces $H^s_p(\Omega), 1 < p < \infty$, Lebesgue spaces $L^p(\Omega,\mu), \ell_p, 1 < p < \infty$, vector-valued Lebesgue spaces $L^p(\Omega,\mu;X)$ where X is a UMD space, Hardy spaces, Lorentz and Orlicz spaces, any von Neumann algebra, and the Schatten–von Neumann classes $C_p(H); 1 < p < 1$; of operators on Hilbert spaces. On the other hand, the space of continuous functions $C(K)$ does not have the UMD property.

We need to recall the notion of M-bounded sequence (MR-bounded sequence) of operators.

Definition 2.3 ([31])**.** We say that a sequence $\{T_k\}_{k\in\mathbb{Z}} \subset \mathcal{B}(X,Y)$ is M-bounded of order n ($n \in \mathbb{N} \cup \{0\}$), if

$$\sup_{0\leq l\leq n} \sup_{k\in\mathbb{Z}} \|k^l \Delta^l T_k\| < \infty, \tag{2.2}$$

where

$$\Delta^0 T_k := T_k, \quad \Delta T_k := \Delta^1 T_k := T_{k+1} - T_k$$

and for $n \in \mathbb{N}$ with $n \geq 2$ we have

$$\Delta^n T_k := \Delta(\Delta^{n-1} T_k).$$

Remark 2.4.

(i) Given $\{M_k\}_{k\in\mathbb{Z}}$ and $\{N_k\}_{k\in\mathbb{Z}}$ be such that they are both M-bounded of order n, then the sum is also M-bounded of the same order. Moreover, if $\{M_k\}_{k\in\mathbb{Z}}$ and $\{N_k\}_{k\in\mathbb{Z}}$ are sequences in $\mathcal{B}(Y,Z)$ and $\mathcal{B}(X,Y)$ that are M-bounded of order n, then $\{M_k\, N_k\}_{k\in\mathbb{Z}} \subset \mathcal{B}(X,Z)$ is also M-bounded of the same order.

(ii) If we replace condition (2.2) in Definition 2.3 by the condition that the set

$$\{k^l \Delta^l M_k : k \in \mathbb{Z}\}, \tag{2.3}$$

is R-bounded for each $0 \leq l \leq n$, then we say that $\{M_k\}_{k\in\mathbb{Z}} \subset \mathcal{B}(X,Y)$ is MR-bounded of order n.

We also recall the definition of n-regular scalar sequences which was first considered in [31].

Definition 2.5. A sequence $\{c_k\}_{k\in\mathbb{Z}} \subset \mathbb{C}$ is called n-regular if the set $\{k^p \frac{\Delta^p c_k}{c_k}\}_{k\in\mathbb{Z}}$ is bounded for all $p = 1,\ldots,n$.

We finally recall the following result recently shown in [19] which provides an important criterion for MR-boundedness in the context of maximal regularity for abstract evolution equations.

Theorem 2.6. *Let $T : D(T) \subset X \to X$ be a closed linear operator defined in a Banach space X. For each $k \in \mathbb{Z}$ let $H_k : X \to D(T)$ be a sequence of bounded and linear operators such that $0 \in \rho(H_k)$ for all $k \in \mathbb{Z}$. Suppose that $(s_k)_{k\in\mathbb{Z}} \subset \mathbb{C}$ is a 1-regular sequence and denote*

$$M_k := s_k T H_k, \tag{2.4}$$

and

$$L_k := (H_k^{-1} - H_{k+1}^{-1})H_k. \tag{2.5}$$

If $\{M_k : k \in \mathbb{Z}\}$ and $\{kL_k : k \in \mathbb{Z}\}$ are R-bounded (uniformly bounded) sets, then $\{M_k : k \in \mathbb{Z}\}$ is MR-bounded (M-bounded) of order 1. If, in addition, $(s_k)_{k\in\mathbb{Z}}$ is 2-regular and the set $\{k^2\Delta L_k : k \in \mathbb{Z}\}$ is R-bounded (uniformly bounded), then $\{M_k : k \in \mathbb{Z}\}$ is MR-bounded (M-bounded) of order 2.

3 Well-posedness in L^p-spaces

Let $1 \le p < \infty$ and X be a Banach space. In this section, we want to give optimal conditions that can describe the well-posedness of the problem

$$u''''(t) + \alpha u'''(t) + \beta u''(t) + \gamma A u''(t) + \delta A u'(t) + \rho A u(t) = f(t), \quad t \in \mathbb{T} := [0, 2\pi] \tag{3.1}$$

in 2π-periodic vector valued L^p-spaces. In other words, we want to obtain a complete characterization on the existence, uniqueness and well-posedness of the problem only in terms of the data of the problem. Here A is a closed linear operator with domain $D(A)$.

We now introduce the notion of the following set denoted as $\rho_s(A)$ as follows:

$$\rho_s(A) := \Big\{ s \in \mathbb{R} : s^4 - \alpha i s^3 - \beta s^2 - \gamma s^2 A + \delta i s A + \rho A : [D(A)] \to X \\ \text{is invertible and } [s^4 - \alpha i s^3 - \beta s^2 - \gamma s^2 A + \delta i s A + \rho A]^{-1} \in \mathcal{B}(X) \Big\}, \tag{3.2}$$

where $[D(A)]$ denotes a Banach space under the norm $\|x\|_{[D(A)]} := \|x\| + \|Ax\|$.
For any $n \in \mathbb{N}$ and $1 \le p < \infty$ we define the vector-valued function spaces [7, Definition 2.4]:

$$W^{n,p}_{per}(\mathbb{T}, X) := \{u \in L^p(\mathbb{T}, X) : \text{ there exists } v \in L^p(\mathbb{T}, X),\ \hat{v}(k) = (ik)^n \hat{u}(k) \text{ for all } k \in \mathbb{Z}\}.$$

Remark 3.1. It is important to point out that the following properties hold

(i) Given $n, m \in \mathbb{N}$, if $n \le m$ then $W^{m,p}_{per}(\mathbb{T}, X) \subset W^{n,p}_{per}(\mathbb{T}, X)$.

(ii) If $u \in W^{n,p}_{per}(\mathbb{T}, X)$ then for all $0 \le k \le n-1$ it follows that $u^{(k)}(0) = u^{(k)}(2\pi)$.

Note that [4]:

$$W^{n,p}_{per}(\mathbb{T}, X) = \{u \in L^p(\mathbb{T}, X) : \ u \text{ is } n\text{-times differentiable a.e.}, \\ u^{(n)} \in L^p(\mathbb{T}, X) \ \text{ and } u^{(k)}(0) = u^{(k)}(2\pi), 0 \le k \le n-1\}.$$

We refer to [4, Lemma 2.1] and [7] for more information about these spaces. In order to consider maximal regularity for our problem we need to define the following space:

$$S_p(A) := W^{4,p}_{per}(\mathbb{T}, X) \cap W^{2,p}_{per}(\mathbb{T}, [D(A)]).$$

The space $S_p(A)$ is a Banach space with the norm

$$\|u\|_{S_p(A)} := \|Au\|_p + \|Au'\|_p + \|Au''\|_p + \|u\|_p + \|u''\|_p + \|u'''\|_p + \|u''''\|_p.$$

We now introduce the following definition.

Definition 3.2. Let $1 \le p < \infty$ and $f \in L^p(\mathbb{T}, X)$ be given. We say that $u \in S_p(A)$ is a strong L^p-solution of equation (3.1) if it satisfies (3.1) for almost all $t \in \mathbb{T}$. We say that equation (3.1) is strongly L^p-well-posed if for each $f \in L^p(\mathbb{T}, X)$, there exists a unique strong L^p-solution of equation (3.1).

