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Ground state sign-changing solutions and infinitely many solutions for fractional logarithmic Schrödinger equations in bounded domains

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Abstract. We consider a class of fractional logarithmic Schrödinger equation in bounded domains. First, by means of the constraint variational method, quantitative deformation lemma and some new inequalities, the positive ground state solutions and ground state sign-changing solutions are obtained. These inequalities are derived from the special properties of fractional logarithmic equations and are critical for us to obtain our main results. Moreover, we show that the energy of any sign-changing solution is strictly larger than twice the ground state energy. Finally, we obtain that the equation has infinitely many nontrivial solutions. Our result complements the existing ones to fractional Schrödinger problems when the nonlinearity is sign-changing and satisfies neither the monotonicity condition nor Ambrosetti–Rabinowitz condition.

Keywords: logarithmic Schrödinger equation, fractional Laplacian, sign-changing solutions, non-Nehari method, infinitely many solutions.

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1 Introduction

In this paper, we consider the following fractional Schrödinger equation with logarithmic nonlinearity:

$$\begin{cases} (-\Delta)^{\alpha}u + V(x)u = |u|^{p-2}u\ln u^2, & x\in\Omega, \\ u = 0, & x\in\mathbb{R}^N\setminus\Omega, \end{cases} \tag{1.1}$$

where $\alpha\in(0,1)$, $N>2\alpha$ and $2<p<2^*_\alpha := \frac{2N}{N-2\alpha}$, $(-\Delta)^\alpha$ denotes the fractional Laplacian operator, Ω is a bounded domain with smooth boundary in $\mathbb{R}^N$ and $V:\Omega\mapsto\mathbb{R}$ satisfy

(V_1) $V\in\mathcal{C}(\Omega,\mathbb{R})$.

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(V_2) $\inf \sigma((-\Delta)^\alpha + V(x)) > 0$, where $\sigma((-\Delta)^\alpha + V)$ is the spectrum of the operator $(-\Delta)^\alpha + V$.

The general form of problem (1.1) can be given by

$$(-\Delta)^\alpha u + V(x)u = f(x,u), \quad \text{in } \mathbb{R}^N, \tag{1.2}$$

which arises in the study of standing waves to the time-dependent Schrödinger equation

$$i\frac{\partial \psi}{\partial t} = (-\Delta)^\alpha \psi + M(x)\psi - F(x,\psi), \tag{1.3}$$

where $\psi : \mathbb{R}^N \times (0,+\infty) \mapsto \mathbb{R}$. This equation is of particular interest in fractional quantum mechanics for the study of particles on stochastic field modelled by Lévy processes. A path integral over the Lévy flights paths and a fractional Schrödinger equation of fractional quantum mechanics are formulated by Laskin [16] from the idea of Feynman and Hibbs path integrals. We call ψ a standing waves solution if it possesses the form $\psi(x,t) = e^{i\omega t}u(x)$. Then ψ is a standing waves solution for (1.3) if and only if u solves (1.2) with $V(x) = M(x) - \omega$. Our goal is to study the case for logarithmic nonlinearity $F(x,\psi) = |\psi|^{p-2}\psi \log |\psi|^2$. Here, the fractional Laplacian operator $(-\Delta)^\alpha$ can be characterized as the singular integral (see, for example [11])

$$(-\Delta)^\alpha u(x) = C(N,\alpha)\,\mathrm{P.V.}\int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2\alpha}}\,\mathrm{d}y, \tag{1.4}$$

for all $x \in \mathbb{R}^N$, where $C(N,\alpha)$ is a normalization constant and P.V. stands for the principal value. When u has sufficient regularity, the fractional Laplacian has a pointwise expression (see [11, Lemma 3.2])

$$(-\Delta)^\alpha u(x) = -\frac{1}{2}C(N,\alpha)\int_{\mathbb{R}^N} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{N+2\alpha}}\,\mathrm{d}y, \qquad \forall x \in \mathbb{R}^N.$$

Equation (1.1) and (1.2) admit applications related to quantum mechanics, phase transitions and minimal surfaces etc. (see [11] and the references therein). There are much attention by various scholars, especially on existence of ground state solution, multiple solutions, semiclassical states and the concentration behavior of positive solutions, see for example [3,9,20,24], and the references therein. When $\alpha = 1$, Chen et al. [5] proved the existence of ground state sign-changing solutions of problem (1.2) with $f(x,u) = Q(x)|u|^{p-2}u \ln u^2$. When $p = 2$, Pietro d'Avenia et al. [9] obtained the existence of infinitely many weak solutions of problem (1.1). If $\alpha = 1$ and $p = 2$, then the problem (1.1) reduces to the classical logarithmic Schrödinger equation

$$-\Delta u + V(x)u = u \ln u^2. \tag{1.5}$$

More recently, many scholars focused on the problem (1.5), such as the existence of ground state solution, multiple solutions, semiclassical states and the concentration behavior of positive solutions, see for example [1,2,8,18,25], and the references therein.

In 2014, Chang et al. [4] proved the existence of a nodal solution of (1.2) with $V(x) = 0$ in bounded domain. They assume that the nonlinearity $f(x,t)$ satisfies the following Ambrosetti–Rabinowitz condition and monotonicity condition:

(AR) There exists $\mu \in (2, 2^*_\alpha)$ such that

$$0 < \mu F(x,t) \le t f(x,t)$$

for a.e. $x \in \Omega$ and all $t \neq 0$, where $F(x,t) = \int_0^t f(x,\tau)d\tau$.

(NC) $t \mapsto f(x,t)/|t|$ is strictly increasing on $(-\infty,0)\cup(0,+\infty)$ for every $x \in \Omega$.

