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Hyers–Ulam stability for a partial difference equation

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Abstract. Under the exponential trichotomy condition we study the Hyers–Ulam stability for the linear partial difference equation:

$$x_{n+1,m} = A_n x_{n,m} + B_{n,m} x_{n,m+1} + f(x_{n,m}), \qquad n, m \in \mathbb{Z}$$

where A_n is a $k \times k$ matrix whose elements are sequences of n, $B_{n,m}$ is a $k \times k$ matrix whose elements are double sequences of m, n and $f : \mathbb{R}^k \to \mathbb{R}^k$ is a vector function. We also investigate the Hyers–Ulam stability in the case where the matrices $A_n, B_{n,m}$ and the vector function $f = f_{n,m}$ are constant.

Keywords: partial difference equations, Hyers–Ulam stability, exponential dichotomy.
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1 Introduction

Partial difference equations is an area which deals with difference equations with several variables. Some classical results in the area can be found, for example, in books [4, 9, 12, 14]. Despite the fact that the study of partial difference equations is pretty much complicated, both theoretically and technically, there are some investigations on solvability, stability and other topics related to the equations (see, for example, [5–7, 10, 13, 15, 25, 29, 31, 34, 40, 41] and the related references therein). Many partial difference equations are obtained from some problems in combinatorics, probability, discrete mathematics and other related areas of mathematics and science (see, for example, [11, 22, 43]).

In [8] the authors studied the so-called μ-exponentially weighted shadowing property of the equation

$$x_{m+1} = L_m x_m + f_m(x_m), \qquad m \in \mathbb{Z},$$

where L_m is a sequence of linear operators, f_m is a sequence of nonlinear operators $m \in \mathbb{Z}$ assuming that the linear equation

$$x_{m+1} = L_m x_m$$

has an exponential dichotomy and the sequence f_m is uniformly Lipschitz continuous.

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Inspired by the above work, as well as some applications of solvability methods for difference equations, here we investigate Hyers–Ulam stability for the nonhomogenous linear partial difference equation of the form:

$$x_{n+1,m} = A_n x_{n,m} + B_{n,m} x_{n,m+1} + f(x_{n,m}), \qquad n, m \in \mathbb{Z}, \tag{1.1}$$

where A_n is a $k \times k$ invertible matrix whose elements are sequences of n, $B_{n,m}$ is a $k \times k$ matrix whose elements are double sequences of m, n and $f : \mathbb{R}^k \to \mathbb{R}^k$ is a vector function.

In what follows we denote by $|\cdot|$ any convenient norm either of a vector or of a matrix. We say that the linear difference equation

$$x_{v+1} = C_v x_v, \qquad v \in \mathbb{Z}, \ldots \tag{1.2}$$

where C_v is an invertible matrix has an exponential trichotomy (see [16, 17]) if there exist constants $K > 0$, $0 < p < 1$ and projections P_1, P_2, P_3 ($P_i^2 = P_i, i = 1, 2, 3$), $P_1 + P_2 + P_3 = 1$ such that

$$\begin{aligned}
&|X_v P_1 X_s^{-1}| \leq K p^{v-s}, && v \geq s, \ s, v \in \mathbb{Z} \\
&|X_v P_2 X_s^{-1}| \leq K p^{s-v}, && s \geq v, \ s, v \in \mathbb{Z} \\
&|X_v P_3 X_s^{-1}| \leq K p^{v-s}, && v \geq s \geq 0 \\
&|X_v P_3 X_s^{-1}| \leq K p^{s-v}, && 0 \geq s \geq v
\end{aligned} \tag{1.3}$$

where X_v is a fundamental matrix solution of (1.2) given by

$$X_v = \begin{cases} \left(\prod_{s=0}^{v-1} C_{v-s-1} \right) C, & v \geq 0 \\ \left(\prod_{s=v}^{-1} C_s^{-1} \right) C, & v \leq 0, \end{cases}$$

and C is a constant matrix. We regard that $X_0 = C$.