As a very important consequence, we obtain the following: There exists a constant $C > 0$ such that for each $f \in L^p(\mathbb{T}, X)$, we have

$$\|u\|_{S_p(A)} \le C\|f\|_{L^p}.$$

Before we provide our main result, we need the following two theorems from [4] that establish the equivalence between R-boundedness and the fact of being an L^p-multiplier. They will be needed in order to characterize L^p-well-posedness for equation (3.1).

Theorem 3.3. *Let X, Y be UMD spaces. If a sequence $\{M_k\}_{k\in\mathbb{Z}} \subset \mathcal{B}(X, Y)$ is MR-bounded of order 1, then $(M_k)_{k\in\mathbb{Z}}$ defines an L^p-Fourier multiplier whenever $1 < p < \infty$.*

Theorem 3.4. *Let X, Y be Banach spaces, $1 \le p < \infty$ and let $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{B}(X, Y)$ be an L^p-Fourier multiplier. Then the set $\{M_k : k \in \mathbb{Z}\}$ is R-bounded.*

Let A be a closed linear operator such that $\mathbb{Z} \subset \rho_s(A)$. We denote

$$N_k := [a_k - \alpha b_k - \beta c_k - \gamma c_k A + \delta ikA + \rho A]^{-1}, \quad a_k = k^4, \quad b_k = ik^3, \quad c_k = k^2, \quad k \in \mathbb{Z}, \tag{3.3}$$

where $\alpha, \beta, \gamma, \delta, \rho \in \mathbb{R}$ are fixed constants.

The following proposition will be an important tool for proving the main result of this section.

Proposition 3.5. *Let A be a closed linear operator defined on a UMD space X and $\alpha, \beta, \gamma, \delta, \rho \in \mathbb{R}$. If $\mathbb{Z} \subset \rho_s(A)$ and $\{k^4 N_k : k \in \mathbb{Z}\}$ and $\{k^2 A N_k : k \in \mathbb{Z}\}$ are R-bounded sets, then $(k^4 N_k)_{k\in\mathbb{Z}}$, $(ik^3 N_k)_{k\in\mathbb{Z}}$, $(k^2 N_k)_{k\in\mathbb{Z}}$, $(k^2 A N_k)_{k\in\mathbb{Z}}$, $(kAN_k)_{k\in\mathbb{Z}}$ and $(AN_k)_{k\in\mathbb{Z}}$ are L^p-Fourier multipliers.*

Proof. We first point out that the R-boundedness of $\{k^4 N_k : k \in \mathbb{Z}\}$ immediately implies the R-boundedness of the sets $\{ik^3 N_k : k \in \mathbb{Z}\}$ and $\{k^2 N_k : k \in \mathbb{Z}\}$. Similarly, if by hypothesis $\{k^2 A N_k : k \in \mathbb{Z}\}$ is R-bounded then the sets $\{kAN_k : k \in \mathbb{Z}\}$ and $\{AN_k : k \in \mathbb{Z}\}$ are so. Let $M_k := k^4 N_k$. In order to show that M_k is an L^p-multiplier we only need to show that $\{k\Delta M_k : k \in \mathbb{Z}\}$ is R-bounded. We apply Theorem 2.6 with $s_k = k^4$, which is 1-regular, $H_k = N_k$ and $T = I$. By hypothesis $\{M_k : k \in \mathbb{Z}\}$ is R-bounded, then we only need to show that $\{kL_k : k \in \mathbb{Z}\}$ is R-bounded. Indeed, we have

$$\begin{aligned} kL_k &= k(N_k^{-1} - N_{k+1}^{-1})N_k \\ &= k[-\Delta a_k + \alpha\Delta b_k + \beta\Delta c_k + \gamma\Delta c_k A - \delta i A]N_k \\ &= -\frac{k\Delta a_k}{a_k} M_k + \alpha\frac{k\Delta b_k}{b_k}(b_k N_k) + \beta\frac{k\Delta c_k}{c_k}(c_k N_k) + \gamma\frac{k\Delta c_k}{c_k}(c_k A N_k) - \delta ikAN_k. \end{aligned}$$

By hypothesis then it follows that $\{kL_k : k \in \mathbb{Z}\}$ is R-bounded. The R-boundedness of $\{k\Delta(ik^3 N_k)\}_{k\in\mathbb{Z}}$ and $\{k\Delta(k^2 N_k)\}_{k\in\mathbb{Z}}$ follows similarly applying Theorem 2.6 with $s_k = ik^3$, $T = I$ and $H_k = N_k$ in the first case, $s_k = k^2$, $T = I$ and $H_k = N_k$ in the second case. As a consequence of Theorem 3.3 they are L^p-Fourier multipliers. On the other hand, the R-boundedness of $\{k\Delta(k^2 A N_k)\}_{k\in\mathbb{Z}}$, $\{k\Delta(kAN_k)\}_{k\in\mathbb{Z}}$ and $\{k\Delta(AN_k)\}_{k\in\mathbb{Z}}$ also follows from Theorem 2.6 with $s_k = k^2$, $T = A$ and $H_k = N_k$ in the first case, $s_k = k$, $T = A$ and $H_k = N_k$ in the second case and $s_k = 1$, $T = A$ and $H_k = N_k$ in the last case. □

We now show the main result of this section that provides a computable criterion to characterize the well-posedness of equation (3.1).

Theorem 3.6. *Let $1 < p < \infty$ and $\alpha, \beta, \gamma, \delta, \rho \in \mathbb{R}$ be given with $(\gamma, \delta, \rho) \neq (0,0,0)$. Assume that A is a closed linear operator defined on a UMD space X. The following assertions are equivalent:*

(i) Equation (3.1) is strongly L^p-well-posed;

(ii) $\mathbb{Z} \subset \rho_s(A)$ and the set $\{k^4 N_k : k \in \mathbb{Z}\}$ is R-bounded.

Proof. We first prove $(i) \Longrightarrow (ii)$. Given $k \in \mathbb{Z}$ and $y \in X$ we define the function $f \in L^p(\mathbb{T}, X)$ as $f(t) = e^{ikt}y$. It is not difficult to check that $\hat{f}(k) = y$ and 0 otherwise. By hypothesis, equation (3.1) is L^p-well-posed and then there exists a unique $u \in S_p(A)$ which solves equation (3.1). If we take

Fourier transform in both sides of (3.1) we get:

$$[a_k - \alpha b_k - \beta c_k - \gamma c_k A + \delta ikA + \rho A]\hat{u}(k) = y, \tag{3.4}$$

and

$$[a_n - \alpha b_n - \beta c_n - \gamma c_n A + \delta inA + \rho A]\hat{u}(n) = 0, \quad n \neq k. \tag{3.5}$$

This shows that $[a_k - \alpha b_k - \beta c_k - \gamma c_k A + \delta ikA + \rho A]$ is surjective. On the other hand, let $x \in D(A)$ be such that

$$[a_k - \alpha b_k - \beta c_k - \gamma c_k A + \delta ikA + \rho A]x = 0.$$

We define $u \in S_p(A)$ as $u(t) = e^{ikt}x$ for $t \in \mathbb{T}$. It is not difficult to see that u is a solution for equation (3.1) when $f = 0$. By uniqueness, then it necessarily follows that $x = 0$ and then $[a_k - \alpha b_k - \beta c_k - \gamma c_k A + \delta ikA + \rho A]$ is bijective from $D(A)$ onto X. Moreover, $[a_k - \alpha b_k - \beta c_k - \gamma c_k A + \delta ikA + \rho A]^{-1} \in \mathcal{B}(X)$. Indeed, given $y \in X$ and $k \in \mathbb{Z}$ let $f(t) = e^{ikt}y$ and let u be the corresponding solution of (3.1) for f. Then $\hat{u}(k) = [a_k - \alpha b_k - \beta c_k - \gamma c_k A + \delta ikA + \rho A]^{-1}y$ and 0 otherwise.

