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Asymptotic behavior of solutions of quasilinear differential-algebraic equations

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Abstract. This paper is concerned with the asymptotic behavior of solutions of linear differential-algebraic equations (DAEs) under small nonlinear perturbations. Some results on the asymptotic behavior of solutions which are well known for ordinary differential equations are extended to DAEs. The main tools are the projector-based decoupling and the contractive mapping principle. Under certain assumptions on the linear part and the nonlinear term, asymptotic behavior of solutions are characterized. As the main result, a Perron type theorem that establishes the exponential growth rate of solutions is formulated.

Keywords: quasilinear differential-algebraic equation, asymptotic behavior, index, projector, contractive mapping.

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1 Introduction

Qualitative theory and numerical analysis of differential-algebraic equations (DAEs) have been extensively studied since the 80's, see for example the monographs [8,9,11] and the references therein. It is well known that DAEs play an important role in mathematical modeling and arise in many real-life applications such as multibody mechanics, electronic circuit design, chemical engineering, etc, see [4,9,10]. Since the derivative cannot be solved explicitly, DAEs are also called singular (or generalized) systems of differential equations. DAEs are generalizations of ordinary differential equations (ODEs) whose qualitative theory is well known, see [6,7]. Roughly speaking, DAEs are mixed systems of implicit differential and algebraic equations, which may involve hidden constraints as well. The facts that the systems are coupled and the dynamics is constrained makes the analysis and numerical treatment of DAEs more complicated. Even the existence and uniqueness of solutions for linear DAEs can be established only under extra restrictive assumptions. Furthermore, solutions of DAEs may be very sensitive to changes in the system data.

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In this paper, we study the asymptotic behavior of solutions of linear DAEs under nonlinear perturbations

$$Ex'(t) = Ax(t) + f(t, x(t), x'(t)), \qquad t \in \mathbb{I} = [0, \infty), \tag{1.1}$$

where $E, A \in \mathbb{C}^{n\times n}$, $f : \mathbb{I} \times \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n$ is continuous, and E is assumed to be singular. The question is that if a nonlinear perturbation f is supposed to be sufficiently small in some sense, how certain solutions of the quasilinear DAE (1.1) behave asymptotically comparing to those of the unperturbed linear DAE as t tends to infinity. In [12], asymptotic integration of solutions of linear DAEs with coefficients subjected to linear time-varying perturbations was studied. If the perturbations are small enough in some sense, then the exponential growth rate of solutions is established. It is known that the exponential growth rate of solutions of linear systems characterized by Lyapunov exponents plays a very important role in the qualitative study of dynamical systems, see [1]. Characterizations of Lyapunov exponents were extended from linear time-varying ODEs to linear time-varying DAEs in [13–15]. In particular, the stability of Lyapunov exponents is investigated when the coefficients are subjected to structured perturbations in [14, 15]. For some other remarkable results on the asymptotics and stability of solutions for DAEs, see [3, 5, 16, 18].

Our aim is to extend some classical results which are well known for quasilinear ordinary differential equations [6, 7] to quasilinear DAEs. One of the most important results for quasilinear systems is the Perron type theorem which was established for ODEs a long time ago, see [7, Theorem 5, p. 97]. Recently, extensions of this result to functional differential equations [17] and nonautonomous ODEs [2] were done. Unlike the approach in [12], in order to characterize the asymptotic behavior of solutions of (1.1), in this work we use the projector-based approach. Conditions for the pencil (E, A) and perturbation f are given so that the asymptotic behavior of solutions of (1.1) is shown to be related to those of the corresponding linear DAE. The paper is organized as follows. In the next section, we briefly introduce the projector-based analysis of linear DAEs and recall some classical results for quasilinear ODEs. In Section 3, the existence and uniqueness of solutions for the initial value problem for DAE (1.1) are established. A simple example is also given for illustrating the feasibility of the assumptions. Then, in Section 4, the asymptotic behavior of solutions is characterized under certain assumptions. As the main result, a Perron type theorem that establishes the exponential growth rate of solutions is formulated. A discussion and some open questions will close the paper.

