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Multiple nonsymmetric nodal solutions for quasilinear Schrödinger system

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Abstract. In this paper, we consider the quasilinear Schrödinger system in $\mathbb{R}^N$ ($N \geq 3$):

$$\begin{cases} -\Delta u + A(x)u - \dfrac{1}{2}\Delta(u^2)u = \dfrac{2\alpha}{\alpha+\beta}|u|^{\alpha-2}u|v|^{\beta}, \\ -\Delta v + Bv - \dfrac{1}{2}\Delta(v^2)v = \dfrac{2\beta}{\alpha+\beta}|u|^{\alpha}|v|^{\beta-2}v, \end{cases}$$

where $\alpha, \beta > 1$, $2 < \alpha + \beta < \frac{4N}{N-2}$, $B > 0$ is a constant. By using a constrained minimization on Nehari–Pohožaev set, for any given integer $s \geq 2$, we construct a non-radially symmetrical nodal solution with its $2s$ nodal domains.

Keywords: quasilinear Schrödinger system, Nehari–Pohožaev set, non-radially symmetrical nodal solutions.

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1 Introduction

We study the following quasilinear Schrödinger system

$$\begin{cases} -\Delta u + A(x)u - \dfrac{1}{2}\Delta(u^2)u = \dfrac{2\alpha}{\alpha+\beta}|u|^{\alpha-2}u|v|^{\beta}, \\ -\Delta v + Bv - \dfrac{1}{2}\Delta(v^2)v = \dfrac{2\beta}{\alpha+\beta}|u|^{\alpha}|v|^{\beta-2}v, \end{cases} \tag{1.1}$$

where $u(x) \to 0$, $v(x) \to 0$ as $|x| \to \infty$, $N \geq 3$, $u := u(x)$, $v := v(x)$ be real valued functions on $\mathbb{R}^N$, α, $\beta > 1$, $2 < \alpha + \beta < \frac{4N}{N-2}$, $B > 0$ is a constant. In the last two decades, much attention has been devoted to the quasilinear Schrödinger equation of the form

$$-\Delta u + V(x)u - \frac{1}{2}u\Delta(u^2) = |u|^{p-2}u, \qquad x \in \mathbb{R}^N. \tag{1.2}$$

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The equation (1.2) is related to the existence of standing waves of the following quasilinear Schrödinger equation

$$i\partial_t z = -\Delta z + V(x)z - l(|z|^2)z - \frac{1}{2}\Delta g(|z|^2)g'(|z|^2)z, \qquad x \in \mathbb{R}^N, \tag{1.3}$$

where V is a given potential, l and g are real functions. The equation (1.3) has been used as models in several areas of physics corresponding to various types of g. The superfluid film equation in plasma physics has this structure for $g(s) = s$ [9]. In the case $g(s) = (1+s)^{\frac{1}{2}}$, the equation (1.3) models the self-channeling of a high-power ultra short laser in matter [19]. The equation (1.3) also appears in fluid mechanics [9, 10], in the theory of Heidelberg ferromagnetism and magnus [11], in dissipative quantum mechanics and in condensed matter theory [14]. When considering the case $g(s) = s$, one obtains a corresponding equation of elliptic type like (1.2). For more detailed mathematical and physical interpretation of equations like (1.2), we refer to [1,3,4,12,18,21] and the references therein.

In recent years, there has been increasing interest in studying problem (1.2), see for examples, [5,6,8,15,16,24,25] and the references therein. More precisely, by the Mountain Pass Theorem and the principle of symmetric criticality, Severo [22] obtained symmetric and non-symmetric solutions for quasilinear Schrödinger equation (1.2). In [13], when $4 \leq p < \frac{4N}{N-2}$, Liu, Wang and Wang established the existence results of a positive ground state solution and a sign-changing ground state solution were given by using the Nehari method for (1.2). Based on the method of perturbation and invariant sets of descending flow, Zhang and Liu [27] studied the nonautonomous case of (1.2), they obtained the existence of infinitely many sign-changing solutions for $4 < p < \frac{4N}{N-2}$. With the help of Nehari method and change of variables, Deng, Peng and Wang [7] considered

$$-\Delta u + V(x)u - u\Delta(u^2) = \lambda|u|^{p-2}u + |u|^{\frac{4N}{N-2}-2}u, \qquad x \in \mathbb{R}^N, \tag{1.4}$$

and proved that (1.4) has at least one pair of k-node solutions if either $N \geq 6$ and $4 < p < \frac{4N}{N-2}$ or $3 \leq N < 6$ and $\frac{2(N+2)}{N-2} < q < \frac{4N}{N-2}$. In addition, problem (1.4) still has at least one pair of k-node solutions if $3 \leq N < 6$, $4 < q \leq \frac{2(N+2)}{N-2}$ and λ sufficiently large. Note that all sign-changing solutions obtained in [7, 13, 27] are only valid for $4 < p < \frac{4N}{N-2}$. When $2 < p < \frac{4N}{N-2}$, Ruiz and Siciliano [20] showed equation (1.2) has a ground states solution via Nehari–Pohožaev type constraint and concentration-compactness lemma, Wu and Wu [26] obtained the existence of radial solutions for (1.2) by using change of variables.

It is natural to pose a series of interesting questions: whether we can find an unified approach to obtain sign-changing solutions for the full subcritical range of $2 < \alpha + \beta < \frac{4N}{N-2}$? Further, whether we can extend these results to system of the quasilinear Schrödinger system? To answer these two questions, we adopt an action of finite subgroup G of $O(2)$ from Szulkin and Waliullah [23] and look for the existence of non-radially symmetrical nodal solutions for quasilinear Schrödinger system (1.1).