This implies $u(t) = -e^{-ikt}[a_k - \alpha b_k - \beta c_k - \gamma c_k A + \delta ikA + \rho A]^{-1}y$ by uniqueness. As a consequence, there exists a positive constant $C > 0$ independent of y and k such that

$$\|u\|_{S_p(A)} \leq C\|f\|_{L^p},$$

which implies

$$\|[a_k - \alpha b_k - \beta c_k - \gamma c_k A + \delta ikA + \rho A]^{-1}\| \leq C$$

for all $k \in \mathbb{Z}$. This proves the claim. We have shown that $\mathbb{Z} \subset \rho_s(A)$. Let $M_k = k^4 N_k$ with $k \in \mathbb{Z}$, where N_k is defined in (3.3). To finish this implication it only remains to show that $(M_k)_{k \in \mathbb{Z}}$ is L^p-Fourier multiplier. Given $f \in L^p(\mathbb{T}, X)$, there exists $u \in S_p(A)$ which is a solution of equation (3.1) by assumption. Taking Fourier transforms on both sides of (3.1), we get that $\hat{u}(k) \in D(A)$ and

$$[a_k - \alpha b_k - \beta c_k - \gamma c_k A + \delta ikA + \rho A]\hat{u}(k) = \hat{f}(k), \qquad k \in \mathbb{Z}.$$

Due to the invertibility of $[a_k - \alpha b_k - \beta c_k - \gamma c_k A + \delta ikA + \rho A]$ we can assert that $\hat{u}(k) = N_k \hat{f}(k), k \in \mathbb{Z}$. As $u \in S_p(A)$ we obtain that

$$\widehat{[u'''']}(k) = k^4 \hat{u}(k) = k^4 N_k \hat{f}(k) = M_k \hat{f}(k).$$

Finally, since $u'''' \in L^p(\mathbb{T}, X)$ we get that $(M_k)_{k\in\mathbb{Z}}$ is L^p-Fourier multipliers and, by Theorem 3.4, we conclude that the set $\{M_k : k \in \mathbb{Z}\}$ is R-bounded, proving (ii).

Let now show $(ii) \implies (i)$. We assume that $\mathbb{Z} \subset \rho_s(A)$ and the set $\{k^4 N_k : k \in \mathbb{Z}\}$ is R-bounded. A simple calculation shows the following identity

$$k^2 AN_k = \frac{k^2}{\gamma k^2 - \rho - i\delta k}\left[1 - \frac{\beta}{k^2} - \frac{i\alpha}{k}\right] k^4 N_k - \frac{k^2}{\gamma k^2 - \rho - i\delta k}, \qquad k \in \mathbb{Z}\setminus\{0\}, \tag{3.6}$$

proving that the set $\{k^2 AN_k : k \in \mathbb{Z}\}$ is R-bounded, too. Let $M_k = k^4 N_k$ and $S_k = k^2 AN_k$. It follows from Proposition 3.5 that $(M_k)_{k\in\mathbb{Z}}$, $(ik^3 N_k)_{k\in\mathbb{Z}}$ and $(k^2 N_k)_{k\in\mathbb{Z}}$ are L^p-Fourier multipliers.

Note that the R-boundedness of the set $\{k^4 N_k\}_{k\in\mathbb{Z}}$ implies that $\{kN_k\}_{k\in\mathbb{Z}}$ is R-bounded and then the set $\{k(N_{k+1} - N_k)\}$ is also R-bounded. It follows from Theorem 3.3 that $\{N_k\}_{k\in\mathbb{Z}}$ is an L^p-Fourier multiplier. In particular, $N_k \in \mathcal{B}(X, [D(A)])$.

Then, for all $f \in L^p(\mathbb{T}, X)$ there exist $w, u_1, u_2, u_3 \in L^p(\mathbb{T}, [D(A)])$ satisfying:

$$\hat{w}(k) = N_k \hat{f}(k), \hat{u}_1(k) = M_k \hat{f}(k), \hat{u}_2(k) = -ik^3 N_k \hat{f}(k), \hat{u}_3(k) = -k^2 N_k \hat{f}(k).$$

Consequently, $\hat{u}_1(k) = k^4 \hat{w}(k)$ when $k \in \mathbb{Z}$. This implies that $w \in W^{4,p}_{per}(\mathbb{T}; [D(A)])$ [4, Lemma 2.1] and $w''''(t) = u_1(t)$ a.e. [4, Lemma 3.1]. In particular, $w'''' \in L^p(\mathbb{T}, [D(A)])$. Similarly, we obtain:

$$\hat{u}_2(k) = (ik)^3 \hat{w}(k) = \widehat{w'''}(k), \quad \hat{u}_3(k) = (ik)^2 \hat{w}(k) = \widehat{w''}(k)$$

and then $w'''(t) = u_2(t)$ and $w''(t) = u_3(t)$. In particular, $w'', w''' \in L^p(\mathbb{T}, [D(A)])$.

By hypothesis and Proposition 3.5, it follows that $\{S_k\}_{k\in Z}$, $\{kAN_k\}_{k\in Z}$ and $\{AN_k\}_{k\in Z}$ are L^p-Fourier multipliers, and then we can ensure that there exist $u_4, u_5, u_6 \in L^p(\mathbb{T}, X)$ such that

$$\hat{u}_4(k) = -k^2 AN_k \hat{f}(k) = A\widehat{w''}(k) = \widehat{Aw''}(k),$$

and

$$\hat{u}_5(k) = ikAN_k \hat{f}(k) = A\widehat{w'}(k) = \widehat{Aw'}(k),$$

as well as

$$\hat{u}_6(k) = AN_k \hat{f}(k) = A\hat{w}(k) = \widehat{Aw}(k),$$

where we have used that A is closed. It follows from [4, Lemma 3.1] that $w(t), w'(t), w''(t) \in D(A)$ and $Aw''(t) = u_4(t)$, $Aw'(t) = u_5(t)$ and $Aw'(t) = u_6(t)$. In addition, $Aw, Aw', Aw'' \in L^p(\mathbb{T}, X)$. As a consequence, $w \in S_p(A)$. Moreover, the following identity holds:

$$I_X = k^4 N_k - \alpha i k^3 N_k - \beta k^2 N_k - \gamma k^2 AN_k + \delta i k AN_k + \rho AN_k, \tag{3.7}$$

and then we obtain

$$\begin{aligned} \hat{f}(k) &= [k^4 N_k - \alpha i k^3 N_k - \beta k^2 N_k - \gamma k^2 AN_k + \delta i k AN_k + \rho AN_k]\hat{f}(k) \\ &= \widehat{w''''}(k) + \alpha\widehat{w'''}(k) + \beta\widehat{w'}(k) + \gamma\widehat{Aw''}(k) + \delta\widehat{Aw'}(k) + \rho\widehat{Aw}(k). \end{aligned}$$

This implies that

$$w''''(t) + \alpha w'''(t) + \beta w''(t) + \gamma Aw''(t) + \delta Aw'(t) + \rho Aw(t) = f(t),$$

by the uniqueness theorem (see [4, p. 314]). It only remains to prove that the solution is unique. Indeed, for a given $w \in S_p(A)$ that satisfies equation (3.1) for $f = 0$, if we take Fourier transform we get that $[a_k - \alpha b_k - \beta c_k - \gamma c_k A + \delta i k A + \rho A]\hat{w}(k) = 0$ for all $k \in \mathbb{Z}$. Hence $w = 0$ since $\mathbb{Z} \subset \rho_s(A)$. Thus, equation (3.1) is strongly L^p-well-posed. □

We point out that L^p-well-posedness does not depend on the parameter p, that is, if equation (3.1) is strongly L^p-well-posed for some $1 < p < \infty$, then it is strongly L^p-well-posed for all $1 < p < \infty$.