F. G. Rodrigo, et al. [13] considered the existence of sign-changing solution for (1.2) with $V(x)=0$ and $f(x,u)=\lambda g(x,u)+|u|^{2^*_\alpha}u$, where $g(x,u)$ satisfies the conditions (AR) and (NC). When $f(x,u)$ satisfies a monotonicity condition, Deng et al. [10] dealt with the least energy sign-changing solutions for fractional elliptic equations (1.2) in bounded domain. Ji [15] concerned with the existence of the least energy sign-changing solutions for a class of fractional Schrödinger–Poisson system when $f(x,t)$ satisfies the following monotonicity condition:

(F) $t \mapsto f(x,t)/t^3$ is strictly increasing on $(-\infty,0)\cup(0,+\infty)$ for every $x \in \mathbb{R}^3$.

For more discussions on the existence of sign-changing solutions, we refer the readers to other references, such as [6,7,14,22,23] and so on.

However, the logarithmic nonlinearity $f(x,u)=|u|^{p-2}u\ln u^2$ is sign-changing and satisfies neither the condition (AR) nor monotonicity condition (NC). In addition, the nonlocal operator brings some new difficulties, such as

$$\int_{\mathbb{R}^N}|(-\Delta)^{\frac{\alpha}{2}}u(x)|^2\mathrm{d}x \neq \int_{\mathbb{R}^N}|(-\Delta)^{\frac{\alpha}{2}}u^+(x)|^2\mathrm{d}x+\int_{\mathbb{R}^N}|(-\Delta)^{\frac{\alpha}{2}}u^-(x)|^2\mathrm{d}x,$$

where

$$u^+(x):=\max\{u(x),0\} \quad \text{and} \quad u^-(x):=\min\{u(x),0\}.$$

But, most methods for local problem heavily rely on the decompositions

$$\int_{\mathbb{R}^N}|\nabla u(x)|^2\mathrm{d}x=\int_{\mathbb{R}^N}|\nabla u^+(x)|^2\mathrm{d}x+\int_{\mathbb{R}^N}|\nabla u^-(x)|^2\mathrm{d}x.$$

Thus, these classic methods do not work for equation (1.1). Therefore, combining constraint variational method, quantitative deformation lemma, non-Nehari manifold method and some new energy inequalities, we will establish the existence of positive ground state solutions and ground state sign-changing solutions for (1.1). Finally, we analysis that the existence of infinitely many nontrivial solutions. To the best of our knowledge, there seem no results concerned with sign-changing solutions for fractional problem (1.1).

Before stating our main results, we introduce some useful results of fractional Sobolev spaces. For $0<\alpha<1$, the fractional Sobolev space is defined as

$$H_0^\alpha(\Omega):=\{u\in L^2(\Omega):[u]_\alpha<\infty, u=0 \ a.e. \text{ in } \mathbb{R}^N\setminus\Omega\},$$

where the Gagliardo seminorm $[u]_\alpha$ is given by

$$[u]_\alpha=\iint_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2\alpha}}\,\mathrm{d}x\,\mathrm{d}y.$$

It is well known that $H_0^\alpha(\Omega)$ is a Hilbert space endowed with the standard inner product

$$\langle u,v\rangle=\iint_{\mathbb{R}^{2N}}\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2\alpha}}\,\mathrm{d}x\,\mathrm{d}y+\int_\Omega u(x)v(x)\,\mathrm{d}x,$$

and the correspondent induced norm

$$\|u\|_{H_0^\alpha(\Omega)}=\sqrt{\langle u,u\rangle}. \tag{1.6}$$

In light of the Propositions 3.4 and 3.6 in [11], we have

$$\|(-\Delta)^{\frac{\alpha}{2}} u\|_2^2 = \frac{1}{2} C(n,\alpha) \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2\alpha}} \,\mathrm{d}x\,\mathrm{d}y,$$

where $\hat{u}$ stands for the Fourier transform of u, $\xi \in \mathbb{R}^N$ and $C(n,\alpha) = \left(\int_{\mathbb{R}^N} \frac{1-\cos \xi_1}{|\xi|^{n+2\alpha}} \,\mathrm{d}x\right)^{-1}$. As a consequence, the norms on $H^\alpha(\Omega)$ defined below

$$u \mapsto \left(\int_\Omega u(x)^2 \,\mathrm{d}x + \int_{\mathbb{R}^N} |(-\Delta)^{\frac{\alpha}{2}} u(x)|^2 \mathrm{d}x\right)^{\frac{1}{2}}$$

$$u \mapsto \left(\int_\Omega u(x)^2 \,\mathrm{d}x + \iint_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2\alpha}} \,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{2}}$$

are equivalent. To find solutions of (1.1), we will use a variational approach. Hence, we will associate a suitable functional to our problem. More precisely, the energy functional associated with problem (1.1) is given by $\Psi : H \mapsto \mathbb{R}$ defined as follows

$$\Psi(u) := \frac{1}{2} \int_{\mathbb{R}^N} \left(|(-\Delta)^{\frac{\alpha}{2}} u|^2 + V(x)u^2\right) \,\mathrm{d}x + \frac{2}{p^2} \int_\Omega |u|^p \,\mathrm{d}x - \frac{1}{p} \int_\Omega |u|^p \ln u^2 \,\mathrm{d}x. \tag{1.7}$$

We define the suitable subspace of $H_0^\alpha(\Omega)$,

$$H := \left\{u \in H_0^\alpha(\Omega) : \int_\Omega V(x)u^2 < +\infty\right\}.$$

In view of assumptions (V_1) and (V_2), it is not hard to check that H is a Hilbert space endowed with the inner product

$$\langle u, v\rangle_H = \int_{\mathbb{R}^N} (-\Delta)^{\frac{\alpha}{2}} u (-\Delta)^{\frac{\alpha}{2}} v \,\mathrm{d}x + \int_\Omega V(x)uv \,\mathrm{d}x,$$

and the induced norm $\|u\|^2 = \langle u, u\rangle_H$, which is equivalent to $\|u\|_{H_0^\alpha(\Omega)}$.

The basic property of Sobolev space H that we need is summarized in the following lemma.