2 Preliminaries

2.1 Projector-based analysis for linear DAEs

Consider the linear time-invariant homogeneous DAEs of the form

$$Ex'(t) = Ax(t), \qquad t \in \mathbb{I}, \tag{2.1}$$

where $E, A \in \mathbb{C}^{n\times n}$, E is singular and $x : \mathbb{I} \to \mathbb{C}^n$. As in the classical theory of ODEs, the search for solutions of (2.1) having the form $e^{\lambda t}x_0$ naturally leads to the generalized eigenvalue problem defined by $\det(\lambda E - A) = 0$, and therefore drives the analysis of homogeneous linear time-invariant DAEs to the theory of matrix pencils, see [8–10], where the Kronecker index is used for the analysis of DAEs (2.1).

The matrix pencil $\{E, A\}$ is said to be *regular* if there exists $\lambda \in \mathbb{C}$ such that the determinant $\det(\lambda E - A)$ is nonzero. Otherwise, if $\det(\lambda E - A) = 0$ for all $\lambda \in C$, then we say that $\{E, A\}$ is irregular or non-regular. If $\{E, A\}$ is regular, then $\lambda \in C$ is a (generalized finite) eigenvalue of $\{E, A\}$ and a nonzero vector ζ is the associated eigenvector if $\lambda E\zeta = A\zeta$. It is known that the system (2.1) is solvable if and only if the matrix pencil $\{E, A\}$ is regular [4,8,9]. The following theorem is known as the Kronecker–Weierstraß canonical form, which plays an important role in the analysis of linear constant-coefficient DAEs.

Theorem 2.1. *Suppose that $\{E, A\}$ is a regular pencil. Then, there exist nonsingular matrices G and H such that*

$$GEH = \begin{bmatrix} I_{n_1} & 0 \\ 0 & N \end{bmatrix}, \qquad GAH = \begin{bmatrix} J_{n_1} & 0 \\ 0 & I_{n_2} \end{bmatrix}, \tag{2.2}$$

where $n_1 + n_2 = n$, J_{n_1} is a $n_1 \times n_1$ matrix and N is a matrix of nilpotency index k, i.e. $N^k = 0$, but $N^{k-1} \neq 0$. If N is a zero matrix, then we define $k = 1$.

The *Kronecker index* of the pencil $\{E, A\}$ is defined by the nilpotency index of the matrix N in (2.2).

Now, we suppose that the matrix pencil $\{E, A\}$ is of index one (in the Kronecker sense) and $\operatorname{rank} E = d < n$. Let Q be any projector onto $\ker E$. Then, we have the following result, which is also presented as the definition of tractability index one, see [8].

Proposition 2.2. *Let $E \in \mathbb{C}^{n\times n}$ be a singular matrix, and Q be an arbitrary projector onto* $\ker E$. *Then, the matrix pencil $\{E, A\}$ is regular with Kronecker index one if and only if the matrix $E_1 = E - AQ$ is non-singular.*

Let us define $P = I - Q$, which is a projector, too. It is easy to show that

$$E_1^{-1}E = P, \qquad E_1^{-1}AQ = -Q.$$

Multiplying (2.1) with PE_1^{-1}, QE_1^{-1} and using the relation $x = Px + Qx$, we obtain

$$\begin{aligned} (Px)' &= PE_1^{-1}APx, \\ 0 &= QE_1^{-1}APx - Qx. \end{aligned} \tag{2.3}$$

Denoting $u = Px$ and $v = Qx$, then the first equation of the system (2.3) can be rewritten as

$$u' = PE_1^{-1}Au. \tag{2.4}$$

In addition, we can rewrite the second equation of the system (2.3) as

$$v = QE_1^{-1}Au. \tag{2.5}$$

Thus, equations (2.4) and (2.5) yield a decoupling of the DAE (2.1) in terms of the differential component u and the algebraic one v. The equation (2.4) is called an inherent ODE for the DAE (2.1). The linear subspace $\operatorname{im} P$ is invariant with respect to this equation. Indeed, an initial condition $u_0 \in \operatorname{im} P$ implies $Qu_0 = 0$. Since $(Qu)' = Qu' = QPE_1^{-1}Au = 0$, we obtain $Qu(t) = 0$ i.e., $u(t) \in \operatorname{im} P$ for all t. Solutions of the DAE (2.1) will be described in terms of solutions u of (2.4) lying in the invariant subspace $\operatorname{im} P$. A projector onto $\ker E$ along $S = \{x \in \mathbb{C}^n \big| Ax \in \operatorname{im} E\}$ is called the canonical projector. For index one DAE (2.4), specially, if we choose Q being the canonical projector, then we have $v = 0$, see [8,10].