4 Well-posedness in Besov and Triebel–Lizorkin spaces

In this section, we now analyze the well-posedness of equation (3.1) in periodic Besov spaces $B^s_{p,q}(\mathbb{T}, X)$ and periodic Triebel–Lizorkin spaces $F^s_{p,q}(\mathbb{T}, X)$. The definition and properties of vector-valued periodic Besov spaces can be found in [5].

Given $1 \leq p, q \leq \infty$ and $s > 0$, we define the maximal regularity space that describes the strongly $B^s_{p,q}$-well-posedness of the equation (3.1) by

$$S_{p,q,s}(A) := B^{s+4}_{p,q}(\mathbb{T}, X) \cap B^{s+2}_{p,q}(\mathbb{T}, [D(A)]).$$

The vectorial space $S_{p,q,s}(A)$ is a Banach space with the norm

$$\|u\|_{S_{p,q,s}(A)} := \|u''\|_{B^s_{p,q}} + \|u'''\|_{B^s_{p,q}} + \|u''''\|_{B^s_{p,q}} + \|Au\|_{B^s_{p,q}} + \|Au'\|_{B^s_{p,q}} + \|Au''\|_{B^s_{p,q}}.$$

Analogously to the case L^p we can define the strongly $B^s_{p,q}$-well-posedness for equation (3.1) as follows.

Definition 4.1. Let $1 \leq p, q < \infty$, $s > 0$ and $f \in B^s_{p,q}(\mathbb{T}, X)$ be given. We say that $u \in S_{p,q,s}(A)$ is a strong $B_{p,q}$-solution of (3.1) if it satisfies (3.1) for all $t \in \mathbb{T}$. We say that (3.1) is strongly $B^s_{p,q}$-well-posed if for each $f \in B^s_{p,q}(\mathbb{T}, X)$, there exists a unique strong $B^s_{p,q}$-solution of (3.1).

Note that if (3.1) is strongly $B^s_{p,q}$-well-posed, by the Closed Graph Theorem, there exists a constant $C > 0$ such that for each $f \in B^s_{p,q}(\mathbb{T}, X)$, we have

$$\|u\|_{S_{p,q,s}(A)} \leq C\|f\|_{B^s_{p,q}}.$$

We now introduce the following notion that corresponds to $B^s_{p,q}$-Fourier multiplier (see [4]).

Definition 4.2. Let X, Y be Banach spaces, $1 \leq p, q < \infty$, $s \in \mathbb{R}$ and $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{B}(X, Y)$. We say that $(M_k)_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-Fourier multiplier if, for each $f \in B^s_{p,q}(\mathbb{T}, X)$ there exists $u \in B^s_{p,q}(\mathbb{T}, Y)$ such that

$$\hat{u}(k) = M_k \hat{f}(k)$$

for all $k \in \mathbb{Z}$.

The following theorem contained in [5] states that M-boundedness of order 2 is sufficient for an operator valued symbol to be a $B^s_{p,q}$- Fourier multiplier.

Theorem 4.3. *Let X, Y be Banach spaces. If $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{B}(X, Y)$ is M-bounded of order 2, then for $1 \leq p, q \leq \infty$, $s \in \mathbb{R}$ the set $(M_k)_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-Fourier multiplier.*

The following result provides necessary conditions for certain sets which will be needed to characterize strongly $B^s_{p,q}$-well-posedness.

Proposition 4.4. *Let A be a closed linear operator defined on a UMD space X and $\alpha, \beta, \gamma, \delta, \rho \in \mathbb{R}$. If $\mathbb{Z} \subset \rho_s(A)$ and the sets $\{k^4 N_k : k \in \mathbb{Z}\}$ and $\{k^2 A N_k : k \in \mathbb{Z}\}$ are uniformly bounded, then $(k^4 N_k)_{k\in\mathbb{Z}}$, $(ik^3 N_k)_{k\in\mathbb{Z}}$, $(k^2 N_k)_{k\in\mathbb{Z}}$, $(k^2 A N_k)_{k\in\mathbb{Z}}$, $(kAN_k)_{k\in\mathbb{Z}}$ and $(AN_k)_{k\in\mathbb{Z}}$ are $B^s_{p,q}$-Fourier multipliers.*

Proof. Let $M_k = k^4 N_k$. In order to show that M_k is a $B^s_{p,q}$-Fourier multiplier and according to Theorem 4.3, we need to prove that $\sup_{k\in\mathbb{Z}}(\|M_k\| + \|k\Delta M_k\|) < \infty$ and $\sup_{k\in\mathbb{Z}} \|k^2\Delta^2 M_k\| < \infty$. The first inequality holds as a consequence of the hypothesis and Proposition 3.5. Therefore, we only need to show the second one which will be done applying Theorem 2.6 to $s_k = k^4$, which is clearly a 2-regular sequence, $H_k = N_k$ and $T = I$. By hypothesis, $\sup_{k\in\mathbb{Z}} \|M_k\| < \infty$. Moreover, by Proposition 3.5 it follows that $\sup_{k\in\mathbb{Z}} \|kL_k\| < \infty$, then it only remains to show that $\sup_{k\in\mathbb{Z}} \|k^2\Delta L_k\| < \infty$. Indeed, we have

$$L_k = ({N_k}^{-1} - N_{k+1}^{-1})N_k = [-\Delta a_k + \alpha\Delta b_k + \beta\Delta c_k + \gamma\Delta c_k A - \delta i A]N_k.$$

Then,

$$\begin{aligned}k^2\Delta L_k = k^2&[(a_{k+1} - a_{k+2})N_{k+1} - (a_k - a_{k+1})N_k]\\ &+ \alpha k^2[(b_{k+2} - b_{k+1})N_{k+1} - (b_{k+1} - b_k)N_k]\\ &+ \beta k^2[(c_{k+2} - c_{k+1})N_{k+1} - (c_{k+1} - c_k)N_k]\\ &+ \gamma k^2[(c_{k+2} - c_{k+1})AN_{k+1} - (c_{k+1} - c_k)AN_k]\\ &- \delta i(AN_{k+1} - AN_k),\end{aligned} \tag{4.1}$$

where $a_k = k^4$ and $b_k = ik^3$ and $c_k = k^2$. We only need to prove that each term is bounded. First of all, a simple calculus shows that:

$$(a_{k+1} - a_{k+2})N_{k+1} - (a_k - a_{k+1})N_k = -(\Delta^2 a_k)N_{k+1} + \frac{\Delta a_k}{a_k}\left[(a_k N_k - a_{k+1}N_{k+1}) + N_{k+1}(\Delta a_k)\right].$$

Therefore

$$\begin{aligned}&k^2[(a_{k+1} - a_{k+2})N_{k+1} - (a_k - a_{k+1})N_k] =\\ &\quad -k^2\frac{(\Delta^2 a_k)}{a_k}\frac{a_k}{a_{k+1}}(a_{k+1}N_{k+1}) + k\frac{\Delta a_k}{a_k}\left[k(a_k N_k - a_{k+1}N_{k+1}) + a_{k+1}N_{k+1}\frac{a_k}{a_{k+1}}\left\{\frac{k(\Delta a_k)}{a_k}\right\}^2\right].\end{aligned}$$

Since the sequence a_k is 2-regular, $M_k = a_k N_k$ and $k\Delta M_k$ are bounded, the above identity shows that

$$\sup_{k\in\mathbb{Z}} \|k^2[(a_{k+1} - a_{k+2})N_{k+1} - (a_k - a_{k+1})N_k]\| < \infty.$$

Analogously and following the same procedure as above, using the fact that b_k is also 2-regular, $b_k N_k$ and $k\Delta(b_k N_k)$ are bounded, we obtain that

$$\sup_{k\in\mathbb{Z}} \|k^2[(b_{k+2} - b_{k+1})N_{k+1} - (b_{k+1} - b_k)N_k]\| < \infty.$$

Following the same idea we get that

$$\sup_{k\in\mathbb{Z}} \|k^2[(c_{k+2} - c_{k+1})N_{k+1} - (c_{k+1} - c_k)N_k]\| < \infty$$

and

$$\sup_{k\in\mathbb{Z}} \|k^2[(c_{k+2} - c_{k+1})AN_{k+1} - (c_{k+1} - c_k)AN_k]\| < \infty$$

since c_k is 2-regular and $S_k = c_k N_k$ and $k\Delta S_k$ are bounded in the first case, meanwhile $R_k = c_k A N_k$ and $k\Delta R_k$ are bounded for proving the second inequality. Finally, the fact that $k\Delta R_k$ is bounded immediately implies the boundedness for the last summand $-\delta i(AN_{k+1} - AN_k)$ in (4.1). Consequently, $(k^4 N_k)_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-Fourier multiplier.