Lemma 1.1 ([11])**.** *The embedding $H \hookrightarrow L^p(\Omega)$ is compact for $p \in (2, 2_\alpha^*)$.*

Note that

$$\lim_{t\to 0} \frac{t^{p-1} \ln t^2}{t} = 0 \quad \text{and} \quad \lim_{t\to\infty} \frac{t^{p-1} \ln t^2}{t^{q-1}} = 0,$$

where $q \in (p, 2_\alpha^*)$, thus, for any $\epsilon > 0$, there exists $C_\epsilon > 0$ such that

$$|t|^{p-1} |\ln t^2| \le \epsilon |t| + C_\epsilon |t|^{q-1}, \qquad \forall x \in \Omega,\ t \in \mathbb{R} \setminus \{0\}. \tag{1.8}$$

By (1.8) and a standard argument, it is easy to check that $\Psi \in C^1(H, \mathbb{R})$ and

$$\langle \Psi'(u), v\rangle = \int_{\mathbb{R}^N} (-\Delta)^{\frac{\alpha}{2}} u (-\Delta)^{\frac{\alpha}{2}} v \,\mathrm{d}x + \int_\Omega V(x)uv \,\mathrm{d}x - \int_\Omega |u|^{p-2} uv \ln u^2 \,\mathrm{d}x, \tag{1.9}$$

for any $u, v \in H$.

Definition 1.2. We say that $u \in H$ is a weak solution of (1.1), if u a critical point of the functional Ψ, that is

$$\int_{\mathbb{R}^N} (-\Delta)^{\frac{\alpha}{2}} u (-\Delta)^{\frac{\alpha}{2}} v \, \mathrm{d}x + \int_{\Omega} V(x) uv \, \mathrm{d}x = \int_{\Omega} |u|^{p-2} uv \ln u^2 \, \mathrm{d}x,$$

for all $v \in H$. Moreover, if $u \in H$ is a solution of (1.1) and $u^{\pm} \neq 0$, then u is called a sign-changing solution.

Definition 1.3. The $u \in H$ is called a classical solution of (1.1), if $(-\Delta)^{\alpha} u$ can be written as (1.4) and equation (1.1) is satisfied pointwise in Ω.

Remark 1.4. Since $(u^+, u^-)_{\alpha} := \int_{\mathbb{R}^N} (-\Delta)^{\frac{\alpha}{2}} u^+ (-\Delta)^{\frac{\alpha}{2}} u^- \, \mathrm{d}x > 0$ for $u^{\pm} \neq 0$, it follows from a simple computation that

$$\Psi(u) = \Psi(u^+) + \Psi(u^-) + (u^+, u^-)_{\alpha} > \Psi(u^+) + \Psi(u^-), \tag{1.10}$$

and

$$\langle \Psi'(u), u^{\pm} \rangle = \langle \Psi'(u^{\pm}), u^{\pm} \rangle + (u^+, u^-)_{\alpha} > \langle \Psi'(u^{\pm}), u^{\pm} \rangle. \tag{1.11}$$

Let

$$c := \inf_{u \in \mathcal{N}} \Psi(u) \quad \text{and} \quad m := \inf_{u \in \mathcal{M}} \Psi(u)$$

where

$$\mathcal{N} := \{u \in H \setminus \{0\} | \langle \Psi'(u), u \rangle = 0\},$$

and

$$\mathcal{M} := \{u \in H, u^{\pm} \neq 0 \mid \langle \Psi'(u), u^+ \rangle = \langle \Psi'(u), u^- \rangle = 0\}.$$

The main result of this work can now be stated as follows.

Theorem 1.5. *Assume that* (V_1) *and* (V_2) *hold. Then problem* (1.1) *possesses one positive ground state solution* $\bar{u} \in \mathcal{N}$ *such that* $\Psi(\bar{u}) = c := \inf_{\mathcal{N}} \Psi(u)$.

Theorem 1.6. *Assume that* (V_1) *and* (V_2) *hold. Then problem* (1.1) *has a ground state sign-changing solution* $\tilde{u} \in \mathcal{M}$ *such that* $\Psi(\tilde{u}) = m := \inf_{\mathcal{M}}$. *Moreover,* $m > 2c$.

Theorem 1.6 indicates that the energy of any sign-changing solution of (1.1) is strictly larger than twice of the ground state energy. In terms of the results, Theorem 1.6 is a relatively new result for fractional equations. In terms of processing technology, we adopt some new technique inequalities derived by the variable transformation and the special concave properties of energy functional.

Theorem 1.7. *Suppose that* (V_1) *and* (V_2) *hold. Then problem* (1.1) *possesses infinitely many nontrivial solutions.*

The remaining of the paper is organized as follows: In Section 2, we present some preliminary results and we set up the variational framework to our problem. In Section 3 and 4, we prove our main result. Throughout this paper, the symbol S denote unit sphere, the C, C_1, $C_2, \ldots$ represent several different positive constants.

2 Some preliminary results

In this section, we give some preliminary lemmas which are crucial for proving our results.

For a fixed function $u \in H$ with $u^{\pm} \neq 0$. We define a continuous function $J : [0,\infty) \times [0,\infty) \mapsto \mathbb{R}$ by

$$\begin{aligned} J(s,t) &:= \Psi\left(s^{\frac{1}{p}}u^+ + t^{\frac{1}{p}}u^-\right) \\ &= \frac{1}{2}\|s^{\frac{1}{p}}u^+ + t^{\frac{1}{p}}u^-\|^2 + \frac{2}{p^2}\int_\Omega |s^{\frac{1}{p}}u^+ + t^{\frac{1}{p}}u^-|^p \,\mathrm{d}x \\ &\quad - \frac{1}{p}\int_\Omega |s^{\frac{1}{p}}u^+ + t^{\frac{1}{p}}u^-|^p \ln\left(s^{\frac{1}{p}}u^+ + t^{\frac{1}{p}}u^-\right)^2 \mathrm{d}x. \end{aligned} \tag{2.1}$$

The following lemma is derived from the special properties of fractional logarithmic equations, which is critical to our results.