For the readers' convenience we give a simple example concerning exponential trichotomy for a linear difference equation. Consider equation (1.2) where

$$C_v = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & c_v \end{bmatrix}, \qquad c_v = \begin{cases} 1/2, & v \geq 0 \\ 2, & v < 0. \end{cases}$$

Then if we take $C = I_3$, I_3 the 3×3 indentity matrix, we get

$$X_v = \begin{bmatrix} (1/2)^v & 0 & 0 \\ 0 & 2^v & 0 \\ 0 & 0 & d_v \end{bmatrix}, \qquad d_v = \begin{cases} (1/2)^v, & v \geq 0 \\ 2^v, & v \leq 0. \end{cases}$$

If we take the projections

$$P_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad P_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad P_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

we have $P_1 + P_2 + P_3$ and (1.3) hold with $K = 1$ and $p = 1/2$.

Moreover, we give some details concerning the form of operator T given in Proposition 2.2. In Proposition 1 of [16] the author proved that if equation (1.2) has an exponential trichotomy (1.3) then the inhomogenous ordinary difference equation

$$x_{v+1} = A_v x_v + f_v, \qquad v \in \mathbb{Z},$$

$f_v : \mathbb{Z} \to \mathbb{R}^k$, $|f_v| \le M, v \in \mathbb{Z}$ where M is a positive constant, has at least bounded solution y_v given by

$$\begin{aligned} y_v &= \sum_{s=-\infty}^{-1} X_v P_1 X_{s+1}^{-1} f_s + \sum_{s=0}^{v-1} X_v (I - P_2) X_{s+1}^{-1} f_s - \sum_{s=v}^{\infty} X_v P_2 X_{s+1}^{-1} f_s, \qquad v \ge 0, \\ y_v &= \sum_{s=-\infty}^{v-1} X_v P_1 X_{s+1}^{-1} f_s - \sum_{s=v}^{-1} X_v (I - P_1) X_{s+1}^{-1} f_s - \sum_{s=0}^{\infty} X_v P_2 X_{s+1}^{-1} f_s, \qquad v \le 0. \end{aligned} \tag{1.4}$$

According to [21] we say that (1.1) has the Hyers–Ulam stability if for any $\epsilon > 0$ there exists a $\delta > 0$ such that if $y_{n,m}$ satisfies either

$$|y_{n+1,m} - A_n y_{n,m} - B_{n,m} y_{n,m+1} - f(y_{n,m})| < \delta \tag{1.5}$$

or

$$|y_{n,m+1} + B_{n,m}^{-1} A_n y_{n,m} - B_{n,m}^{-1} y_{n+1,m} + B_{n,m}^{-1} f(y_{n,m})| < \delta \tag{1.6}$$

then there exists a solution $x_{n,m}$ of (1.1) such that

$$|x_{n,m} - y_{n,m}| < \epsilon, \qquad n, m \in \mathbb{Z}. \tag{1.7}$$

Now in this paper assuming that equation (1.2) where $C_n = A_n$ has an exponential trichotomy then, under some assumptions on the matrices $A_n, B_{n,m}$ and the function f, we prove that (1.1) has the Hyers–Ulam stability. In addition, if $B_{n,m} = A_n D_m$, A_n, D_m are invertible matrices and the equation (1.2) where $C_m = -D_m^{-1}$ has an exponential trichotomy, then, under some assumptions on the matrices A_n, D_m and the function f, we prove that equation (1.1) has also the Hyers–Ulam stability. Finally we study the Hyers–Ulam stability in the case where the matrices $A_n, B_{n,m}$ are constants, that is $A_n = A, B_{n,m} = B$ and the function $f = f_{n,m}$ is independent on x that is $f_{n,m} : \mathbb{N} \times \mathbb{N} \to \mathbb{R}^k$.

Roughly speaking the stability of Hyers–Ulam means that for any approximate solution of equation (1.1) there exists a solution of (1.1) which is near the approximate solution. Since this is very important there exists an increasing interest in studying this stability. Therefore there are many papers which deal with this subject (see [1,3,8,21] and the related references therein).

In what follows we denote

$$l^\infty = l^\infty(\mathbb{Z}^2)$$

the space of all double sequences $(z_{n,m}) \subset \mathbb{R}^k$ which are bounded.

In the study we will essentially use a method related to the solvability of the linear difference equation of first order, by which the studied difference equations are transformed to some difference equations of 'integral' type, for which it is easier to apply methods from nonlinear functional analysis. Here it is applied the contraction principle. It should be mentioned that recently appeared many papers on difference equations and systems of difference equations which have been solved by transforming them to some linear solvable ones (see, for example, [2,20,26–28,30,32,33,35–39,42] and the related references therein).