We now consider $M_k = ik^3 N_k$. In order to prove that it is a $B^s_{p,q}$-Fourier multiplier it only remains to show again that $\sup_{k\in\mathbb{Z}} \|k^2\Delta^2 M_k\| < \infty$, which can be do using the second part of Theorem 2.6 with $s_k = ik^3, H_k = N_k$ and $T = I$. By hypothesis and Proposition 3.5 it follows that $\sup_{k\in\mathbb{Z}} \|M_k\| < \infty$ and $\sup_{k\in\mathbb{Z}} \|kL_k\| < \infty$, respectively. The inequality $\sup_{k\in\mathbb{Z}} \|k^2\Delta L_k\| < \infty$ has already been shown since L_k is exactly the same that in the above computation. Therefore, $(ik^3 N_k)_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-Fourier multiplier. Similarly, we obtain that $(k^2 N_k)_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-Fourier multiplier.

Let now $M_k = k^2 AN_k$. From Proposition 3.5 it follows that $\sup_{k\in\mathbb{Z}}(\|M_k\| + \|k\Delta M_k\|) < \infty$. To prove that $\sup_{k\in\mathbb{Z}} \|k^2\Delta^2 M_k\| < \infty$ we apply Theorem 2.6 with $s_k = k^2, H_k = N_k$ and $T = A$. It remains to show that $\sup_{k\in\mathbb{Z}} \|k^2\Delta L_k\| < \infty$, where L_k is the same that in the above calculus. Therefore, $(k^2 AN_k)_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-Fourier multiplier. The same procedure can be applied to $M_k = kAN_k$ with $s_k = k, H_k = N_k$ and $T = A$ and $M_k = AN_k$ with $s_k = 1, H_k = N_k$ and $T = A$. The conclusion then holds and consequently $(kAN_k)_{k\in\mathbb{Z}}$ and $(AN_k)_{k\in\mathbb{Z}}$ are $B^s_{p,q}$-Fourier multipliers. □

We now enunciate the main result of this section. The proof follows essentially the same steps than the one of Theorem 3.6. However, we include here the essential changes of the proof that differ from Theorem 3.6 in order to make it clear to the reader.

Theorem 4.5. *Let $1 \le p, q \le \infty$, $s > 0$ and $\alpha, \beta, \gamma, \delta, \rho \in \mathbb{R}$ be given with $(\gamma, \delta, \rho) \neq (0, 0, 0)$. Assume A is a closed linear operator defined on a Banach space X. The following assertions are equivalent:*

(i) The equation

$$u''''(t) + \alpha u'''(t) + \beta u''(t) + \gamma Au''(t) + \delta Au'(t) + \rho Au(t) = f(t), \qquad t \in [0, 2\pi]$$

is strongly $B^s_{p,q}$-well-posed;

(ii) $\mathbb{Z} \subset \rho_s(A)$ and $\sup_{k\in\mathbb{Z}} \|k^4 N_k\| < \infty$.

Proof. $(i) \implies (ii)$ follows the same lines of Theorem 3.6 and therefore is omitted. We prove $(ii) \implies (i)$. We assume that $\mathbb{Z} \subset \rho_s(A)$ and the set $\{k^4 N_k : k \in \mathbb{Z}\}$ is uniformly bounded. The identity (3.6) shows that the set $\{k^2 AN_k : k \in \mathbb{Z}\}$ is uniformly bounded.

Analogously, the identities $kN_k = \frac{1}{k^3}(k^4 N_k)$ and $k^2 N_k = \frac{1}{k^2}(k^4 N_k)$ show that the sets $\{kN_k : k \in \mathbb{Z}\}$ and $\{k^2 N_k : k \in \mathbb{Z}\}$ are also uniformly bounded. Therefore the sets $\{k(N_{k+1} - N_k)\}_{k\in\mathbb{Z}}$ and $\{k^2(N_{k+2} - 2N_{k+1} + N_k)\}_{k\in\mathbb{Z}}$ are uniformly bounded and hence, by Theorem 4.3, the set $\{N_k\}_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-Fourier multiplier. Moreover, by hypothesis and Proposition 4.4 it follows that $(k^4 N_k)_{k\in\mathbb{Z}}$, $(ik^3 N_k)_{k\in\mathbb{Z}}$, $(k^2 N_k)_{k\in\mathbb{Z}}$, $(k^2 AN_k)_{k\in\mathbb{Z}}$, $(kAN_k)_{k\in\mathbb{Z}}$ and $(AN_k)_{k\in\mathbb{Z}}$ are $B^s_{p,q}$-Fourier multipliers.

Let $f \in B^s_{p,q}(\mathbb{T}, X)$ be given. Since $(k^4 N_k)_{k\in\mathbb{Z}}$, $(ik^3 N_k)_{k\in\mathbb{Z}}$, $(k^2 N_k)_{k\in\mathbb{Z}}$ are $B^s_{p,q}$ multipliers, there exist $w, u_1, u_2, u_3 \in B^s_{p,q}(\mathbb{T}, [D(A)])$ satisfying:

$$\hat{w}(k) = N_k \hat{f}(k), \hat{u}_1(k) = k^4 N_k \hat{f}(k), \hat{u}_2(k) = -ik^3 N_k \hat{f}(k), \hat{u}_3(k) = -k^2 N_k \hat{f}(k). \tag{4.2}$$

Consequently, $\hat{u}_1(k) = k^4 \hat{w}(k)$ when $k \in \mathbb{Z}$. This implies that $w \in B^{s+4}_{p,q}(\mathbb{T}; [D(A)])$ and $w''''(t) = u_1(t)$. In particular, $w'''' \in B^s_{p,q}(\mathbb{T}, [D(A)])$. Similarly, we obtain:

$$\hat{u}_2(k) = (ik)^3 \hat{w}(k) = \widehat{w'''}(k), \qquad \hat{u}_3(k) = (ik)^2 \hat{w}(k) = \widehat{w''}(k),$$

and then $w'''(t) = u_2(t)$ and $w''(t) = u_3(t)$. In particular, $w'', w''' \in B^s_{p,q}(\mathbb{T}, [D(A)])$.