Before stating our main results, we make the following assumptions:

(A_1) $A \in C^1(\mathbb{R}^N, \mathbb{R}^+), 0 < A_0 \leq A(x) \leq A_\infty = \lim_{|x|\to\infty} A(x) < +\infty;$

(A_2) $\nabla A(x) \cdot x \in L^\infty(\mathbb{R}^N)$, $(\alpha+\beta-2)A(x) - \nabla A(x)\cdot x \geq 0;$

(A_3) the map $s \mapsto s^{\frac{N+2}{N+\alpha+\beta}} A(s^{\frac{1}{N+\alpha+\beta}} x)$ is concave for any $x \in \mathbb{R}^N$;

(A_4) $A(x)$ is radially symmetric with respect to the first two coordinates, that is to say, if $(x_1, x_2, x_3, \dots, x_N)$, $(y_1, y_2, y_3, \dots, y_N) \in \mathbb{R}^N$ and $x_1^2 + x_2^2 = y_1^2 + y_2^2$, then

$$A(x_1, x_2, z_3, \dots, z_N) = A(y_1, y_2, z_3, \dots, z_N).$$

It is worth noting that (A_1) is used to derive the existence of a strongly convergent subsequence, while for the system, we only need one such kind of condition in our equations, which seems to be a different phenomenon due to the coupling of u and v. (A_2)–(A_3) once appeared in [20, 26] to obtain the existence of ground states solutions for the quasilinear Schrödinger equation.

Our main result reads as follows.

Theorem 1.1. *Assume that (A_1)–(A_4) hold. For any given integer $s \geq 2$, the problem (1.1) possesses a non-radially symmetrical nodal solution with its $2s$ nodal domains.*

Corollary 1.2. *If $A(x)$ is a positive constant, one can still obtain the same results as Theorem 1.1 for system (1.1).*

Remark 1.3. Since $s \in \mathbb{N}$ is arbitrary, the solution we obtained in Theorem 1.1 is actually a result of multiplicity.

Remark 1.4. As a main novelty with respect to some results in [7, 13, 27], we are able to deal with exponents $\alpha + \beta \in (2, \frac{4N}{N-2})$ and obtain the existence and multiplicity of nodal solution without any radial symmetry.

The rest of the paper is organized as follows. In Section 2, we establish some preliminary results. Theorem 1.1 is proved in Section 3.

2 Preliminaries

Throughout this paper, $\|u\|_{H^1}$ and $|u|_r$ denote the usual norms of $H^1(\mathbb{R}^N)$ and $L^r(\mathbb{R}^N)$ for $r > 1$, respectively. C and C_i $(i = 1, 2, \dots)$ denote (possibly different) positive constants and $\int_{\mathbb{R}^N} g$ denotes the integral $\int_{\mathbb{R}^N} g(z)dz$. The $\to$ and $\rightharpoonup$ denote strong convergence and weak convergence, respectively.

Let $H^1(\mathbb{R}^N)$ be the usual Sobolev space, define $X := H \times H$ with

$$H := \left\{ u \in H^1(\mathbb{R}^N) \mid \int_{\mathbb{R}^N} u^2 |\nabla u|^2 < +\infty \right\}.$$

The term $\int_{\mathbb{R}^N} u^2 |\nabla u|^2$ is not convex and H is not even a vector space. So, the usual min-max techniques cannot be directly applied, nevertheless H is a complete metric space with distance

$$d_H(u, \omega) = \|u - \omega\|_{H^1} + |\nabla u^2 - \nabla \omega^2|_2.$$

Define

$$d_X\big((u, v), (\omega, \nu)\big) := \|u - \omega\|_{H^1} + |\nabla u^2 - \nabla \omega^2|_2 + \|v - \nu\|_{H^1} + |\nabla v^2 - \nabla \nu^2|_2.$$

Then we call $(u, v) \in X$ is a weak solution of (1.1) if for any $\varphi_1, \varphi_2 \in C_0^\infty(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} \left((1 + u^2) \nabla u \nabla \varphi_1 + \left(u |\nabla u|^2 + A(x) u - \frac{2\alpha |u|^{\alpha-2} u |v|^\beta}{\alpha + \beta} \right) \varphi_1 \right) = 0,$$

and

$$\int_{\mathbb{R}^N}\left((1+v^2)\nabla v\nabla\varphi_2+\left(v|\nabla v|^2+Bv-\frac{2\beta|u|^\alpha|v|^{\beta-2}v}{\alpha+\beta}\right)\varphi_2\right)=0.$$

Hence there is a one-to-one correspondence between solutions of (1.1) and critical points of the following functional $I: X\to\mathbb{R}$ defined by

$$\begin{aligned}I(u,v)=&\frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u|^2+|\nabla v|^2+A(x)u^2+Bv^2)\\&+\frac{1}{2}\int_{\mathbb{R}^N}(u^2|\nabla u|^2+v^2|\nabla v|^2)-\frac{2}{\alpha+\beta}\int_{\mathbb{R}^N}|u|^\alpha|v|^\beta.\end{aligned}\tag{2.1}$$

For any $\varphi_1,\varphi_2\in C_0^\infty(\mathbb{R}^N)$, $(u,v)\in X$, and $(u,v)+(\varphi_1,\varphi_2)\in X$, we compute the Gateaux derivative

$$\begin{aligned}\langle I'(u,v),(\varphi_1,\varphi_2)\rangle=&\int_{\mathbb{R}^N}\big((1+u^2)\nabla u\nabla\varphi_1+(1+v^2)\nabla v\nabla\varphi_2+u|\nabla u|^2\varphi_1\\&+v|\nabla v|^2\varphi_2+A(x)u\varphi_1+Bv\varphi_2\big)-\frac{2\alpha}{\alpha+\beta}\int_{\mathbb{R}^N}|u|^{\alpha-2}u|v|^\beta\varphi_1\\&-\frac{2\beta}{\alpha+\beta}\int_{\mathbb{R}^N}|v|^{\beta-2}v|u|^\alpha\varphi_2.\end{aligned}$$

Then, $(u,v)\in X$ is a solution of (1.1) if and only if

$$\langle I'(u,v),(\varphi_1,\varphi_2)\rangle=0,\qquad \varphi_1,\ \varphi_2\in C_0^\infty(\mathbb{R}^N).$$

Motivated by [23], we recall that a subset U of a Banach space $\mathbb{E}$ is called invariant with respect to an action of a group G (or G-invariant) if $gU\subset U$ for all $g\in G$, and a functional $I: U\to\mathbb{R}$ is invariant (or G-invariant) if $I(gu)=I(u)$ for all $g\in G$, $u\in U$. The subspace

$$\mathbb{E}_G:=\{u\in\mathbb{E}\mid gu=u\ \text{ for all } g\in G\}$$

is called the fixed point space of this action.