Finally it should be mentioned that, there is a plenty of papers dealing with solvability or invariants for difference equations (see, for example, [18,19,23,24,28,30,35,36,38,39]).

2 Main results

Firstly we give a proposition which concerns the existence and uniqueness of the solutions of (1.1).

Proposition 2.1.

(i) *For a given sequence c_m there exists a unique solution $x_{n,m}$ of* (1.1) *such that $x_{0,m} = c_m$, $m \in \mathbb{Z}$. Moreover, $x_{n,m}$ satisfies the following relations*

$$x_{n,m} = \begin{cases} X_n X_0^{-1} c_m + \sum_{s=0}^{n-1} X_n X_{s+1}^{-1} \left(B_{s,m} x_{s,m+1} + f(x_{s,m}) \right), & n \geq 0,\ m \in \mathbb{Z} \\ X_n X_0^{-1} c_m - \sum_{s=n}^{-1} X_n X_{s+1}^{-1} \left(B_{s,m} x_{s,m+1} + f(x_{s,m}) \right), & n \leq 0,\ m \in \mathbb{Z} \end{cases} \tag{2.1}$$

where X_n is a fundamental matrix solution of (1.2) *with $C_n = A_n$.*

(ii) *Suppose that $B_{n,m} = A_n D_m$ and A_n, B_m are invertible matrices. Then there exists a unique solution of* (1.1) *such that $x_{n,0} = d_n$, $n \in \mathbb{Z}$ where d_n is given sequence. In addition if*

$$R(x_{n,s}) = D_s^{-1} A_n^{-1} x_{n+1,s} - D_s^{-1} A_n^{-1} f(x_{n,s}),$$

$x_{n,m}$ satisfies the following equalities

$$x_{n,m} = \begin{cases} X_m X_0^{-1} d_n + \sum_{s=0}^{m-1} X_m X_{s+1}^{-1} R(x_{n,s}), & m \geq 0,\ n \in \mathbb{Z} \\ X_m X_0^{-1} d_n - \sum_{s=m}^{-1} X_m X_{s+1}^{-1} R(x_{n,s}), & m \leq 0,\ n \in \mathbb{Z} \end{cases} \tag{2.2}$$

where X_m is a fundamental matrix solution of (1.2) with $C_m = -D_m^{-1}$.

From (1.1) and using the constant variation formula for a fixed m we can prove (2.1). Since from (1.1) we have

$$\begin{aligned} x_{n,m+1} &= -B_{n,m}^{-1} A_n x_{n,m} + B_{n,m}^{-1} x_{n+1,m} - B_{n,m}^{-1} f(x_{n,m}) \\ &= -D_m^{-1} x_{n,m} + D_m^{-1} A_n^{-1} x_{n+1,m} - D_m^{-1} A_n^{-1} f(x_{n,m}), \end{aligned} \tag{2.3}$$

using the constant variation formula for a fixed n we can easily get (2.2).

We prove the Hyers–Ulam stability in the case where equation (1.2) with $C_n = A_n$ or $C_m = -D_m^{-1}$, $B_{n,m} = A_n D_m$ has an exponential trichotomy.

Proposition 2.2. *The following statements are true:*

(i) *Suppose that* (1.2) *with $C_n = A_n$ has an exponential trichotomy* (1.3)*, that there exists a positive number M such that*

$$|B_{n,m}| \leq M, \qquad n, m \in \mathbb{Z}, \tag{2.4}$$

and that $f : \mathbb{R}^k \to \mathbb{R}^k$ is a vector function such that for all $x, y \in \mathbb{R}^k$

$$|f(x) - f(y)| \leq L|x - y|, \tag{2.5}$$

where L is a positive constant. Then if

$$(M + L)\frac{2K(p+1)}{1-p} < 1 \tag{2.6}$$

equation (1.1) *has the Hyers–Ulam stability.*

(ii) *Suppose that $B_{n,m} = A_n D_m$, $n, m \in \mathbb{Z}$ where A_n, D_m are invertible matrices for any $m \in \mathbb{Z}$, that equation (1.2) where $C_m = -D_m^{-1}$ has an exponential trichotomy (1.3), that there exists a positive number M such that*

$$|D_m^{-1} A_n^{-1}| \leq M, \qquad n, m \in \mathbb{Z}. \tag{2.7}$$

and that (2.5) is true. Then if

$$M(1+L)\frac{2K(p+1)}{1-p} < 1, \tag{2.8}$$

equation (1.1) has the Hyers–Ulam stability.