Lemma 2.1. *The $J(s,t)$ defined in (2.1) is strictly concave in $(0,+\infty)^2$ and thus there exists a unique global maximum point in $(0,+\infty)^2$.*

Proof. It follows from (2.1) that

$$\begin{aligned} \frac{\partial J}{\partial s}(s,t) &= \frac{1}{p}s^{\frac{2}{p}-1}\|u^+\|^2 + \frac{1}{p}s^{\frac{1}{p}-1}t^{\frac{1}{p}}(u^+,u^-)_\alpha - \frac{1}{p}\int_\Omega |u^+|^p \ln(u^+)^2 \,\mathrm{d}x \\ &\quad - \frac{1}{p}\int_\Omega |u^+|^p \ln(s^{\frac{2}{p}})\,\mathrm{d}x, \end{aligned} \tag{2.2}$$

$$\begin{aligned} \frac{\partial J}{\partial t}(s,t) &= \frac{1}{p}t^{\frac{2}{p}-1}\|u^-\|^2 + \frac{1}{p}t^{\frac{1}{p}-1}s^{\frac{1}{p}}(u^+,u^-)_\alpha - \frac{1}{p}\int_\Omega |u^-|^p \ln(u^-)^2 \,\mathrm{d}x \\ &\quad - \frac{1}{p}\int_\Omega |u^-|^p \ln(t^{\frac{2}{p}})\,\mathrm{d}x, \end{aligned} \tag{2.3}$$

$$\frac{\partial^2 J}{\partial s^2}(s,t) = \frac{2-p}{p^2}s^{\frac{2}{p}-2}\|u^+\|^2 + \frac{1-p}{p^2}s^{\frac{1}{p}-2}t^{\frac{1}{p}}(u^+,u^-)_\alpha - \frac{2}{p^2 s}\int_\Omega |u^+|^p\,\mathrm{d}x, \tag{2.4}$$

$$\frac{\partial^2 J}{\partial t^2}(s,t) = \frac{2-p}{p^2}t^{\frac{2}{p}-2}\|u^-\|^2 + \frac{1-p}{p^2}t^{\frac{1}{p}-2}s^{\frac{1}{p}}(u^+,u^-)_\alpha - \frac{2}{p^2 t}\int_\Omega |u^-|^p\,\mathrm{d}x \tag{2.5}$$

and

$$\frac{\partial^2 J}{\partial s\partial t}(s,t) = \frac{\partial^2 G}{\partial t\partial s}(s,t) = \frac{1}{p^2}s^{\frac{1}{p}-1}t^{\frac{1}{p}-1}(u^+,u^-)_\alpha. \tag{2.6}$$

Therefore, the Hessian matrix $D^2J(s,t)$ is

$$\begin{aligned} D^2J(s,t) &= \begin{pmatrix} \frac{\partial^2 J}{\partial s^2} & \frac{\partial^2 J}{\partial s\partial t} \\ \frac{\partial^2 J}{\partial t\partial s} & \frac{\partial^2 J}{\partial t^2} \end{pmatrix}(s,t) \\ &= \frac{2-p}{p^2}\begin{pmatrix} s^{\frac{2}{p}-2}\|u^+\|^2 + s^{\frac{1}{p}-2}t^{\frac{1}{p}}(u^+,u^-)_\alpha & 0 \\ 0 & t^{\frac{2}{p}-2}\|u^-\|^2 + t^{\frac{1}{p}-2}s^{\frac{1}{p}}(u^+,u^-)_\alpha \end{pmatrix} \\ &\quad + \frac{1}{p^2}(u^+,u^-)_\alpha \begin{pmatrix} -s^{\frac{1}{p}-2}t^{\frac{1}{p}} & s^{\frac{1}{p}-1}t^{\frac{1}{p}-1} \\ s^{\frac{1}{p}-1}t^{\frac{1}{p}-1} & -t^{\frac{1}{p}-2}s^{\frac{1}{p}} \end{pmatrix} \\ &\quad + \frac{2}{p^2}\begin{pmatrix} -\frac{1}{s}\int_\Omega |u^-|^p\,\mathrm{d}x & 0 \\ 0 & -\frac{1}{t}\int_\Omega |u^-|^p\,\mathrm{d}x \end{pmatrix} \\ &=: J_1(s,t) + J_2(s,t) + J_3(s,t). \end{aligned} \tag{2.7}$$

Note that $2 < p < 2^*_\alpha$ and $(u^+, u^-)_\alpha > 0$, it is not difficult to verify that $J_1(s,t)$, $J_2(s,t)$ and $J_3(s,t)$ are negative definite matrices for $s, t > 0$. Thus, $D^2 J(s,t)$ is a negative definite matrix. Since $J(0,0) = 0$ and

$$J(s,t) \to -\infty \quad \text{as } |(s,t)| \to +\infty,$$

which shows that $J(s,t)$ is strictly concave and there exists a unique global maximum point in $(0,+\infty)^2$. We complete the proof. □

In view of Lemma 2.1, we have the following corollaries.

Corollary 2.2. *Assume that $u \in \mathcal{M}$, then*

$$\Psi(u^+ + u^-) = \max_{\widetilde{s},\widetilde{t} \geq 0} \Psi(\widetilde{s}^{\frac{1}{p}} u^+ + \widetilde{t}^{\frac{1}{p}} u^-) > \Psi(s^{\frac{1}{p}} u^+ + t^{\frac{1}{p}} u^-), \tag{2.8}$$

for any $s, t \geq 0$ and $(s,t) \neq (1,1)$.