Let $x=(y,z)=(y_1,y_2,z_1,\dots,z_N)\in\mathbb{R}^N$ and let $O(2)$ be the group of orthogonal transformations acting on $\mathbb{R}^2$ by $(g,y)\mapsto gy$. For any positive integer s we define G_s to be the finite subgroup of $O(2)$ generated by the two elements α and β in $O(2)$, where α is the rotation in the y-plane by the angle $\frac{2\pi}{s}$ and β is the reflection in the line $y_1=0$ if $s=2$, and in the line $y_2=\tan(\pi/s)y_1$ for other s (so in complex notation $w=y_1+iy_2$, $\alpha w=we^{\frac{2\pi i}{s}}$, $\beta w=we^{\frac{2\pi i}{s}}$).

$\forall\, g\in G_s$, $x\in\mathbb{R}^N$, $gx:=(gy,z)$. Define the action of G_s on $H^1(\mathbb{R}^N)$ by setting

$$(g(u,v))x:=(gu,gv)x=(\det(g)ug^{-1}x,\det(g)vg^{-1}x).$$

Define

$$\mathcal{V}:=\{(u,v)\in X\mid (u,v)(gx)=(\det(g)u(x),\det(g)v(x)),\ \ g\in G_s\},$$
$$\mathcal{M}:=\{(u,v)\in\mathcal{V}\backslash\{(0,0)\}\mid \mathcal{G}(u,v)=0\},$$

where $\mathcal{G}: X\to\mathbb{R}$ and

$$\begin{aligned}\mathcal{G}(u,v)=&\frac{N}{2}\int_{\mathbb{R}^N}(|\nabla u|^2+|\nabla v|^2)+\frac{N+2}{2}\int_{\mathbb{R}^N}(A(x)u^2+Bv^2\\&+u^2|\nabla u|^2+v^2|\nabla v|^2)-\frac{2(N+\alpha+\beta)}{\alpha+\beta}\int_{\mathbb{R}^N}|u|^\alpha|v|^\beta.\end{aligned}$$

Let

$$m := \inf_{(u,v)\in\mathcal{M}} I(u,v). \tag{2.2}$$

Then our aim is to prove that m is achieved. In the rest of this section, we will give some properties of the set $\mathcal{M}$.

For any $u \in H^1(\mathbb{R}^N)$, we define $u_t : \mathbb{R}^+ \to H^1(\mathbb{R}^N)$ by:

$$u_t(x) := tu(t^{-1}x).$$

Let $t \in \mathbb{R}^+$ and $(u,v) \in X$. We have that

$$\begin{aligned} I(u_t,v_t) = \frac{t^N}{2}\int_{\mathbb{R}^N}(|\nabla u|^2+|\nabla v|^2) + \frac{t^{N+2}}{2}\int_{\mathbb{R}^N}(A(x)u^2+Bv^2 \\ + u^2|\nabla u|^2+v^2|\nabla v|^2) - \frac{2t^{N+\alpha+\beta}}{\alpha+\beta}\int_{\mathbb{R}^N}|u|^\alpha|v|^\beta. \end{aligned}$$

Denote $h_{uv}(t) := I(u_t,v_t)$. Since $\alpha+\beta > 2$, we see that $h_{uv}(t) > 0$ for $t > 0$ small enough and $h_{uv}(t) \to -\infty$ as $t \to +\infty$, this implies that $h_{uv}(t)$ attains its maximum. Moreover, $h_{uv}(t) : \mathbb{R}^+ \to \mathbb{R}$ is C^1 and

$$\begin{aligned} h'_{uv}(t) = \frac{N}{2}t^{N-1}\int_{\mathbb{R}^N}(|\nabla u|^2+|\nabla v|^2) + \frac{N+2}{2}t^{N+1}\int_{\mathbb{R}^N}(A(x)u^2+Bv^2 \\ + u^2|\nabla u|^2+v^2|\nabla v|^2) - \frac{2(N+\alpha+\beta)}{\alpha+\beta}t^{N+\alpha+\beta-1}\int_{\mathbb{R}^N}|u|^\alpha|v|^\beta. \end{aligned}$$

Lemma 2.1. *If $(u,v) \in X$ is a weak solution of (1.1), then (u,v) satisfies the following $P(u,v) = 0$, where*

$$\begin{aligned} P(u,v) := \frac{N-2}{2}\int_{\mathbb{R}^N}(|\nabla u|^2+|\nabla v|^2+u^2|\nabla u|^2+v^2|\nabla v|^2) \\ + \frac{N}{2}\int_{\mathbb{R}^N}(A(x)u^2+Bv^2) + \frac{1}{2}\int_{\mathbb{R}^N}\nabla A(x)\cdot xu^2 \\ - \frac{2N}{\alpha+\beta}\int_{\mathbb{R}^N}|u|^\alpha|v|^\beta. \end{aligned} \tag{2.3}$$

Proof. The proof is standard, so we omit it here. □

The lemma below shows (2.2) is well defined.