Proof. **(i)** Let ϵ be an arbitrary positive number and δ be a positive number such that

$$\delta < \frac{1 - p - 2K(1+p)(M+L)}{2K(1+p)}\epsilon. \tag{2.9}$$

Suppose that $y_{n,m}$ is a double sequence such that (1.5) is satisfied. Let

$$H(z_{n,m}) = -y_{n+1,m} + A_n y_{n,m} + B_{n,m}(y_{n,m+1} + z_{n,m+1}) + f(y_{n,m} + z_{n,m}). \tag{2.10}$$

Inspired by (1.4) we define the operator T on l^∞ as follows: If $z_{n,m} \in l^\infty$ then we set

$$\begin{aligned} Tz_{n,m} = & \sum_{s=-\infty}^{-1} X_n P_1 X_{s+1}^{-1} H(z_{s,m}) + \sum_{s=0}^{n-1} X_n (I - P_2) X_{s+1}^{-1} H(z_{s,m}) \\ & - \sum_{s=n}^{\infty} X_n P_2 X_{s+1}^{-1} H(z_{s,m}), \qquad n \geq 0,\ m \in \mathbb{Z}. \\ Tz_{n,m} = & \sum_{s=-\infty}^{n-1} X_n P_1 X_{s+1}^{-1} H(z_{s,m}) - \sum_{s=n}^{-1} X_n (I - P_1) X_{s+1}^{-1} H(z_{s,m}) \\ & - \sum_{s=0}^{\infty} X_n P_2 X_{s+1}^{-1} H(z_{s,m}), \qquad n \leq 0,\ m \in \mathbb{Z}. \end{aligned} \tag{2.11}$$

We prove that $T(l^\infty) \subseteq l^\infty$. Let

$$|z|_\infty = \sup\{|z_{n,m}|,\ n, m, \in \mathbb{Z}\}.$$

From (2.10) we obtain

$$\begin{aligned} H(z_{n,m}) = & -y_{n+1,m} + A_n y_{n,m} + B_{n,m} y_{n,m+1} + f(y_{n,m}) \\ & + B_{n,m} z_{n,m+1} + f(y_{n,m} + z_{n,m}) - f(y_{n,m}). \end{aligned} \tag{2.12}$$

Then from (1.5), (2.4), (2.5) and (2.12) we have

$$|H(z_{n,m})| \leq \delta + (M+L)|z|_\infty. \tag{2.13}$$

Therefore from (1.3), (2.13) and since $I - P_2 = P_1 + P_3$ for $n \geq 0$, $m \in \mathbb{Z}$ we get

$$\begin{aligned} |Tz_{n,m}| & \leq \left(\sum_{s=-\infty}^{-1} Kp^{n-s-1} + 2\sum_{s=0}^{n-1} Kp^{n-s-1} + \sum_{s=n}^{\infty} Kp^{-n+s+1} \right)(\delta + (M+L)|z|_\infty) \\ & \leq \frac{2K(1+p)}{1-p}\Big(\delta + (M+L)|z|_\infty\Big). \end{aligned} \tag{2.14}$$

Furthermore from (1.3), (2.13) and since $I - P_1 = P_2 + P_3$ for $n \leq 0$, $m \in \mathbb{Z}$ we have

$$\begin{aligned} |Tz_{n,m}| &\leq \left(\sum_{s=-\infty}^{n-1} Kp^{n-s-1} + 2\sum_{s=n}^{-1} Kp^{-n+s+1} + \sum_{s=0}^{\infty} Kp^{-n+s+1} \right) (\delta + (M+L)|z|_\infty) \\ &\leq \frac{2K(1+p)}{1-p} \Big(\delta + (M+L)|z|_\infty \Big). \end{aligned} \tag{2.15}$$