Proof. Let $J : [0,\infty) \times [0,\infty) \to \mathbb{R}$ be defined in (2.1). Since $u \in \mathcal{M}$, then $\langle \Psi'(u), u^+ \rangle = \langle \Psi'(u), u^- \rangle = 0$. This, combined with (2.2) and (2.3), implies that

$$\frac{\partial J}{\partial s}(1,1) = 0 \quad \text{and} \quad \frac{\partial J}{\partial t}(1,1) = 0.$$

Then, by the strict concavity of J in Lemma 2.1, (2.8) follows immediately, which is the desired conclusion. □

Since $\langle \Psi'(u), u^+ \rangle = p\frac{\partial J}{\partial s}(1,1)$ and $\langle \Psi'(u), u^- \rangle = p\frac{\partial J}{\partial t}(1,1)$, the following corollary can be directly derived from Lemma 2.1.

Corollary 2.3. *If $u \in H$ with $u^\pm \neq 0$, there exists a unique pair $(s_u, t_u) \in \mathbb{R}^+ \times \mathbb{R}^+$ such that $s_u^{\frac{1}{p}} u^+ + t_u^{\frac{1}{p}} u^- \in \mathcal{M}$.*

Corollary 2.4. *Assume that $u \in \mathcal{N}$, then*

$$\Psi(u) = \max_{t \geq 0} \Psi(t^{\frac{1}{p}} u) > \Psi(\widetilde{t}^{\frac{1}{p}} u), \tag{2.9}$$

for any $\widetilde{t} \geq 0$ and $\widetilde{t} \neq 1$.

Proof. By setting $s = t$ in (2.1), we can deduce similarly that

$$\widetilde{J}(t) = \Psi(t^{\frac{1}{p}} u)$$

is strictly concave in $(0,+\infty)$ and has a unique global maximum point. This, together with $u \in \mathcal{N}$, implies the desired conclusion. □

The following corollary directly follows from the Corollary 2.4 and [19, Proposition 8].

Corollary 2.5. *For any $u \in H \setminus \{0\}$, there exists a unique $t = t(u) > 0$ such that $tu \in \mathcal{N}$. Moreover, the map $\hat{\pi} : H \setminus \{0\} \mapsto \mathcal{N}$ is continuous for $\hat{\pi}(u) = t(u)u$ and $\pi := \hat{\pi}|_S$ defines a homeomorphism between the unit sphere S of H with $\mathcal{N}$.*

In view of Corollaries 2.2, 2.3, 2.4 and 2.5, we have the following results.

Lemma 2.6. *The following equalities hold true:*

$$\inf_{\mathcal{N}} \Psi(u) =: c = \inf_{u\in E, u\neq 0} \max_{t\geq 0} \Psi(t^{\frac{1}{p}} u)$$

and

$$\inf_{\mathcal{M}} \Psi(u) =: m = \inf_{u\in E, u^{\pm}\neq 0} \max_{s,t\geq 0} \Psi(s^{\frac{1}{p}} u^{+} + t^{\frac{1}{p}} u^{-}).$$

Proof. We only prove the second equality because the other case is similar. On the one hand, it follows from Corollary 2.2 that

$$\inf_{u\in E, u^{\pm}\neq 0} \max_{s,t\geq 0} \Psi(s^{\frac{1}{p}} u^{+} + t^{\frac{1}{p}} u^{-}) \leq \inf_{u\in\mathcal{M}} \max_{s,t\geq 0} \Psi(s^{\frac{1}{p}} u^{+} + t^{\frac{1}{p}} u^{-}) = \inf_{u\in\mathcal{M}} \Psi(u) = m. \tag{2.10}$$

On the other hand, for any $u \in H$ with $u^{\pm} \neq 0$, by Corollary 2.3, we have

$$\max_{s,t\geq 0} \Psi(s^{\frac{1}{p}} u^{+} + t^{\frac{1}{p}} u^{-}) \geq \Psi(s_u^{\frac{1}{p}} u^{+} + t_u^{\frac{1}{p}} u^{-}) \geq \inf_{v\in\mathcal{M}} \Psi(v) = m. \tag{2.11}$$

Thus, the conclution directly follows from (2.10) and (2.11). □

Proposition 2.7. *For any $u \in \mathcal{M}$, there exists $\varrho > 0$ such that $\|u^{\pm}\|_q \geq \varrho$.*

Proof. Since $u \subset \mathcal{M}$, we have $\langle \Psi'(u), u^{\pm}\rangle = 0$, that is

$$\int_{\mathbb{R}^N} (-\Delta)^{\frac{\alpha}{2}} u (-\Delta)^{\frac{\alpha}{2}} u^{\pm} \,\mathrm{d}x + \int_{\Omega} V(x)|u^{\pm}|^2 \,\mathrm{d}x = \int_{\Omega} |u^{\pm}|^p \ln |u^{\pm}|^2 \,\mathrm{d}x.$$

Then, by (1.8), $(u^{+}, u^{-})_{\alpha} > 0$ and the Sobolev inequality, we have

$$\begin{aligned} \|u^{\pm}\|^2 &\leq \int_{\Omega} |u^{\pm}|^p \ln (u^{\pm})^2 \,\mathrm{d}x \\ &\leq \frac{1}{2}\|u^{\pm}\|^2 + C_1 \|u^{\pm}\|^2 \|u^{\pm}\|_q^{q-2}, \end{aligned}$$

for some $C_1 > 0$ independent of u. Thus there exists a constant $\varrho > 0$ such that $\|u^{\pm}\|_q \geq \varrho$. □

Proposition 2.8. *For any $u \in \mathcal{N}$, there exists $\gamma > 0$ such that $\|u\|_q \geq \gamma$.*

Proof. By (1.8) and the Sobolev inequality, for any $u \in \mathcal{N}$, we deduce that

$$\begin{aligned} \|u\|^2 &= \int_{\Omega} |u|^p \ln u^2 \,\mathrm{d}x \\ &\leq \frac{1}{2}\|u\|^2 + C_2 \|u\|^2 \|u\|_q^{q-2}. \end{aligned}$$

for some $C_2 > 0$ independent of u. Then there exists $\gamma > 0$ such that $\|u\|_q \geq \gamma$. □