Lemma 2.2. *For any $(u,v) \in X$ and $u,v \neq 0$, the map h_{uv} attains its maximum at exactly one point $\bar{t}$. Moreover, h_{uv} is positive and increasing for $t \in [0,\bar{t}]$ and decreasing for $t > \bar{t}$. Finally*

$$m = \inf_{(u,v)\in X}\max_{t>0} I(u_t,v_t).$$

Proof. For any $t > 0$, set $s = t^{N+\alpha+\beta}$, we obtain

$$\begin{aligned} h_{uv}(s) = \frac{s^{\frac{N}{N+\alpha+\beta}}}{2}\int_{\mathbb{R}^N}|\nabla u|^2 + \frac{s^{\frac{N}{N+\alpha+\beta}}}{2}\int_{\mathbb{R}^N}|\nabla v|^2 + \frac{s^{\frac{N+2}{N+\alpha+\beta}}}{2}\int_{\mathbb{R}^N}u^2|\nabla u|^2 \\ + \frac{s^{\frac{N+2}{N+\alpha+\beta}}}{2}\int_{\mathbb{R}^N}v^2|\nabla v|^2 + \frac{s^{\frac{N+2}{N+\alpha+\beta}}}{2}\int_{\mathbb{R}^N}A(s^{\frac{1}{N+\alpha+\beta}}x)u^2 \\ + \frac{s^{\frac{N+2}{N+\alpha+\beta}}}{2}\int_{\mathbb{R}^N}Bv^2 - \frac{2s}{\alpha+\beta}\int_{\mathbb{R}^N}|u|^\alpha|v|^\beta. \end{aligned}$$

This is a concave function by condition (A_3) and we already know that it attains its maximum, let $\bar{t}$ be the unique point at which this maximum is achieved. Notice that $\mathcal{G}(u_t, v_t) = th'_{uv}(t)$, then $\bar{t}$ is the unique critical point of h_{uv} and h_{uv} is positive and increasing for $0 < t < \bar{t}$ and decreasing for $t > \bar{t}$. In particular, $\bar{t} \in \mathbb{R}$ is the unique value such that $u_{\bar{t}} \in \mathcal{M}$, and $I(u_{\bar{t}}, v_{\bar{t}})$ reaches a global maximum for $t = \bar{t}$. This finishes the proof. □

Lemma 2.3. *$m > 0$.*

Proof. For every $(u, v) \in \mathcal{M}$, it follows from (A_2) that

$$\begin{aligned} I(u,v) = &\frac{\alpha+\beta}{2(N+\alpha+\beta)} \int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla v|^2) \\ &+ \frac{\alpha+\beta-2}{2(N+\alpha+\beta)} \int_{\mathbb{R}^N} (Bv^2 + u^2|\nabla u|^2 + v^2|\nabla v|^2) \\ &+ \frac{1}{2(N+\alpha+\beta)} \int_{\mathbb{R}^N} \big((\alpha+\beta-2)A(x) - \nabla A(x)\cdot x\big)u^2 \\ > &\ 0. \end{aligned}$$

The proof is complete. □

3 Proof of Theorem 1.1

We need the following variant of the Lions Lemma.

Lemma 3.1. *If $q \in [2, \frac{4N}{N-2})$, $\{u_n\}$ is bounded in X, $r_0 > 0$ is such that for all $r \geq r_0$*

$$\lim_{n\to\infty} \sup_{z\in\mathbb{R}} \int_{B((0,z),r)} |u_n|^q = 0, \tag{3.1}$$

then we have $u_n \to 0$ in $L^p(\mathbb{R}^N)$ for $p \in (2, \frac{4N}{N-2})$.

Proof. By using [24, Lemma 2.2], it remains to prove that for some $r > 0$,

$$\lim_{n\to\infty} \sup_{z\in\mathbb{R}^N} \int_{B(z,r)} |u_n|^q = 0.$$

Suppose that

$$\int_{B(z_n,1)} |u_n|^q \geq c > 0. \tag{3.2}$$

Observe that in the family $\{B(gz_n, 1)\}_{g\in O(2)}$, we find an increasing number of disjoint balls provided that $|(z_n^1, z_n^2)| \to \infty$. Since $\{u_n\}$ is bounded in $L^q(\mathbb{R}^N), q \in [2, \frac{4N}{N-2})$, by (3.2), $|(z_n^1, z_n^2)|$ must be bounded. Then for sufficiently large $r \geq r_0$, one obtains

$$\int_{B((0,z_n^3),r)} |u_n|^q \geq \int_{B(z_n,1)} |u_n|^q \geq c > 0,$$

and we get a contradiction with (3.1). □

Lemma 3.2. *Let $u_n \rightharpoonup u, v_n \rightharpoonup v$ in X, $u_n \to u, v_n \to v$ a.e in $\mathbb{R}^N$. Then*

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^\alpha |v_n|^\beta - \int_{\mathbb{R}^N} |u|^\alpha |v|^\beta = \lim_{n\to\infty} \int_{\mathbb{R}^N} |u_n - u|^\alpha |v_n - v|^\beta.$$

Proof. For $n = 1, 2, \ldots$, we have that

$$\int_{\mathbb{R}^N} |u_n|^\alpha |v_n|^\beta - \int_{\mathbb{R}^N} |u_n - u|^\alpha |v_n - v|^\beta$$
$$= \int_{\mathbb{R}^N} (|u_n|^\alpha - |u_n - u|^\alpha)|v_n|^\beta + \int_{\mathbb{R}^N} |u_n - u|^\alpha (|v_n|^\beta - |v_n - v|^\beta).$$