Relations (2.14) and (2.15) imply that $T(l^\infty) \subseteq l^\infty$. We prove now that T is a contraction on the space S. Let $z_{n,m}, w_{n,m} \in l^\infty$. Using (2.10) we get for $n, m \in \mathbb{Z}$

$$H(z_{n,m}) - H(w_{n,m}) = B_{n,m}(z_{n,m+1} - w_{n,m+1}) + f(y_{n,m} + z_{n,m}) - f(y_{n,m} + w_{n,m})$$

and so from (2.4), (2.5) we have

$$|H(z_{n,m}) - H(w_{n,m})| \leq (M+L)|z-w|_\infty, \qquad n, m \in \mathbb{Z}. \tag{2.16}$$

From (2.11) we have for $n \geq 0$, $m \in \mathbb{Z}$

$$\begin{aligned} Tz_{n,m} - Tw_{n,m} = &\sum_{s=-\infty}^{-1} X_n P_1 X_{s+1}^{-1} \big(H(z_{s,m}) - H(w_{s,m}) \big) + \sum_{s=0}^{n-1} X_n (I - P_2) X_{s+1}^{-1} \big(H(z_{s,m}) - H(w_{s,m}) \big) \\ &- \sum_{s=n}^{\infty} X_n P_2 X_{s+1}^{-1} \big(H(z_{s,m}) - H(w_{s,m}) \big). \end{aligned}$$

Then relations (1.3) and (2.16) for $n \geq 0$ and $m \in \mathbb{Z}$ imply that

$$\begin{aligned} |Tz_{n,m} - Tw_{n,m}| &\leq \left(\sum_{s=-\infty}^{-1} Kp^{n-s-1} + 2\sum_{s=0}^{n-1} Kp^{n-s-1} + \sum_{s=n}^{\infty} Kp^{-n+s+1} \right) (M+L)|z-w|_\infty \\ &\leq \frac{2K(1+p)}{1-p} (M+L)|z-w|_\infty. \end{aligned} \tag{2.17}$$

Moreover from (1.3) and (2.16) for $n \leq 0$ and $m \in \mathbb{Z}$ we get

$$\begin{aligned} |Tz_{n,m} - Tw_{n,m}| &\leq \left(\sum_{s=-\infty}^{n-1} Kp^{n-s-1} + \sum_{s=n}^{-1} Kp^{-n+s+1} + \sum_{s=0}^{\infty} Kp^{-n+s+1} \right) (M+L)|z-w|_\infty \\ &\leq \frac{2K(1+p)}{1-p} (M+L)|z-w|_\infty. \end{aligned} \tag{2.18}$$

So, from (2.6), (2.17) and (2.18) T is a contraction on the complete metric space l^∞. Hence there exists a unique $z_{n,m} \in l^\infty$ such that

$$Tz_{n,m} = z_{n,m}, \qquad n, m \in \mathbb{Z}. \tag{2.19}$$

From (2.10), (2.11) and (2.19) we obtain for $n \geq 0, m \in \mathbb{Z}$

$$\begin{aligned}
z_{n,m} &= \sum_{s=-\infty}^{-1} X_n P_1 X_{s+1}^{-1} H(z_{s,m}) + \sum_{s=0}^{n-1} X_n (I - P_2) X_{s+1}^{-1} H(z_{s,m}) \\
&\quad + \sum_{s=0}^{n-1} X_n P_2 X_{s+1}^{-1} H(z_{s,m}) - \sum_{s=0}^{n-1} X_n P_2 X_{s+1}^{-1} H(z_{s,m}) - \sum_{s=n}^{\infty} X_n P_2 X_{s+1}^{-1} H(z_{s,m}) \\
&= \sum_{s=-\infty}^{-1} X_n P_1 X_{s+1}^{-1} H(z_{s,m}) + \sum_{s=0}^{n-1} X_n X_{s+1}^{-1} H(z_{s,m}) - \sum_{s=0}^{\infty} X_n P_2 X_{s+1}^{-1} H(z_{s,m}) \\
&= X_n X_0^{-1} \sum_{s=-\infty}^{-1} X_0 P_1 X_{s+1}^{-1} H(z_{s,m}) + \sum_{s=0}^{n-1} X_n X_{s+1}^{-1} (-y_{s+1,m} + A_s y_{s,m}) \\
&\quad + \sum_{s=0}^{n-1} X_n X_{s+1}^{-1} \Big(B_{s,m} (y_{s,m+1} + z_{s,m+1}) + f(y_{s,m} + z_{s,m}) \Big) \\
&\quad - X_n X_0^{-1} \sum_{s=0}^{\infty} X_0 P_2 X_{s+1}^{-1} H(z_{s,m}).
\end{aligned} \tag{2.20}$$