By hypothesis and Proposition 4.4 the sets $\{k^2 AN_k\}_{k\in Z}$, $\{kAN_k\}_{k\in Z}$ and $\{AN_k\}_{k\in Z}$ are $B^s_{p,q}$-Fourier multipliers, and then we have that there exist $u_4, u_5, u_6 \in B^s_{p,q}(\mathbb{T}, X)$ such that

$$\begin{aligned} \hat{u}_4(k) &= -k^2 AN_k \hat{f}(k) = A\widehat{w''}(k) = \widehat{Aw''}(k), \\ \hat{u}_5(k) &= ikAN_k \hat{f}(k) = A\widehat{w'}(k) = \widehat{Aw'}(k), \\ \hat{u}_6(k) &= AN_k \hat{f}(k) = A\widehat{w}(k) = \widehat{Aw}(k). \end{aligned} \tag{4.3}$$

where we have used that A is closed. It follows from [4, Lemma 3.1] that $w(t), w'(t), w''(t) \in D(A)$ and $Aw''(t) = u_4(t)$, $Aw'(t) = u_5(t)$ and $Aw'(t) = u_6(t)$ a.e. In addition, Aw, Aw', $Aw'' \in B^s_{p,q}(\mathbb{T}, X)$. Replacing (4.2) - (4.3) in the following identity:

$$\hat{f}(k) = k^4 N_k \hat{f}(k) - \alpha i k^3 N_k \hat{f}(k) - \beta k^2 N_k \hat{f}(k) - \gamma k^2 AN_k \hat{f}(k) + \delta ikAN_k \hat{f}(k) + \rho AN_k \hat{f}(k),$$

we obtain by the uniqueness of the Fourier coefficients that w solves equation (3.1). The uniqueness follows the same lines as in Theorem 3.6. □

We point out that the second assertion in Theorem 4.5 does not depend on the parameters p, q and s, and then strongly $B^s_{p,q}$-well-posedness for equation (3.1) holds for some $1 \le p, q \le \infty$, $s > 0$ if and only if it is strongly $B^s_{p,q}$-well-posed for all $1 \le p, q \le \infty$, $s > 0$. To finish this section, we consider well-posedness in periodic Triebel–Lizorkin spaces $F^s_{p,q}$ with $1 \le p < \infty$, $1 \le q \le \infty$, $s \in \mathbb{R}$. We do not include the formal definition of these spaces but we refer the reader to [14] for the details and properties of these spaces.

Using a similar argument as the one in the proof of Theorem 4.5, we obtain the following characterization of the strongly $F^s_{p,q}$-well-posedness of equation (3.1). In order to prove this result we use the operator-valued Fourier multiplier theorem proved in [14]. We omit the details.

Theorem 4.6. *Let $1 \le p < \infty, 1 \le q \le \infty$, $s > 0$ and $\alpha, \beta, \gamma, \delta, \rho \in \mathbb{R}$ be given with $(\gamma, \delta, \rho) \neq (0,0,0)$. Assume that A is a closed linear operator defined on a Banach space X. The following assertions are equivalent:*

(i) The equation

$$u''''(t) + \alpha u'''(t) + \beta u''(t) + \gamma Au''(t) + \delta Au'(t) + \rho Au(t) = f(t), \quad t \in [0, 2\pi]$$

is strongly $F^s_{p,q}$-well-posed;

(ii) $\mathbb{Z} \subset \rho_s(A)$ and $\sup_{k\in\mathbb{Z}} \|k^4 N_k\| < \infty$.

As it was pointed out for $B^s_{p,q}$-well-posedness, the problem (3.1) is strongly $F^s_{p,q}$-well-posed for all $1 \le p < \infty, 1 \le q \le \infty$, $s > 0$ if it is so for some $1 \le p < \infty, 1 \le q \le \infty$, $s > 0$.

5 Sufficient conditions: L^p–L^q-well-posedness

Based on the previous abstract results, we give in this section a practical criteria to widely solve the following Cauchy problem in L^p–L^q spaces with periodic boundary conditions:

$$\begin{cases} \partial_{tttt} u(x,t) + \alpha \partial_{ttt} u(x,t) + \beta \partial_{tt} u(x,t) + \gamma A_x \partial_{tt} u(x,t) + \delta A_x \partial_t u(x,t) + \rho A_x u(x,t) = f(x,t), \\ u(x,0) = u(x,2\pi),\ \partial_t u(x,0) = \partial_t u(x,2\pi),\ \partial_{tt} u(x,0) = \partial_{tt} u(x,2\pi),\ \partial_{ttt} u(x,0) = \partial_{ttt} u(x,2\pi), \end{cases} \tag{5.1}$$

where $x \in \Omega \subset \mathbb{R}^N$ and $t \in (0, 2\pi)$. We begin with some preliminaries on R-sectorial operators. Given any $\theta \in (0, \pi)$, we denote $\Sigma_\theta := \{z \in \mathbb{C} : |\arg(z)| < \theta, \ z \neq 0\}$. Recall that a closed operator $A : D(A) \subset X \to X$ with dense domain $D(A)$ is said to be R-sectorial of angle θ if the following conditions are satisfied:

(i) $\sigma(A) \subseteq \mathbb{C} \setminus \Sigma_\theta$;

(ii) The set $\{z(z-A)^{-1} : z \in \Sigma_\theta\}$ is R-bounded in $\mathcal{B}(X)$.

The permanence properties for R-sectorial operators are similar to those for sectorial operators. For instance, they behave well under perturbations. Sufficient conditions for R-sectoriality are studied in the monograph [22, Chapter 4]. As a consequence of our main theorem, we obtain the following remarkable result.

Theorem 5.1. *Assume that X is a UMD space, $1 < p < \infty$, $\alpha, \beta, \gamma, \delta, \rho \in (0, \infty)$ and let A be an R-sectorial operator on X of angle $\pi/2$. If $\rho + \beta\gamma < \alpha\delta$ then equation (5.1) is strongly L^p-well-posed.*

Proof. Define $d_k = \frac{(k^4 - \beta k^2) - i\alpha k^3}{(\gamma k^2 - \rho) - i\delta k}$ and we note that

$$\Re(d_k) = \frac{k^2[\gamma k^4 - k^2(\rho + \beta\gamma - \alpha\delta) + \rho\beta]}{(\gamma k^2 - \rho)^2 + \delta^2 k^2} > 0,$$

since $\rho + \beta\gamma < \alpha\delta$. Therefore $d_k \in \Sigma_{\pi/2}$. The R-sectoriality of angle $\pi/2$ of the operator A ensures the invertibility of $d_k I - A$ and the set $\{d_k(d_k - A)^{-1}\}_{k \in \mathbb{Z}}$ is R-bounded. Finally, we note the following identity

$$k^4 N_k = \frac{k^4}{(k^4 - \beta k^2) - i\alpha k^3} d_k (d_k - A)^{-1}, \qquad k \in \mathbb{Z},$$

which proves that the set $\{k^4 N_k\}_{k \in \mathbb{Z}}$ is R-bounded. By Theorem 3.6 we conclude that the problem (3.1) is strongly L^p-well-posed. □

Example 5.2. Let $1 < p < \infty$ and $\alpha, \beta, \gamma, \delta, \rho$ be strictly positive real numbers satisfying $\rho + \beta\gamma < \alpha\delta$. We consider the following equation in a bounded smooth domain $\Omega \subset \mathbb{R}^N$:

$$\begin{cases} [\partial_{tttt}u + \alpha\partial_{ttt}u + \beta\partial_{tt}u + \gamma\Delta\partial_{tt}u + \delta\Delta\partial_t u + \rho\Delta u](x,t) = f(x,t), \text{ for } (x,t) \in \Omega \times (0,2\pi); \\ u(x,t) = 0, \text{ for } (x,t) \in \partial\Omega \times (0,2\pi); \\ u(x,0) = u(x,2\pi), \partial_t u(x,0) = \partial_t u(x,2\pi), \partial_{tt}u(x,0) = \partial_{tt}u(x,2\pi), \partial_{ttt}u(x,0) = \partial_{ttt}u(x,2\pi), \end{cases} \tag{5.2}$$