Since $u_n \rightharpoonup u, v_n \rightharpoonup v$ in $H^1(\mathbb{R}^N)$, from [17, Lemma 2.5], one has

$$\int_{\mathbb{R}^N} (|u_n|^\alpha - |u_n - u|^\alpha - |u|^\alpha)^{\frac{p}{\alpha}} \to 0, \qquad n \to \infty,$$

which means that

$$|u_n|^\alpha - |u_n - u|^\alpha \to |u|^\alpha \quad \text{in } L^{\frac{p}{\alpha}}(\mathbb{R}^N).$$

Using $|v_n|^\beta \rightharpoonup |v|^\beta$ in $L^{\frac{p}{\beta}}(\mathbb{R}^N)$, it follows from $\alpha + \beta = p$ that

$$\int_{\mathbb{R}^N} (|u_n|^\alpha - |u_n - u|^\alpha)|v_n|^\beta \to \int_{\mathbb{R}^N} |u|^\alpha |v|^\beta, \qquad n \to \infty.$$

Similarly,

$$|v_n|^\beta - |v_n - v|^\beta \to |v|^\beta \quad \text{in } L^{\frac{p}{\beta}}(\mathbb{R}^N).$$

As $|u_n - u|^\alpha \rightharpoonup 0$ in $L^{\frac{p}{\alpha}}(\mathbb{R}^N)$, we obtain that

$$\int_{\mathbb{R}^N} |u_n - u|^\alpha (|v_n|^\beta - |v_n - v|^\beta) \to 0, \qquad n \to \infty.$$

This proves the lemma. □

The following lemma is due to Poppenberg, Schmitt and Wang from [18, Lemma 2].

Lemma 3.3. *Assume that $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^N)$. Then*

$$\liminf_{n\to\infty} \int_{\mathbb{R}^N} u_n^2 |\nabla u_n|^2 \geq \liminf_{n\to\infty} \int_{\mathbb{R}^N} (u_n - u)^2 |\nabla u_n - \nabla u|^2 + \int_{\mathbb{R}^N} u^2 |\nabla u|^2. \tag{3.3}$$

Proof. The proof is analogous to that of [18, Lemma 2], so we omit it here. □

Lemma 3.4. *m is achieved at some $(u, v) \in \mathcal{M}$.*

Proof. Let $\{(u_n, v_n)\} \subset \mathcal{M}$ be a sequence such that $I(u_n, v_n) \to m$. Using $(u_n, v_n) \subset \mathcal{M}$ and (A_2), we may obtain

$$\begin{aligned} 1 + m &\geq I(u_n, v_n) \\ &= \frac{\alpha+\beta}{2(N+\alpha+\beta)} \int_{\mathbb{R}^N} (|\nabla u_n|^2 + |\nabla v_n|^2) \\ &\quad + \frac{\alpha+\beta-2}{2(N+\alpha+\beta)} \int_{\mathbb{R}^N} (Bv_n^2 + u_n^2|\nabla u_n|^2 + v_n^2|\nabla v_n|^2) \\ &\quad + \frac{1}{2(N+\alpha+\beta)} \int_{\mathbb{R}^N} ((\alpha+\beta-2)A(x) - \nabla A(x)\cdot x) u_n^2, \end{aligned}$$

which implies that $\{u_n\}$, $\{v_n\}$, $\{u_n^2\}$ and $\{v_n^2\}$ are bounded in $H^1(\mathbb{R}^N)$, then, there exists a subsequence of (u_n, v_n), still denoted by (u_n, v_n) such that $(u_n, v_n) \rightharpoonup (u, v)$ in X. Then $\{u_n\}$ and $\{v_n\}$ are bounded in $L^{\alpha+\beta}(\mathbb{R}^N)$. The proof consists of three steps.

Step 1.

$$\int_{\mathbb{R}^N} |u_n|^\alpha |v_n|^\beta \not\to 0.$$

It follows from Lemma 2.3 that

$$\begin{aligned} I(u_n, v_n) = &\int_{\mathbb{R}^N} \bigg(\frac{1}{2}(|\nabla u_n|^2 + |\nabla v_n|^2 + A(x)u_n^2 + Bv_n^2 \\ &+ u_n^2|\nabla u_n|^2 + v_n^2|\nabla v_n|^2) - \frac{2}{\alpha+\beta}|u_n|^\alpha|v_n|^\beta\bigg) \\ \to\ & m > 0, \end{aligned}$$

then

$$\int_{\mathbb{R}^N} (|\nabla u_n|^2 + |\nabla v_n|^2 + A(x)u_n^2 + Bv_n^2 + u_n^2|\nabla u_n|^2 + v_n^2|\nabla v_n|^2) \not\to 0.$$

By Lemma 2.2, for $t > 1$,

$$\begin{aligned} m &\leftarrow I(u_n, v_n) \\ &\geq I\big((u_n)_t, (v_n)_t\big) \\ &= \frac{t^N}{2}\int_{\mathbb{R}^N}(|\nabla u_n|^2 + |\nabla v_n|^2) + \frac{t^{N+2}}{2}\int_{\mathbb{R}^N}(A(tx)u_n^2 + Bv_n^2) \\ &\quad + \frac{t^{N+2}}{2}\int_{\mathbb{R}^N}(u_n^2|\nabla u_n|^2 + v_n^2|\nabla v_n|^2) - \frac{2t^{N+\alpha+\beta}}{\alpha+\beta}\int_{\mathbb{R}^N}|u_n|^\alpha|v_n|^\beta \\ &\geq \frac{t^N}{2}\int_{\mathbb{R}^N}(|\nabla u_n|^2 + |\nabla v_n|^2 + A_0u_n^2 + Bv_n^2 + u_n^2|\nabla u_n|^2 + v_n^2|\nabla v_n|^2) \\ &\quad - \frac{2t^{N+\alpha+\beta}}{\alpha+\beta}\int_{\mathbb{R}^N}|u_n|^\alpha|v_n|^\beta \\ &\geq \frac{t^N}{2}\delta - \frac{2t^{N+\alpha+\beta}}{\alpha+\beta}\int_{\mathbb{R}^N}|u_n|^\alpha|v_n|^\beta, \end{aligned}$$