Then for $n = 0$ we get

$$z_{0,m} = \sum_{s=-\infty}^{-1} X_0 P_1 X_{s+1}^{-1} H(z_{s,m}) - \sum_{s=0}^{\infty} X_0 P_2 X_{s+1}^{-1} H(z_{s,m}). \tag{2.21}$$

We claim that

$$y_{n,m} = X_n X_0^{-1} y_{0,m} + \sum_{s=0}^{n-1} X_n X_{s+1}^{-1} (y_{s+1,m} - A_s y_{s,m}), \qquad n \geq 0,\ m \in \mathbb{Z}. \tag{2.22}$$

It is obvious that (2.22) is true for $n = 0$. Suppose that (2.22) holds for a fixed n. Then

$$\begin{aligned}
X_{n+1} X_0^{-1} y_{0,m} &+ \sum_{s=0}^{n} X_{n+1} X_{s+1}^{-1} (y_{s+1,m} - A_s y_{s,m}) \\
&= A_n X_n X_0^{-1} y_{0,m} + y_{n+1,m} - A_n y_{n,m} + A_n \sum_{s=0}^{n-1} X_n X_{s+1}^{-1} (y_{s+1,m} - A_s y_{s,m}) \\
&= A_n y_{n,m} + y_{n+1,m} - A_n y_{n,m} = y_{n+1,m}.
\end{aligned}$$

Therefore (2.22) is true for every n. Using (2.20), (2.21) and (2.22) we obtain for $n \geq 0, m \in \mathbb{Z}$.

$$z_{n,m} + y_{n,m} = X_n X_0^{-1} (y_{0,m} + z_{0,m}) + \sum_{s=0}^{n-1} X_n X_{s+1}^{-1} \Big(B_{s,m} (y_{s,m+1} + z_{s,m+1}) + f(y_{s,m} + z_{s,m}) \Big).$$

Then if $x_{n,m} = z_{n,m} + y_{n,m}$ from (2.1) we have that $x_{n,m}$, $n \geq 0, m \in \mathbb{Z}$ is a solution of (1.1). So, from (2.9), (2.14) and (2.19) we have

$$|x - y|_\infty = |z|_\infty \leq \frac{2K(1+p)\delta}{1 - p - 2K(p+1)(M+L)} < \epsilon.$$

In addition from (2.11) and (2.19) we have for $n \le 0, m \in \mathbb{Z}$

$$\begin{aligned} z_{n,m} &= \sum_{s=-\infty}^{n-1} X_n P_1 X_{s+1}^{-1} H(z_{s,m}) - \sum_{s=n}^{-1} X_n (I - P_1) X_{s+1}^{-1} H(z_{s,m}) \\ &\quad - \sum_{s=n}^{-1} X_n P_1 X_{s+1}^{-1} H(z_{s,m}) + \sum_{s=n}^{-1} X_n P_1 X_{s+1}^{-1} H(z_{s,m}) - \sum_{s=0}^{\infty} X_n P_2 X_{s+1}^{-1} H(z_{s,m}) \\ &= \sum_{s=-\infty}^{-1} X_n P_1 X_{s+1}^{-1} H(z_{s,m}) - \sum_{s=n}^{-1} X_n X_{s+1}^{-1} H(z_{s,m}) - \sum_{s=0}^{\infty} X_n P_2 X_{s+1}^{-1} H(z_{s,m}) \\ &= X_n X_0^{-1} \sum_{s=-\infty}^{-1} X_0 P_1 X_{s+1}^{-1} H(z_{s,m}) - \sum_{s=n}^{-1} X_n X_{s+1}^{-1} \Big(-y_{s+1,m} + A_s y_{s,m} \Big) \\ &\quad - \sum_{s=n}^{-1} X_n X_{s+1}^{-1} \Big(B_{s,m} (y_{s,m+1} + z_{s,m+1}) + f(y_{s,m} + z_{s,m}) \Big) \\ &\quad - X_n X_0^{-1} \sum_{s=0}^{\infty} X_0 P_2 X_{s+1}^{-1} H(z_{s,m}). \end{aligned} \tag{2.23}$$

So, for $n = 0$ we get (2.21). Moreover, arguing as in (2.22) we can show that

$$y_{n,m} = X_n X_0^{-1} y_{0,m} - \sum_{s=n}^{-1} X_n X_{s+1}^{-1} (y_{s+1,m} - A_s y_{s,m}), \qquad n \le 0, \ m \in \mathbb{Z}. \tag{2.24}$$

Therefore from (2.1), (2.23) and (2.24) we can prove that $x_{n,m} = y_{n,m} + z_{n,m}$ is a solution of (1.1). Using (2.9), (2.15) the proof of **(i)** is completed.