Using $E = EP$, let us reformulate (2.1) as

$$E(Px)'(t) = Ax(t). \tag{2.6}$$

This makes sense to look for solutions defined in the space

$$C^1_P(\mathbb{I}, \mathbb{C}^n) = \{x \in C(\mathbb{I}, \mathbb{C}^n) \mid Px \in C^1(\mathbb{I}, \mathbb{C}^n)\} \supset C^1(\mathbb{I}, \mathbb{C}^n),$$

where $C^1(\mathbb{I}, \mathbb{C}^n)$ denotes the space of continuously differentiable functions defined on $\mathbb{I}$. Therefore, $x(\cdot) \in C^1_P(\mathbb{I}, \mathbb{C}^n)$ is a solution of (2.1) if and only if it can be written as

$$x(t) = u(t) + v(t), \tag{2.7}$$

where $u \in C^1(\mathbb{I}, \mathbb{C}^n)$ is a solution of (2.4) in the invariant space $\operatorname{im} P$ and $v \in C(\mathbb{I}, \mathbb{C}^n)$ given by (2.5), see [8, 11]. It is sufficient to assign an initial condition to the differential component u. The initial value for the algebraic component v follows from the algebraic constraint, namely $v(0) = QE_1^{-1}Au(0)$.

Now, let us construct a fundamental solution for DAE (2.1) as follows. In space $\operatorname{im} P$, we consider an orthogonal basis $u_1, u_2, \ldots, u_d$. Clearly, $U = (u_1, u_2, \ldots, u_d)$ is an $n \times d$-size matrix and $U^T U = I_d$.

Suppose that $Y(t)$ is a fundamental matrix of the inherent ODE (2.4) restricted on $\operatorname{im} P$, which is defined by the solution of the matrix-valued IVP

$$\begin{aligned} Y'(t) &= PE_1^{-1}AY(t), \\ Y(0) &= U. \end{aligned} \tag{2.8}$$

It is easy to verify that $Y(t) = e^{tPE_1^{-1}A}U$ and the columns of $Y(t)$ are linearly independent solutions of the equation (2.4) restricted on $\operatorname{im} P$. Then, we define a fundamental matrix X of the DAE (2.1) by

$$X(t) = (I + QE_1^{-1}A)Y(t). \tag{2.9}$$

We note that we can also obtain another important associated ODE, the so-called essentially underlying ODE as follows. Let us introduce the change of variables $u(t) = Uw(t)$. Then, w satisfies

$$w'(t) = U^T PE_1^{-1}AUw(t). \tag{2.10}$$

It is also easy to verify that $Z(t) = U^T Y(t)$ is the normalized fundamental solution of the EUODE (2.10) and we also have the representation $Y(t) = UZ(t)$.

The following lemma, which is an extension of Lemma 3.1 [18], characterizes the spectra of the eigenvalue problems associated with the DAE (2.1), the inherent ODE (2.4), and the essentially underlying ODE (2.10).

Lemma 2.3. *Let a regular index-1 pencil $\{E, A\}$ be given and Q denotes an arbitrary projector onto* $\ker E$. *Further,* $M := PE_1^{-1}A$, $N := U^T PE_1^{-1}AU$, $d :=$ rank $E = n - \dim(\ker E)$. *Then* $\deg(\det(\lambda E - A)) = d$, *i.e., $\{E, A\}$ has d finite eigenvalues, say $\lambda_1, \ldots, \lambda_d$. Moreover, $\lambda_1, \ldots, \lambda_d$ belong also to the spectrum of M and they are exactly the same as the eigenvalues of N. The remaining eigenvalues of M are zero.*

If Q is chosen being the canonical projector, then the eigenvectors associated with the finite eigenvalues belong to $\operatorname{im} P$ and the eigenvectors associated with the other zero eigenvalues of M span $\ker P$, see [18, Lemma 3.1].