where δ is a fixed constant. It suffices to choose $t > 1$ so that $\frac{t^N\delta}{2} > 2m$ to get a lower bound for

$$\int_{\mathbb{R}^N} |u_n|^\alpha|v_n|^\beta.$$

Therefore, we may assume (passing to a subsequence, if necessary) that

$$\int_{\mathbb{R}^N} |u_n|^\alpha|v_n|^\beta \to D \in (0, \infty). \tag{3.4}$$

Step 2. $u \neq 0$. By using (3.4) and Hölder's inequality, we can assume (passing to a subsequence, if necessary) that

$$\int_{\mathbb{R}^N} |u_n|^{\alpha+\beta} > \delta > 0.$$

By Lemma 3.1, there exist $\delta > 0$ and $\{z_n\} \subset \mathbb{R}$ such that

$$\limsup_{n\to+\infty} \int_{B((0,z_n),r)} |u_n|^{\alpha+\beta} > \delta > 0. \tag{3.5}$$

Define

$$y = (x_1, x_2), \qquad z = (x_3, \ldots, x_N),$$

$$w_n(x) = w_n(y,z) = u_n(y, z + z_n),$$

and

$$\sigma_n(x) = \sigma_n(y,z) = v_n(y, z + z_n),$$

then $w_n \rightharpoonup w, \sigma_n \rightharpoonup \sigma$ in X. In this case, by (A_4), we may obtain $I(u_n, v_n) = I(w_n, \sigma_n)$. By using (3.5) and $w_n \to w$ in $L^{\alpha+\beta}_{loc}(\mathbb{R}^N)$, one has

$$\begin{aligned} 0 < \delta < \limsup_{n\to+\infty} \int_{B((0,z_n),r)} |u_n|^{\alpha+\beta} \\ = \limsup_{n\to+\infty} \int_{B((0,0),r)} |w_n|^{\alpha+\beta} \\ = \int_{B((0,0),r)} |w|^{\alpha+\beta}, \end{aligned}$$

which implies $w \neq 0$, and then $u \neq 0$.

Step 3. We claim that $(u,v) \in \mathcal{M}$. Indeed, if $(u,v) \notin \mathcal{M}$, we discuss three cases:

Case 1: $\mathcal{G}(u,v) < 0$. By Lemma 2.2, there exists $t \in (0,1)$ such that $(u_t, v_t) \in \mathcal{M}$, it follows from (A_2), $(u_n, v_n) \in \mathcal{M}$ and Fatou's Lemma that

$$\begin{aligned} m &= \liminf_{n\to+\infty} \left(I(u_n,v_n) - \frac{1}{N+\alpha+\beta}\mathcal{G}(u_n,v_n) \right) \\ &= \liminf_{n\to+\infty} \left(\frac{\alpha+\beta}{2(N+\alpha+\beta)} \int_{\mathbb{R}^N} (|\nabla u_n|^2 + |\nabla v_n|^2) \right. \\ &\quad + \frac{1}{2(N+\alpha+\beta)} \int_{\mathbb{R}^N} ((\alpha+\beta-2)A(x) - \nabla A(x)\cdot x) u_n^2 \\ &\quad \left. + \frac{\alpha+\beta-2}{2(N+\alpha+\beta)} \int_{\mathbb{R}^N} (Bv_n^2 + u_n^2|\nabla u_n|^2 + v_n^2|\nabla v_n|^2) \right) \\ &\geq \frac{\alpha+\beta}{2(N+\alpha+\beta)} \int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla v|^2) \\ &\quad + \frac{1}{2(N+\alpha+\beta)} \int_{\mathbb{R}^N} ((\alpha+\beta-2)A(x) - \nabla A(x)\cdot x) u^2 \\ &\quad + \frac{\alpha+\beta-2}{2(N+\alpha+\beta)} \int_{\mathbb{R}^N} (Bv^2 + u^2|\nabla u|^2 + v^2|\nabla v|^2) \\ &> \frac{\alpha+\beta}{2(N+\alpha+\beta)} t^N \int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla v|^2) \\ &\quad + \frac{t^{N+2}}{2(N+\alpha+\beta)} \int_{\mathbb{R}^N} ((\alpha+\beta-2)A(x) - \nabla A(x)\cdot x) u^2 \\ &\quad + \frac{\alpha+\beta-2}{2(N+\alpha+\beta)} t^{N+\alpha+\beta} \int_{\mathbb{R}^N} (Bv^2 + u^2|\nabla u|^2 + v^2|\nabla v|^2) \\ &= I(u_t,v_t) - \frac{1}{N+\alpha+\beta}\mathcal{G}(u_t,v_t) \\ &\geq m, \end{aligned}$$

which is a contradiction.

Case 2: $\mathcal{G}(u,v) > 0$. Set $\xi_n := u_n - u, \gamma_n := v_n - v$, by Lemma 3.2, the Brézis–Lieb Lemma [2], (3.3), (A_1) and (B_1), we may obtain

$$\mathcal{G}(u_n,v_n) \geq \mathcal{G}(u,v) + \mathcal{G}(\xi_n,\gamma_n) + o_n(1). \tag{3.6}$$