(ii) Let ϵ be an arbitrary positive number and δ be a positive number such that

$$\delta < \frac{1 - p - 2KM(1+p)(1+L)}{2K(1+p)} \epsilon. \tag{2.25}$$

Suppose that $y_{n,m}$ is a double sequence such that (1.6) is satisfied. Then using (2.2), (2.3), (2.5), (2.7), (2.8), (2.25) and arguing as in the case **(i)** we can prove **(ii)**. □

In what follows we study the Hyers–Ulam stability for the equation

$$x_{n+1,m} = A x_{n,m} + B x_{n,m+1} + f_{n,m}, \qquad n, m \in \mathbb{N} \tag{2.26}$$

where A, B are $k \times k$ are constant matrices and $f_{n,m} : \mathbb{N} \times \mathbb{N} \to \mathbb{R}^k$ is a double sequence. Firstly we give a formula for the solutions of (2.26).

Let $x_{n,m}$ be a double sequence. Then we define the operators E_1, E_2 as follows:

$$E_1 x_{n,m} = x_{n+1,m}, \qquad E_2 x_{n,m} = x_{n,m+1}.$$

Proposition 2.3. *Consider the partial difference equations* (2.26)*. Then the following statements are true:*

(i) *There exists a unique solution $x_{n,m}$ of* (2.26) *with $x_{0,m} = c_m$, c_m is a given sequence. Moreover $x_{n,m}$ is given by*

$$x_{n,m} = (A + BE_2)^n c_m + \sum_{s=0}^{n-1} (A + BE_2)^{n-s-1} f_{s,m}. \tag{2.27}$$

(ii) *Let B be an invertible matrix. There exists a unique solution $x_{n,m}$ of (2.26) where $x_{n,0} = d_n$, d_n is a given sequence. Furthermore $x_{n,m}$ is given by*

$$x_{n,m} = (-B^{-1}A + B^{-1}E_1)^m d_n + \sum_{s=0}^{m-1} (-B^{-1}A + B^{-1}E_1)^{m-s-1}(-B^{-1}) f_{n,s}. \tag{2.28}$$

Proof. **(i)** From (2.26) we get

$$x_{n+1,m} = Ax_{n,m} + BE_2 x_{n,m} + f_{n,m} = (A + BE_2)x_{n,m} + f_{n,m}, \qquad n, m \in \mathbb{N}. \tag{2.29}$$

Then from (2.29), for a fixed $m \in \mathbb{N}$ by the constant variation formula we get (2.27). So, the proof of part **(i)** is completed.

(ii) From (2.26) we get for a fixed $n \in \mathbb{N}$

$$\begin{aligned} x_{n,m+1} &= B^{-1}x_{n+1,m} - B^{-1}Ax_{n,m} - B^{-1}f_{n,m} \\ &= (B^{-1}E_1 - B^{-1}A)x_{n,m} - B^{-1}f_{n,m}. \end{aligned}$$

Then by the constant variation formula we take (2.28). This completes the proof of the proposition. □

Proposition 2.4. *Suppose that A, B are $k \times k$ matrices. Suppose that either*

$$|A| + |B| < 1 \tag{2.30}$$

or if B is invertible and

$$|B^{-1}| + |B^{-1}A| < 1. \tag{2.31}$$

Then equation (2.26) has the Hyers–Ulam stability.