where Δ denotes the Laplacian operator. By [26, Appendix] we have that the L^q realization Δ_q in $X = L^q(\Omega)$ of Δ is an R-sectorial operator in X with arbitrary angle $\theta \in (0, \pi)$, and that Δ_q coincides with Δ in the domain $D(\Delta_q)$ of Δ_q. Therefore, we can denote $(\Delta_q, D(\Delta_q))$ by $(\Delta, D_q(\Delta))$. Thus, Theorem 5.1 implies that for any given $f \in L^p(\mathbb{T}, L^q(\Omega))$ the solution u of the problem (5.2) written in abstract form as:

$$\begin{cases} [\partial_{tttt}u + \alpha\partial_{ttt}u + \beta\partial_{tt}u + \gamma\Delta\partial_{tt}u + \delta\Delta\partial_t u + \rho\Delta u](t) = f(t), \text{ for } t \in (0,2\pi); \\ u(x,0) = u(x,2\pi), \partial_t u(x,0) = \partial_t u(x,2\pi), \partial_{tt}u(x,0) = \partial_{tt}u(x,2\pi), \partial_{ttt}u(x,0) = \partial_{ttt}u(x,2\pi), \end{cases}$$

exists, is unique and belongs to the space $W^{4,p}_{per}(\mathbb{T}, L^q(\Omega)) \cap W^{2,p}_{per}(\mathbb{T}, [D(\Delta_q)])$. Moreover, for any $1 < p, q < \infty$ the estimate

$$\|u\|_{L^p(\mathbb{T},L^q(\Omega))} + \|u''\|_{W^{2,p}_{per}(\mathbb{T},L^q(\Omega))} + \|u'''\|_{W^{3,p}_{per}(\mathbb{T},L^q(\Omega))} + \|u''''\|_{W^{4,p}_{per}(\mathbb{T},L^q(\Omega))} \\ + \|Au\|_{L^p(\mathbb{T},[D(\Delta_q)])} + \|Au'\|_{W^{1,p}_{per}(\mathbb{T},[D(\Delta_q)])} + \|Au''\|_{W^{2,p}_{per}(\mathbb{T},[D(\Delta_q)])} \leq C\|f\|_{L^p(\mathbb{T},L^q(\Omega))}$$

holds.

We finish with the following example that considers the fractional Laplacian operator.

Example 5.3. Let $1 < p < \infty$, $\frac{1}{2} < s < 1$ and $\alpha, \beta, \gamma, \delta, \rho$ be strictly positive real numbers satisfying $\rho + \beta\gamma < \alpha\delta$. Consider the following nonlocal equation in a bounded smooth domain $\Omega \subset \mathbb{R}^N$:

$$\begin{cases} [\partial_{tttt}u + \alpha\partial_{ttt}u + \beta\partial_{tt}u \\ -\gamma(-\Delta)^s\partial_{tt}u - \delta(-\Delta)^s\partial_t u - \rho(-\Delta)^s u](x,t) = f(x,t), \text{ for } (x,t) \in \mathbb{R}^N \times (0,2\pi); \\ u(x,t) = 0, \text{ for } (x,t) \in \partial\Omega \times (0,2\pi); \\ u(x,0) = u(x,2\pi), \partial_t u(x,0) = \partial_t u(x,2\pi), \partial_{tt}u(x,0) = \partial_{tt}u(x,2\pi), \partial_{ttt}u(x,0) = \partial_{ttt}u(x,2\pi), \end{cases} \tag{5.3}$$

where the fractional Laplacian $-(-\Delta)^s$ is defined by

$$(-\Delta)^s v := \mathcal{F}^{-1}_{\xi}(\,|\xi|\,(\mathcal{F}v)(\xi)\,), \qquad v \in H^{1,q}(\Omega).$$

For $X = L^q(\Omega)$ and $D_q((-\Delta)^s) := H^{1,q}(\Omega)$, $1 < q < \infty$, the fractional operator $-(-\Delta)^s : H^{1,q}(\Omega) \to L^q(\Omega)$ is also R-sectorial of angle θ for an arbitrary $\theta \in (0, s\pi)$, see [1, Proposition 2.2]. Hence, by Theorem 5.1, for any $f \in L^p(\mathbb{T}, L^q(\Omega))$ there exists a unique solution $u \in W^{4,p}_{per}(\mathbb{T}, L^q(\Omega)) \cap W^{2,p}_{per}(\mathbb{T}, H^{1,q}(\Omega))$ of the problem (5.3) and satisfies the following maximal regularity estimate

$$\|u\|_{L^p(\mathbb{T},L^q(\Omega))} + \|u''\|_{W^{2,p}_{per}(\mathbb{T},L^q(\Omega))} + \|u'''\|_{W^{3,p}_{per}(\mathbb{T},L^q(\Omega))} + \|u''''\|_{W^{4,p}_{per}(\mathbb{T},L^q(\Omega))} \\ + \|Au\|_{L^p(\mathbb{T},H^{1,q}(\Omega))} + \|Au'\|_{W^{1,p}_{per}(\mathbb{T},H^{1,q}(\Omega))} + \|Au''\|_{W^{2,p}_{per}(\mathbb{T},H^{1,q}(\Omega))} \leq C\|f\|_{L^p(\mathbb{T},L^q(\Omega))}.$$

Analogous examples hold for the cases of the scales of Besov and Triebel–Lizorkin spaces, replacing R-sectorial operator by sectorial operator and R-boundedness by uniform boundedness. For instance, from Theorem 4.5 we obtain the following result.

Theorem 5.4. *Let X be a Banach space, $1 < p < \infty$, $\alpha, \beta, \gamma, \delta, \rho \in (0,\infty)$ and let A be a sectorial operator on X of angle $\pi/2$. If $\rho + \beta\gamma < \alpha\delta$ then equation (5.1) is strongly $B^s_{p,q}$-well-posed.*

Acknowledgements

The first author is partially supported by FONDECYT grant number 1180041 and DICYT, Universidad de Santiago de Chile, USACH. The second author is supported by MEC, grants MTM2016-75963-P and PID2019-105011GB-I00.

References

[1] G. Akrivis, B. Li, Maximum norm analysis of implicit–explicit backward difference formulas for nonlinear parabolic equations, *IMA J. Numer. Anal.* **38**(2018), No. 1, 75–101. https://doi.org/10.1093/imanum/drx008; MR3800015

[2] H. Amann, *Linear and quasilinear parabolic problems,* Monographs in Mathematics, Vol. 89, Basel, Birkhäuser Verlag, 1995. https://doi.org/10.1007/978-3-030-11763-4; MR3930629

[3] W. Arendt, Semigroups and evolution equations: functional calculus, regularity and kernel estimates, in: *Evolutionary equations*, Vol. 1, Handbook of Differential Equations, North-Holland, Amsterdam, 2004, pp. 1–85. MR2103696

[4] W. Arendt, S. Bu, The operator-valued Marcinkiewicz multiplier theorem and maximal regularity, *Math. Z.* **240**(2002), No. 2, 311–343. https://doi.org/10.1007/s002090100384; MR1900314

[5] W. Arendt, S. Bu, Operator-valued Fourier multipliers on periodic Besov spaces and applications, *Proc. Edinburgh Math. Soc. (2)* **47**(2004), 15–33. https://doi.org/10.1017/S0013091502000378; MR2064734

[6] S. K. Bose, G. C. Gorain, Exact controllability and boundary stabilization of flexural vibrations of an internally damped flexible space structure, *Appl. Math. Comput.* **126**(2002), No. 2–3, 341–360. https://doi.org/10.1016/S0096-3003(00)00112-0; MR1879167

[7] S. Bu, G. Cai, Periodic solutions of third-order degenerate differential equations in vector-valued functional spaces, *Israel J. Math.* **212**(2016), 163–188. https://doi.org/10.1007/s11856-016-1282-0; MR3504322