Proof. Let λ_k be an arbitrary finite eigenvalue of matrix pencil $\{E, A\}$ and ξ_k be an associated eigenvector. From the equality $\lambda_k E\xi_k = A\xi_k$ with $\xi_k = P\xi_k + Q\xi_k$, it is easy to see that

$$\lambda_k P\xi_k = PE_1^{-1}AP\xi_k,$$
$$QE_1^{-1}AP\xi_k = Q\xi_k.$$

This means that ξ_k is an eigenvector corresponding to the eigenvalue λ_k of the pencil $\{E, A\}$ if and only if $P\xi_k$ is an eigenvector corresponding to the eigenvalue λ_k of the matrix $PE_1^{-1}A$. Furthermore, let us define the vector ζ_k by $P\xi_k = U\zeta_k$. Then, we obtain $\lambda_k U\zeta_k = PE_1^{-1}APU\zeta_k$. It follows that $\lambda_k\zeta_k = U^T PE_1^{-1}APU\zeta_k$. This means that λ_k is an eigenvalue and ζ_k is a corresponding eigenvector of N. □

Remark 2.4. It is quite obvious to see that all the solutions of DAE (2.1) are bounded if and only if all the solutions of the inherent ODE (2.4) (and also those of the essential underlying ODE (2.10)) are so. It is also well known that this happens if and only if all the finite eigenvalues of pencil $\{E, A\}$ have non-positive real parts and any eigenvalue with zero real part must be semi-simple.

2.2 Preliminary results for quasilinear ODEs

Consider a special case of (2.1), namely the case of well-known quasilinear ODE

$$x'(t) = Ax(t) + h(t, x), \tag{2.11}$$

i.e., $E = I$, $f(t, x, y) \equiv h(t, x)$, where I is the identity matrix. According to the stability theory of ODEs, if the spectrum $\sigma\{I, A\}$ belong to $\mathbb{C}^-$ and the nonlinear term is sufficiently small in some sense, then the trivial solution is asymptotically stable in Lyapunov sense. This result was extended to DAEs in [16].

Next, we recall some other well-known results on the asymptotic behavior of solutions in the theory of ODEs, see [6,7].

Proposition 2.5 ([6, Problem 1, p. 344])**.** *Let all solutions of the linear system with constant coefficients $y' = Ay$ be bounded for $t \geq 0$, that is, let $\|e^{tA}\| \leq M$, $t \geq 0$, for some constant M. Let h be continuous and let there exist a constant k and a function $\alpha(t)$ such that*

$$\|h(t, x)\| \leq \alpha(t)\|x\| \quad \textit{for } \|x\| \leq k \textit{ and } t \geq 0, \tag{2.12}$$

and let

$$\int_0^\infty \alpha(t)dt < \infty. \tag{2.13}$$

Then, there exists a constant M_1 such that any solution x of the system (2.11) *satisfies*

$$\|x(t)\| < M_1\|x(0)\| \quad \textit{if } \|x(0)\| \leq \frac{k}{M_1}.$$

Proposition 2.6 ([6, Problem 2, p. 345])**.** *Let the assumptions of Proposition 2.5 be satisfied. It is clear that $e^{tA} = X_1(t) + X_2(t)$, where $X_1(t)$ contains elements which are sums of exponential terms $e^{i\lambda_j t}$ for real λ_j and*

$$\|X_1(t)\| \leq K_1, \quad -\infty < t < \infty, \textit{ and}$$
$$\|X_2(t)\| \leq K_2 e^{-\sigma t}, \quad 0 \leq t < \infty$$

for some positive constants $\sigma > 0$, K_1 and K_2.