Proof. Suppose firstly that (2.30) is satisfied. Let ϵ be an arbitrary number and $\delta = \epsilon(1 - (|A| + |B|))$. Let $y_{n,m}$ be a double sequence such that

$$|y_{n+1,m} - Ay_{n,m} - By_{n,m+1} - f_{n,m}| < \delta. \tag{2.32}$$

We set

$$y_{n+1,m} - Ay_{n,m} - By_{n,m+1} - f_{n,m} = Q_{n,m}.$$

Then, from (2.32), it is obvious that

$$|Q_{n,m}| < \delta, \qquad n, m \in \mathbb{N}. \tag{2.33}$$

Arguing as in the case **(i)** of Proposition 2.3 we obtain

$$y_{n,m} = (A + BE_2)^n y_{0,m} + \sum_{s=0}^{n-1} (A + BE_2)^{n-s-1}(f_{s,m} + Q_{s,m}). \tag{2.34}$$

Let $x_{n,m}$ be a solution of (2.26) with $x_{0,m} = y_{0,m}$. Then from (2.27) and (2.34) we have

$$x_{n,m} - y_{n,m} = -\sum_{s=0}^{n-1} (A + BE_2)^{n-s-1} Q_{s,m}. \tag{2.35}$$

Relations (2.30), (2.33) and (2.35) imply that

$$
\begin{aligned}
|x_{n,m}-y_{n,m}| &\le \sum_{s=0}^{n-1}(|A|+|B|E_2)^{n-s-1}|Q_{s,m}| \\
&= \sum_{s=0}^{n-1}\sum_{k=0}^{n-s-1}\frac{(n-s-1)!}{k!(n-s-1-k)!}|A|^{n-s-1-k}|B|^k E_2^k|Q_{s,m}| \\
&= \sum_{s=0}^{n-1}\sum_{k=0}^{n-s-1}\frac{(n-s-1)!}{k!(n-s-1-k)!}|A|^{n-s-1-k}|B|^k|Q_{s,m+k}| \\
&< \delta\sum_{s=0}^{n-1}(|A|+|B|)^{n-s-1} \le \frac{\delta}{1-(|A|+|B|)} = \epsilon.
\end{aligned}
$$

This completes the proof of case **(i)**.

Suppose that (2.31) is fulfilled. Let ϵ be a positive number and $\delta = \epsilon(1-(|B^{-1}|+|B^{-1}A|))$. Let $y_{n,m}$ be a double sequence such that

$$
|y_{n,m+1}+B^{-1}Ay_{n,m}-B^{-1}y_{n+1,m}+B^{-1}f_{n,m}|<\delta. \tag{2.36}
$$

We set

$$
y_{n,m+1}+B^{-1}Ay_{n,m}-B^{-1}y_{n+1,m}+B^{-1}f_{n,m}=\widehat{Q}_{n,m}.
$$

Then using the same argument as in the case **(ii)** of Proposition 2.3 we get,

$$
y_{n,m}=(-B^{-1}A+B^{-1}E_1)^m y_{n,0}+\sum_{s=0}^{m-1}(-B^{-1}A+B^{-1}E_1)^{m-s-1}(\widehat{Q}_{n,s}-B^{-1}f_{n,s}). \tag{2.37}
$$

Let $x_{n,m}$ be a solution of (2.26) with $x_{n,0}=y_{n,0}$. Then from (2.28) and (2.37) we obtain

$$
x_{n,m}-y_{n,m}=-\sum_{s=0}^{m-1}(-B^{-1}A+B^{-1}E_1)^{m-s-1}\widehat{Q}_{n,s}.
$$

Hence from (2.31) and (2.33) we get

$$
\begin{aligned}
|x_{n,m}-y_{n,m}| &\le \sum_{s=0}^{m-1}(|B^{-1}A|+|B^{-1}|E_1)^{m-s-1}|\widehat{Q}_{n,s}| \\
&= \sum_{s=0}^{m-1}\sum_{k=0}^{m-s-1}\frac{(m-s-1)!}{k!(m-s-1-k)!}|B^{-1}A|^{m-s-1-k}|B^{-1}|^k E_1^k|\widehat{Q}_{n,s}| \\
&= \sum_{s=0}^{m-1}\sum_{k=0}^{m-s-1}\frac{(m-s-1)!}{k!(m-s-1-k)!}|B^{-1}A|^{m-s-1-k}|B^{-1}|^k|\widehat{Q}_{n+k,s}| \\
&= \delta\sum_{s=0}^{m-1}(|B^{-1}|+|B^{-1}A|)^{m-s-1} < \frac{\delta}{1-(|B^{-1}|+|B^{-1}A|)} = \epsilon.
\end{aligned}
$$

This completes the proof of the proposition. □

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