Lemma 2.9. *$c > 0$ and $m > 0$ can be achieved.*

Proof. We only prove that $m > 0$ and is achieved since the other case is similar. Let $\{u_n\} \in \mathcal{M}$ be such that $\Psi(u_n) \to m$. By (1.7) and (1.9), one has

$$\begin{aligned} m + o(1) &= \Psi(u_n) - \frac{1}{p}\langle \Psi'(u_n), u_n\rangle \\ &= \left(\frac{1}{2} - \frac{1}{p}\right) \|u_n\|^2 + \frac{2}{p^2} \int_{\Omega} |u_n|^p \,\mathrm{d}x \\ &\geq \left(\frac{1}{2} - \frac{1}{p}\right) \|u_n\|^2. \end{aligned}$$

This shows that $\{u_n\}$ is bounded. Thus, passing to a subsequence, we may assume that $u_n^{\pm} \rightharpoonup \hat{u}^{\pm}$ weakly in H and $u_n^{\pm} \to \hat{u}^{\pm}$ strongly in $L^s(\Omega)$ for $2 \le s < 2_\alpha^*$. Since $\{u_n\} \subset \mathcal{M}$, then it follows from Proposition 2.7 that there exists a constant $\varrho > 0$ such that $\|u_n^{\pm}\|_q \ge \varrho$. By the compactness of the embedding $H \hookrightarrow L^s(\Omega)$ for $2 \le s < 2_\alpha^*$, we have

$$\|\hat{u}^{\pm}\|_q = \lim_{n\to\infty} \|u_n^{\pm}\|_q \ge \varrho,$$

which shows $\hat{u}^{\pm} \neq 0$. By (1.8), (1.9), the Theorem A.2 in [21], the weak semicontinuity of norm and the Lebesgue dominated convergence theorem, we have

$$\begin{aligned}
\|\hat{u}^{\pm}\|^2 + \int_{\mathbb{R}^N} (-\Delta)^{\frac{\alpha}{2}} \hat{u}^{\mp} (-\Delta)^{\frac{\alpha}{2}} \hat{u}^{\pm}\, \mathrm{d}x &\le \liminf_{n\to\infty} \left(\|u_n^{\pm}\|^2 + \int_{\mathbb{R}^N} (-\Delta)^{\frac{\alpha}{2}} u_n^{\mp} (-\Delta)^{\frac{\alpha}{2}} u_n^{\pm}\, \mathrm{d}x \right) \\
&= \liminf_{n\to\infty} \int_\Omega |u_n^{\pm}|^p \ln(u_n^{\pm})^2\, \mathrm{d}x \\
&= \int_\Omega |\hat{u}^{\pm}|^p \ln(\hat{u}^{\pm})^2\, \mathrm{d}x,
\end{aligned} \tag{2.12}$$

which implies

$$\langle \Psi'(\hat{u}), \hat{u}^{\pm} \rangle \le 0. \tag{2.13}$$

According to Corollary 2.3, there exist $\hat{s}, \hat{t} > 0$ such that $\hat{s}^{\frac{1}{p}} \hat{u}^+ + \hat{t}^{\frac{1}{p}} \hat{u}^- \in \mathcal{M}$ and

$$\Psi(\hat{s}^{\frac{1}{p}} \hat{u}^+ + \hat{t}^{\frac{1}{p}} \hat{u}^-) \ge m. \tag{2.14}$$

By the concavity of $\hat{J}(s,t) := \Psi(s^{\frac{1}{p}} \hat{u}^+ + t^{\frac{1}{p}} \hat{u}^-)$ for $s, t \ge 0$ and the Taylor expansion, for some $\theta \in (0,1)$, we have

$$\begin{aligned}
\hat{J}(\hat{s}, \hat{t}) &= \hat{J}(1,1) + \hat{J}_s'(1,1)(\hat{s}-1) + \hat{J}_t'(1,1)(\hat{t}-1) \\
&\quad + \frac{1}{2!}((\hat{s}-1), (\hat{t}-1)) D^2 \hat{J}(1+\theta(\hat{s}-1), 1+\theta(\hat{t}-1))((\hat{s}-1), (\hat{t}-1))^{\mathrm{T}} \\
&\le \hat{J}(1,1) + \hat{J}_s'(1,1)(\hat{s}-1) + \hat{J}_t'(1,1)(\hat{t}-1).
\end{aligned} \tag{2.15}$$

That is

$$\Psi(\hat{u}) \ge \Psi(\hat{s}^{\frac{1}{p}} \hat{u}^+ + \hat{t}^{\frac{1}{p}} \hat{u}^-) - \frac{1}{p}(\hat{s}-1)\langle \Psi'(\hat{u}), \hat{u}^+ \rangle - \frac{1}{p}(\hat{t}-1)\langle \Psi'(\hat{u}), \hat{u}^- \rangle. \tag{2.16}$$

Therefore, it follows from (1.7), (1.9), (2.12), (2.13), (2.14), (2.16), Lemma 2.1, Corollary 2.2 and the weak semicontinuity of norm that

$$\begin{aligned}
m &= \lim_{n\to\infty} \left(\Psi(u_n) - \frac{1}{p} \langle \Psi'(u_n), u_n \rangle \right) \\
&= \lim_{n\to\infty} \left((\frac{1}{2} - \frac{1}{p}) \|u_n\|^2 + \frac{2}{p^2} \int_\Omega |u_n|^p\, \mathrm{d}x \right) \\
&\ge (\frac{1}{2} - \frac{1}{p}) \|\hat{u}\|^2 + \frac{2}{p^2} \int_\Omega |\hat{u}|^p\, \mathrm{d}x \\
&= \Psi(\hat{u}) - \frac{1}{p} \langle \Psi'(\hat{u}), \hat{u} \rangle \\
&\ge \Psi(\hat{s}^{\frac{1}{p}} \hat{u}^+ + \hat{t}^{\frac{1}{p}} \hat{u}^-) - \frac{\hat{s}}{p} \langle \Psi'(\hat{u}), \hat{u}^+ \rangle - \frac{\hat{t}}{p} \langle \Psi'(\hat{u}), \hat{u}^- \rangle \\
&\ge m,
\end{aligned}$$

which implies that

$$\langle \Psi'(\hat{u}), \hat{u}^{\pm} \rangle = 0 \quad \text{and} \quad \Psi(\hat{u}) = m. \tag{2.17}$$