[8] S. Bu, G. Cai, Well-posedness of second-order degenerate differential equations with finite delay in vector-valued function spaces, *Pacific J. Math.* **288**(2017), No. 1, 27–46. https://doi.org/10.2140/pjm.2017.288.27; MR3667761

[9] S. Bu, G. Cai, Periodic solutions of third-order integro-differential equations in vector-valued functional spaces, *J. Evol. Equ.* **17**(2017), No. 2, 749–780. https://doi.org/10.1007/s00028-016-0335-5; MR3665228

[10] S. Bu, G. Cai, Periodic solutions of fractional degenerate differential equations with delay in Banach spaces, *Israel J. Math.* **232**(2019), No. 2, 695–717. https://doi.org/10.1007/s11856-019-1884-4; MR3990956

[11] S. Bu, G. Cai, Well-posedness of fractional integro-differential equations in vector-valued functional spaces, *Math. Nachr.* **292**(2019), No. 5, 969–982. https://doi.org/10.1002/mana.201800104; MR3953838

[12] S. Bu, G. Cai, Periodic solutions of second order degenerate differential equations with finite delay in Banach spaces, *J. Fourier Anal. Appl.* **25**(2019), No. 1, 32–50. https://doi.org/10.1007/s00041-017-9560-8; MR3901916

[13] S. Bu, G. Cai, Solutions of third order degenerate equations with infinite delay in Banach spaces, *Banach J. Math. Anal.* **14**(2020), No. 3, 1201–1221. https://doi.org/10.1007/s43037-020-00058-x; MR4123328

[14] S. Bu, J. Kim, Operator-valued Fourier multipliers on periodic Triebel spaces, *Acta Math. Sin. (Engl. Ser.)* **21**(2005), 1049–1056. https://doi.org/10.1007/s10114-004-0453-9; MR2176315

[15] D. L. Burkhölder, Martingales and Fourier analysis in Banach spaces, in: G. Letta, M. Pratelli (Eds.), *Probability and analysis (Varenna, 1985)*, Lecture Notes in Mathematics, Vol. 1206, Berlin, Springer, 1986, pp. 61–108. https://doi.org/10.1007/BFb0076300; MR0864712

[16] G. Cai, S. Bu, Periodic solutions of third-order degenerate differential equations in vector-valued functional spaces, *Israel J. Math.* **212**(2016), No. 1, 163–188. https://doi.org/10.1007/s11856-016-1282-0; MR3504322

[17] K. E. Chlouverakis, J. C. Sprott, Chaotic hyperjerk systems, *Chaos Solitons Fractals* **28**(2006), 739–746. https://doi.org/10.1016/j.chaos.2005.08.019; MR2204007

[18] Ph. Clément, S. O. Londen, G. Simonett, Quasilinear evolutionary equations and continuous interpolation spaces, *J. Differential Equations* **196**(2004), No. 2, 418–447. https://doi.org/10.1016/j.jde.2003.07.014; MR2028114

[19] J. A. Conejero, C. Lizama, M. Murillo-Arcila, J. B. Seoane-Sepúlveda, Well-posedness for degenerate third order equations with delay and applications to inverse problems, *Israel J. Math.* **229**(2019), 219–254. https://doi.org/10.1007/s11856-018-1796-8; MR3905603

[20] F. Dell'Oro, I. Lasiecka, V. Pata, The Moore–Gibson–Thompson equation with memory in the critical case, *J. Differential Equations* **261**(2016), 4188–4222. https://doi.org/10.1016/j.jde.2016.06.025; MR3532069

[21] F. Dell'Oro, V. Pata, On a fourth-order equation of Moore–Gibson–Thompson type, *Milan J. Math.* **85**(2017), 215–234. https://doi.org/10.1007/s00032-017-0270-0; MR3735562

[22] R. Denk, M. Hieber, J. Prüss, $\mathcal{R}$-boundedness, Fourier multipliers and problems of elliptic and parabolic type, *Mem. Amer. Math. Soc.* **166**(2003), No. 788. https://doi.org/10.1090/memo/0788; MR2006641

[23] D. Eager, A. M. Pendrill, N. Reistad, Beyond velocity and acceleration: jerk, snap and higher derivatives, *Eur. J. Phys.* **37**(2016), 065008, 11 pp. https://doi.org/10.1088/0143-0807/37/6/065008

[24] G. Figliolini, C. Lanni, Jerk and jounce relevance for the kinematic performance of long–dwell mechanisms, *Mech. Mach. Sci.* **73**(2019), 219–228. https://doi.org/10.1007/978-3-030-20131-9_22

[25] M. F. Hamilton, D. T. Blackstock, Nonlinear acoustics, Academic Press, New York, 1998.

[26] B. Jin, B. Li, Z. Zhou, Discrete maximal regularity of time-stepping schemes for fractional evolution equations. *Numer. Math.* **138**(2018), No. 1, 101–131. https://doi.org/10.1007/s00211-017-0904-8; MR3745012

[27] P. M. Jordan, Nonlinear acoustic phenomena in viscous thermally relaxing fluids: Shock bifurcation and the emergence of diffusive solitons, *J. Acoust. Soc. Am.* **124**(2008), No. 4, 2491–2491. https://doi.org/10.1121/1.4782790

[28] B. Kaltenbacher, I. Lasiecka, R. Marchand, Wellposedness and exponential decay rates for the Moore–Gibson–Thompson equation arising in high intensity ultrasound, *Control Cybernet.* **40**(2011), No. 4, 971–988. MR2977496

[29] B. Kaltenbacher, I. Lasiecka, M. Pospieszalska, Well-posedness and exponential decay of the energy in the nonlinear Moore–Gibson–Thompson equation arising in high intensity ultrasound, *Math. Models Meth. Appl. Sci.* **22**(2012), 1250035, 34 pp. https://doi.org/10.1142/S0218202512500352; MR2974173

[30] N. Kalton, L. Weis, The $\mathcal{H}^\infty$ calculus and sums of closed operators, *Math. Ann.* **321**(2001), 319–345. https://doi.org/10.1007/s002080100231; MR1866491

[31] V. Keyantuo, C. Lizama, Fourier multipliers and integro-differential equations in Banach spaces, *J. London Math. Soc. (2)* **69**(2004), 737–750. https://doi.org/10.1112/S0024610704005198; MR2050043

[32] I. Lasiecka, X. Wang, Moore–Gibson–Thompson equation with memory, part I: Exponential decay of energy, *Z. Angew. Math. Phys.* **67**(2016), Art. No. 17, 23 pp. https://doi.org/10.1007/s00033-015-0597-8; MR3480524

[33] I. Lasiecka, X. Wang, Moore–Gibson–Thompson equation with memory, part II: General decay of energy, *J. Differential Equations* **259**(2015), 7610–7635. https://doi.org/10.1016/j.jde.2015.08.052; MR3401607

[34] W. Liu, Z. Chen, Z. Tu, New general decay result for a fourth-order Moore–Gibson–Thompson equation with memory, *Electron. Res. Arch.* **28**(2020), No. 1, 433–457. https://doi.org/10.3934/era.2020025

[35] R. Marchand, T. Mcdevitt, R. Triggiani, An abstract semigroup approach to the third-order Moore–Gibson–Thompson partial differential equation arising in high-intensity ultrasound: structural decomposition, spectral analysis, exponential stability, *Math. Meth. Appl. Sci.* **35**(2012), 1896–1929. https://doi.org/10.1002/mma.1576; MR2982472

[36] V. Poblete, J. C. Pozo, Periodic solutions of an abstract third-order differential equation, *Studia Math.* **215**(2013), 195–219. https://doi.org/10.4064/sm215-3-1; MR3080779

[37] M. Visser, Jerk, snap and the cosmological equation of state, *Class. Quantum Gravity* **21**(2004), No. 11, 2603–2615. https://doi.org/10.1088/0264-9381/21/11/006