Then

$$\limsup_{n\to\infty} \mathcal{G}(\xi_n,\gamma_n) < 0.$$

By Lemma 2.2, there exists $t_n \in (0,1)$ such that $((\xi_n)_{t_n},(\gamma_n)_{t_n}) \in \mathcal{M}$. Furthermore, one has that

$$\limsup_{n\to\infty} t_n < 1,$$

otherwise, along a subsequence, $t_n \to 1$ and hence

$$\mathcal{G}(\xi_n,\gamma_n) = \mathcal{G}((\xi_n)_{t_n},(\gamma_n)_{t_n}) + o_n(1) = o_n(1),$$

a contradiction. It follows from $(u_n,v_n) \in \mathcal{M}$, (3.6), (A_2) that

$$\begin{aligned}
m + o_n(1) &= I(u_n,v_n) - \frac{1}{N+\alpha+\beta}\mathcal{G}(u_n,v_n)\\
&= \frac{\alpha+\beta}{2(N+\alpha+\beta)}\int_{\mathbb{R}^N}(|\nabla u_n|^2+|\nabla v_n|^2)\\
&\quad + \frac{1}{2(N+\alpha+\beta)}\int_{\mathbb{R}^N}\big((\alpha+\beta-2)A(x)-\nabla A(x)\cdot x\big)u_n^2\\
&\quad + \frac{\alpha+\beta-2}{2(N+\alpha+\beta)}\int_{\mathbb{R}^N}(Bv_n^2+u_n^2|\nabla u_n|^2+v_n^2|\nabla v_n|^2)\\
&\geq \frac{\alpha+\beta}{2(N+\alpha+\beta)}\int_{\mathbb{R}^N}(|\nabla u|^2+|\nabla v|^2+|\nabla\xi_n|^2+|\nabla\gamma_n|^2)\\
&\quad + \frac{1}{2(N+\alpha+\beta)}\int_{\mathbb{R}^N}\big((\alpha+\beta-2)A(x)-\nabla A(x)\cdot x\big)(u^2+\xi_n^2)\\
&\quad + \frac{\alpha+\beta-2}{2(N+\alpha+\beta)}\int_{\mathbb{R}^N}(Bv^2+u^2|\nabla u|^2+v^2|\nabla v|^2+\gamma_n^2+\xi_n^2|\nabla\xi_n|^2\\
&\quad + \gamma_n^2|\nabla\gamma_n|^2)\\
&> \frac{\alpha+\beta}{2(N+\alpha+\beta)}\int_{\mathbb{R}^N}(|\nabla u|^2+|\nabla v|^2+|\nabla\xi_n|^2+|\nabla\gamma_n|^2)\\
&\quad + \frac{1}{2(N+\alpha+\beta)}\int_{\mathbb{R}^N}\big((\alpha+\beta-2)A(x)-\nabla A(x)\cdot x\big)(u^2+t_n^{N+2}\xi_n^2)\\
&\quad + \frac{\alpha+\beta-2}{2(N+\alpha+\beta)}\int_{\mathbb{R}^N}(Bv^2+u^2|\nabla u|^2+v^2|\nabla v|^2+t_n^{N+2}\gamma_n^2\\
&\quad + t_n^{N+2}\xi_n^2|\nabla\xi_n|^2+t_n^{N+2}\gamma_n^2|\nabla\gamma_n|^2)\\
&= I\big((\xi_n)_{t_n},(\gamma_n)_{t_n}\big) + \frac{\alpha+\beta}{2(N+\alpha+\beta)}\int_{\mathbb{R}^N}(|\nabla u|^2+|\nabla v|^2)\\
&\quad + \frac{1}{2(N+\alpha+\beta)}\int_{\mathbb{R}^N}\big((\alpha+\beta-2)A(x)-\nabla A(x)\cdot x\big)u^2\\
&\quad + \frac{\alpha+\beta-2}{2(N+\alpha+\beta)}\int_{\mathbb{R}^N}(Bv^2+u^2|\nabla u|^2+v^2|\nabla v|^2)\\
&\geq m,
\end{aligned}$$

which is also a contradiction.

Therefore, $(u,v) \in \mathcal{M}$. By using Lebesgue's dominated convergence theorem, Fatou's

Lemma, (A_2) and $(u_n, v_n) \in \mathcal{M}$, we may get

$$
\begin{aligned}
m &= I(u,v) - \frac{1}{N+\alpha+\beta}\mathcal{G}(u,v) \\
&= \frac{\alpha+\beta}{2(N+\alpha+\beta)}\int_{\mathbb{R}^N}(|\nabla u|^2 + |\nabla v|^2) \\
&\quad + \frac{1}{2(N+\alpha+\beta)}\int_{\mathbb{R}^N}\big((\alpha+\beta-2)A(x) - \nabla A(x)\cdot x\big)u^2 \\
&\quad + \frac{\alpha+\beta-2}{2(N+\alpha+\beta)}\int_{\mathbb{R}^N}(Bv^2 + u^2|\nabla u|^2 + v^2|\nabla v|^2) \\
&\le \liminf_{n\to+\infty}\Bigg(\frac{\alpha+\beta}{2(N+\alpha+\beta)}\int_{\mathbb{R}^N}(|\nabla u_n|^2 + |\nabla v_n|^2) \\
&\quad + \frac{1}{2(N+\alpha+\beta)}\int_{\mathbb{R}^N}\big((\alpha+\beta-2)A(x) - \nabla A(x)\cdot x\big)u_n^2 \\
&\quad + \frac{\alpha+\beta-2}{2(N+\alpha+\beta)}\int_{\mathbb{R}^N}(Bv_n^2 + u_n^2|\nabla u_n|^2 + v_n^2|\nabla v_n|^2)\Bigg) \\
&= \liminf_{n\to+\infty}\left(I(u_n,v_n) - \frac{1}{N+\alpha+\beta}\mathcal{G}(u_n,v_n)\right) \\
&= m,
\end{aligned}
$$

which implies that $(u_n, v_n) \to (u, v)$ in X and $I(u, v) = m$. □

Having a minimum of $I|_{\mathcal{M}}$, the fact that it is indeed a solution of (1.1), is based on a general idea used in [13, Lemma 2.5].