Therefore, $\hat{u} \in \mathcal{M}$ and $\Psi(\hat{u}) = m$. Since $\hat{u}^{\pm} \neq 0$, then by (1.7), (1.9) and (2.17), we have

$$\begin{aligned} m = \Psi(\hat{u}) &= \frac{1}{2}\|\hat{u}\|^2 + \frac{2}{p^2}\int_\Omega |\hat{u}|^p \,\mathrm{d}x - \frac{1}{p}\int_\Omega |\hat{u}|^p \ln \hat{u}^2 \,\mathrm{d}x \\ &\geq \frac{1}{2}\|\hat{u}\|^2 - \frac{1}{p}\int_\Omega |\hat{u}|^p \ln \hat{u}^2 \,\mathrm{d}x \\ &= \left(\frac{1}{2} - \frac{1}{p}\right)\|\hat{u}^+\|^2 + \frac{1}{p}\langle \Psi'(\hat{u}), \hat{u} \rangle \\ &\geq \left(\frac{1}{2} - \frac{1}{p}\right)\|\hat{u}^+\|^2 + \left(\frac{1}{2} - \frac{1}{p}\right)\|\hat{u}^-\|^2 + \frac{1}{p}\langle \Psi'(\hat{u}), \hat{u}^+ \rangle + \frac{1}{p}\langle \Psi'(\hat{u}), \hat{u}^- \rangle \\ &> 0. \end{aligned}$$

That is $m > 0$. The proof is completed. □

Lemma 2.10. *The minimizers of* $\inf_{\mathcal{N}} \Psi(u)$ *and* $\inf_{\mathcal{M}} \Psi(u)$ *are critical points of* Ψ.

Proof. We prove it by contradiction. Assume that $\widetilde{u} \in \mathcal{M}$, $\Psi(\widetilde{u}) = m$ and $\Psi'(\widetilde{u}) \neq 0$. Then there exists $\delta > 0$, $\mu > 0$ such that $\|\Psi'(v)\| \geq \mu$, for $\|v - \widetilde{u}\| \leq 3\delta$. Let $D = (\frac{1}{2}, \frac{3}{2}) \times (\frac{1}{2}, \frac{3}{2})$. By Lemma 2.1, we have

$$\beta := \max_{s,t \in \partial D} \Psi\left(s^{\frac{1}{p}}\widetilde{u}^+ + t^{\frac{1}{p}}\widetilde{u}^-\right) < m. \tag{2.18}$$

Applying the classical deformation [21, Lemma 2.3] with $\varepsilon := \min\{(m-\beta)/3, \mu\delta/8\}$ and $S := B_\delta(\widetilde{u})$, there exists a deformation $\eta \in C([0,1] \times H, H)$ such that

(a) $\eta(1, u) = u$, if $u \notin \Psi^{-1}(m - 2\varepsilon, m + 2\varepsilon)$,

(b) $\eta(1, \Psi^{m+\varepsilon} \cap S) \subset \Psi^{m-\varepsilon}$,

(c) $\Psi(\eta(1, u)) \leq u$, $\forall u \in H$.

Corollary 2.2 implies that $\Psi(s^{\frac{1}{p}}\widetilde{u}^+ + t^{\frac{1}{p}}\widetilde{u}^-) \leq \Psi(\widetilde{u}) = m$, for $s > 0, t > 0$. Then it follows from (b) that

$$\Psi\left(\eta\left(1, s^{\frac{1}{p}}\widetilde{u}^+ + t^{\frac{1}{p}}\widetilde{u}^-\right)\right) \leq m - \varepsilon, \tag{2.19}$$

for $s > 0, t > 0$ and $|s-1|^2 + |t-1|^2 < \delta^2/\|\widetilde{u}\|^2$. Furthermore, using Lemma 2.1 and (c), we derive that

$$\Psi\left(\eta\left(1, s^{\frac{1}{p}}\widetilde{u}^+ + t^{\frac{1}{p}}\widetilde{u}^-\right)\right) \leq \Psi\left(s^{\frac{1}{p}}\widetilde{u}^+ + t^{\frac{1}{p}}\widetilde{u}^-\right) < \Psi(\widetilde{u}) = m, \tag{2.20}$$

for $s > 0, t > 0$ and $|s-1|^2 + |t-1|^2 \geq \delta^2/\|\widetilde{u}\|^2$. Thus, from (2.19) and (2.20), we obtain

$$\max_{s,t \in \overline{D}} \Psi\left(\eta\left(1, s^{\frac{1}{p}}\widetilde{u}^+ + t^{\frac{1}{p}}\widetilde{u}^-\right)\right) < m.$$

Define $g(s,t) = s^{\frac{1}{p}}\widetilde{u}^+ + t^{\frac{1}{p}}\widetilde{u}^-$. To complete the proof it suffices to prove that

$$\eta(1, g(D)) \cap \mathcal{M} \neq \emptyset, \tag{2.21}$$

which implies $\max_{s,t\in\overline{D}}\Psi(\eta(1,s^{\frac{1}{p}}\widetilde{u}^{+}+t^{\frac{1}{p}}\widetilde{u}^{-}))\geq m$ and it contradicts (2.21). Let us define $\kappa(s,t):=\eta(1,g(s,t))$ and

$$\phi(s,t):=\left(\frac{1}{ps}\langle\Psi'(\kappa(s,t)),(\kappa(s,t))^{+}\rangle,\frac{1}{pt}\langle\Psi'(\kappa(s,t)),(\kappa(s,t))^{-}\rangle\right).$$