Proof of Theorem 1.1. Let $(\tilde{u}, \tilde{v}) \in \mathcal{M}$ be a minimizer of the functional $I|_{\mathcal{M}}$. We show that $I'(\tilde{u}, \tilde{v}) = 0$. By Lemma 2.2,

$$
I(\tilde{u},\tilde{v}) = \inf_{(u,v)\in X}\max_{t>0} I(u_t, v_t) = m.
$$

We argue by contradiction by assuming that $(\tilde{u},\ \tilde{v})$ is not a weak solution of (1.1). Then, we can chose $\phi_1,\ \phi_2 \in C_0^\infty(\mathbb{R}^N) \cap \mathcal{V}$ such that

$$
\begin{aligned}
\langle I'(\tilde{u},\tilde{v}), (\phi_1,\phi_2)\rangle = \int_{\mathbb{R}^N}\Bigg(&\nabla\tilde{u}\nabla\phi_1 + \nabla\tilde{v}\nabla\phi_2 + \nabla(\tilde{u}^2)\nabla(\tilde{u}\phi_1) + \nabla(\tilde{v}^2)\nabla(\tilde{v}\phi_2) \\
&+ A(x)\tilde{u}\phi_1 + B\tilde{v}\phi_2 - \frac{2\alpha}{\alpha+\beta}|\tilde{u}|^{\alpha-2}\tilde{u}|\tilde{v}|^{\beta}\phi_1 - \frac{2\beta}{\alpha+\beta}|\tilde{v}|^{\beta-2}\tilde{v}|\tilde{u}|^{\alpha}\phi_2\Bigg) \\
< \ -1.&
\end{aligned}
$$

Then we fix $\varepsilon > 0$ sufficiently small such that

$$
\big\langle I'(\tilde{u}_t + \sigma\phi_1, \tilde{v}_t + \sigma\phi_2), (\phi_1,\phi_2)\big\rangle \le -\frac{1}{2}, \qquad \forall\ |t-1|, \|\sigma| \le \varepsilon
$$

and introduce a cut-off function $0 \le \zeta \le 1$ such that $\zeta(t) = 1$ for $|t-1| \le \frac{\varepsilon}{2}$ and $\zeta(t) = 0$ for $|t-1| \ge \varepsilon$. For $t \ge 0$, we define

$$
\gamma_1(t) := \begin{cases} \tilde{u}_t, & \text{if } |t-1| \ge \varepsilon, \\ \tilde{u}_t + \varepsilon\zeta(t)\phi_1, & \text{if } |t-1| < \varepsilon, \end{cases}
$$

$$\gamma_2(t) := \begin{cases} \tilde{v}_t, & \text{if } |t-1| \geq \varepsilon, \\ \tilde{v}_t + \varepsilon\zeta(t)\phi_2, & \text{if } |t-1| < \varepsilon. \end{cases}$$

Note that $\gamma_1(t)$ and $\gamma_2(t)$ are continuous curve in the metric space (X,d) and, eventually choosing a smaller ε, we get that for $|t-1| < \varepsilon$,

$$d_X\big((\gamma_1(t),\gamma_2(t)),(0,0)\big) > 0.$$

Claim: $\sup_{t\geq 0} I(\gamma_1(t),\gamma_2(t)) < m$.

Indeed, if $|t-1| \geq \varepsilon$, then $I(\gamma_1(t),\gamma_2(t)) = I(\tilde{u}_t,\tilde{v}_t) < I(u,v) = m$. If $|t-1| < \varepsilon$, by using the mean value theorem to the C^1 map $[0,\varepsilon] \ni \sigma \mapsto I(\tilde{u}_t + \sigma\zeta(t)\phi_1, \tilde{v}_t + \sigma\zeta(t)\phi_2) \in \mathbb{R}$, we find, for a suitable $\bar{\sigma} \in (0,\varepsilon)$,

$$\begin{aligned} & I(\tilde{u}_t + \sigma\zeta(t)\phi_1, \tilde{v}_t + \sigma\zeta(t)\phi_2) \\ &\quad = I(\tilde{u}_t,\tilde{v}_t) + \langle I'(\tilde{u}_t + \bar{\sigma}\zeta(t)\phi_1, \tilde{v}_t + \bar{\sigma}\zeta(t)\phi_2), (\zeta(t)\phi_1, \zeta(t)\phi_2)\rangle \\ &\quad \leq I(\tilde{u}_t,\tilde{v}_t) - \frac{1}{2}\zeta(t) \\ &\quad < m. \end{aligned}$$

To conclude, we observe that $\mathcal{G}(\gamma_1(1-\varepsilon),\gamma_2(1-\varepsilon)) > 0$ and $\mathcal{G}(\gamma_1(1+\varepsilon),\gamma_2(1+\varepsilon)) < 0$. By the continuity of the map $t \mapsto \mathcal{G}(\gamma_1(t),\gamma_2(t))$ there exists $t_0 \in (1-\varepsilon, 1+\varepsilon)$ such that $\mathcal{G}(\gamma_1(t_0),\gamma_2(t_0)) = 0$. Namely,

$$(\gamma_1(t_0),\gamma_2(t_0)) = (\tilde{u}_{t_0} + \varepsilon\zeta(t_0)\phi_1, \tilde{v}_{t_0} + \varepsilon\zeta(t_0)\phi_2) \in \mathcal{M}$$

and $I(\gamma_1(t_0),\gamma_2(t_0)) < m$, this is a contradiction.

In addition, from the definition of $\mathcal{V}$ and the fact that $\det(\eta) = -1, (u(\eta x), v(\eta x)) = (\det(\eta)u(x), \det(\eta)v(x)) = (-u(x), -v(x))$. So (u,v) will change sign when (y_1,y_2) cross perpendicularly the half lines $y_2 = \pm y_1 \frac{\tan \pi j}{s}$ $(y_1 \geq 0)$, $j = 1,2,\ldots,s$. Hence (u,v) is a nodal solution with at least $2s$ nodal domains. □

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