Since $\kappa(s,t)|_{\partial D}=g(s,t)$, we have

$$\frac{1}{ps}\langle\Psi'(g(s,t)),s^{\frac{1}{p}}u^{+}\rangle=J_s'(s,t),\quad \text{on } \partial D,$$

and

$$\frac{1}{pt}\langle\Psi'(g(s,t)),t^{\frac{1}{p}}u^{+}\rangle=J_t'(s,t),\quad \text{on } \partial D.$$

Therefore, by the homotopy invariance of Brouwer's degree, we can deduce from (2.7) that

$$\begin{aligned}\deg(\phi,D,(0,0))&=\deg((J_s',J_t'),D,(0,0))\\&=\operatorname{sgn}\left(\det\begin{pmatrix}J_s'\\J_t'\end{pmatrix}(1,1)\right)=1,\end{aligned}$$

which implies that $\phi(\bar{s},\bar{t})=0$ for some $(\bar{s},\bar{t})\in D$, that is $\kappa(\bar{s},\bar{t})=\eta(1,g(\bar{s},\bar{t}))\in\mathcal{M}$, which is a contradiction.

The proof of $\inf_{\mathcal{N}}\Psi(u)$ is critical points of Ψ is similar to above argument and hence is omitted here. □

3 Proof of Theorems 1.5 and 1.6

We first prove Theorem 1.5. According to 2.9 and 2.10, there exists $\bar{u}\in\mathcal{N}$ such that $\Psi(\bar{u})=c$ and $\Psi'(\bar{u})=0$. Now, we only need to prove that u is a positive solution of problem (1.1). Indeed, replacing $\Psi(u)$ with the functional

$$\Psi^{+}(u):=\frac{1}{2}\int_{\mathbb{R}^N}\left(|(-\Delta)^{\frac{\alpha}{2}}u|^2+V(x)u^2\right)\,\mathrm{d}x+\frac{2}{p^2}\int_{\Omega}|(u^{+}|^{p}\,\mathrm{d}x-\frac{1}{p}\int_{\Omega}|u^{+}|^{p}\ln(u^{+})^{2}\,\mathrm{d}x.$$

In this way we can get a solution u such that

$$(-\Delta)^{\alpha}u+V(x)u=|u^{+}|^{p-2}u^{+}\ln(u^{+})^{2}\quad\text{in }\Omega.\tag{3.1}$$

Testing equation (3.1) with u^{-}, we obtain

$$\int_{\mathbb{R}^N}(-\Delta)^{\frac{\alpha}{2}}u(-\Delta)^{\frac{\alpha}{2}}u^{-}\,\mathrm{d}x+\int_{\Omega}V(x)|u^{-}|^{2}\,\mathrm{d}x=0.\tag{3.2}$$

On the other hand,

$$\begin{aligned}\int_{\mathbb{R}^N}(-\Delta)^{\frac{\alpha}{2}}u(-\Delta)^{\frac{\alpha}{2}}u^{-}\,\mathrm{d}x&=\int_{\mathbb{R}^N}|(-\Delta)^{\frac{\alpha}{2}}u^{-}|^{2}\,\mathrm{d}x+\int_{\mathbb{R}^N}(-\Delta)^{\frac{\alpha}{2}}u^{+}(-\Delta)^{\frac{\alpha}{2}}u^{-}\,\mathrm{d}x\\&=\int_{\mathbb{R}^N}|(-\Delta)^{\frac{\alpha}{2}}u^{-}|^{2}\,\mathrm{d}x+(u^{+},u^{-})_{\alpha}\geq 0.\end{aligned}\tag{3.3}$$

Thus, it follows from (3.2) and (3.3), we have $u^{-}=0$ and $u\geq 0$. Since $|t|^{p-1}|\ln t^{2}|\leq|t|+C_q|t|^{q-1},\forall x\in\Omega,t\in\mathbb{R}\setminus\{0\}$, for $q\in(2,2_{\alpha}^{*})$, by the regularity theorem [12, Lemma 3.4], we can obtain that $u \in C^{0,\mu}$ for some $\mu \in (0,1)$. Therefore, using the maximum principle [17, Proposition 2.17], we obtain $u \equiv 0$ in Ω, a contradiction. Thus, u is a positive solution of problem (1.1).

Finally, we prove Theorem 1.6. We conclude from Lemma 2.9 and Lemma 2.10 that problem (1.1) has a sign-changing solution $\tilde{u} \in \mathcal{M}$ such that $\Psi(\tilde{u}) = m$ and $\Psi'(\tilde{u}) = 0$. It remains to prove that $\Psi(\tilde{u}) = m := \inf_{\mathcal{M}} \Psi(u) > 2c$. Indeed, by (1.10), Corollary 2.2 and Lemma 2.6, we have

$$\begin{aligned} m = \Psi(\tilde{u}) &= \max_{s,t\geq 0} \Psi(s^{\frac{1}{p}}\tilde{u}^+ + t^{\frac{1}{p}}\tilde{u}^-) \\ &> \max_{s\geq 0} \Psi(s^{\frac{1}{p}}\tilde{u}^+) + \max_{t\geq 0} \Psi(t^{\frac{1}{p}}\tilde{u}^-) \geq 2c. \end{aligned}$$

The proof is completed.

4 Infinitely many solutions

In the following, we analysis the existence of infinitely many nontrivial solutions for problem (1.1).

Define $\hat{\varphi} : H \mapsto \mathbb{R}$ and $\varphi : S \mapsto \mathbb{R}$ by $\hat{\varphi}(u) = \Psi(\hat{\pi}(u))$ and $\varphi := \hat{\varphi}|_S$, respectively. Clearly, $\hat{\varphi}$ and φ are even since Ψ is even. It is not difficult to verify that φ is bounded from below in S and φ satisfies the Palais–Smale condition on S. Hence, arguing as [19], the functional Ψ has infinitely many critical points, which shows that (1.1) has infinitely many nontrivial solutions. The Theorem 1.7 is proved.

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