Then, corresponding to any solution x of (2.11), there is a constant vector p such that

$$x(t) - X_1(t)p \to 0 \quad \text{as } t \to \infty.$$

Theorem 2.7 ([7, Theorem 5, p. 97])**.** *Suppose that $x(t)$ is a bounded solution of (2.11) and*

$$\|h(t, x(t))\| \le \alpha(t)\|x(t)\|, \tag{2.14}$$

for $t \ge 0$, where $\alpha : [0, \infty) \longrightarrow [0, \infty)$ is a continuous nonnegative function satisfying

$$\int_t^{t+1} \alpha(s)ds \to 0 \quad \text{as } t \to \infty, \tag{2.15}$$

then either $x(t) = 0$ for all large t or

$$\mu = \lim_{t\to\infty} \frac{\log \|x(t)\|}{t} \tag{2.16}$$

exists and is equal to the real part of one of the eigenvalues of the matrix A.

Obviously, if $\alpha(t) \to 0$ as $t \to \infty$, or $\int_\sigma^\infty \alpha^p(s)ds < \infty$ for some $p \in [1, \infty)$, then the condition (2.15) holds. This theorem is known as a Perron type theorem for ODEs.

3 Existence and uniqueness of solutions for quasilinear DAEs

Throughout the remainder of the paper, we consider the quasilinear DAE (1.1) where it is assumed that the matrix pencil $\{E, A\}$ is regular of index-1, f is continuous, and the Jacobian $f_y(t, x, y)$ exists.

Clearly, (1.1) generalizes the well-understood case of ODEs (2.11). Now, we focus on the case of singular E, i.e. the equation (1.1) is a DAE. To make sure that the nonlinear perturbation in (1.1) plays a proper role, we need a technical assumption

$$\ker E \subseteq \ker f_y(t, x, y), \qquad (t, x, y) \in \mathbb{I} \times \mathbb{C}^n \times \mathbb{C}^n. \tag{3.1}$$

It is shown, e.g., in [8], that (3.1) is sufficient for implying the identity

$$f(t, x, y) = f(t, x, Py). \tag{3.2}$$

This suggests a more proper reformulation of equation (1.1) as follows

$$E(Px)'(t) = Ax(t) + f(t, x(t), (Px)'(t)). \tag{3.3}$$

This is just a special case of DAEs with properly stated derivative discussed in [10]. We look for solutions of (3.3) that belong to the class $C^1_P(\mathbb{I}, \mathbb{C}^m)$. It is worth mentioning that this class is independent of the choice of projector P, see [8].

First, we establish the (local) existence and uniqueness of solutions of IVPs for (1.1).

Theorem 3.1. *Let pencil $\{E,\ A\}$ be of index-1 and let f satisfy*

$$\|PE_1^{-1}f(t, x, y) - PE_1^{-1}f(t, \bar{x}, \bar{y})\| \le \alpha_1(t)\|x - \bar{x}\| + \beta_1(t)\|y - \bar{y}\|, \tag{3.4}$$

$$\|QE_1^{-1}f(t, x, y) - QE_1^{-1}f(t, \bar{x}, \bar{y})\| \le \alpha_2(t)\|x - \bar{x}\| + \beta_2(t)\|y - \bar{y}\|, \tag{3.5}$$

for all $t \ge 0$ and $x, \bar{x}, y, \bar{y} \in \mathbb{C}^n$, $\alpha_i(t)$ and $\beta_i(t)$ are non-negative bounded functions ($i = 1, 2$ and $t \ge 0$) such that $\sup_{t\in[0,\infty)} \alpha_2(t) < 1$ and $\sup_{t\in[0,\infty)} \gamma(t) < 1$, where $\gamma(t) = \frac{\alpha_1(t)\beta_2(t)}{1-\alpha_2(t)} + \beta_1(t)$. Then, for any $t_1 \ge 0$ and $x^1 \in \mathbb{C}^n$, there exists a positive a such that the IVP for equation (1.1) with initial condition $P(x(t_1) - x^1) = 0$ has a unique solution defined on $[t_1, t_1 + a)$.

Proof. Since $\{E, A\}$ is regular of index-1, we have $QE_1^{-1}A = -Q, E_1^{-1}E = P$ and $PE_1^{-1}A = PE_1^{-1}AP$. Multiplying both sides of (1.1) with PE_1^{-1}, QE_1^{-1} respectively, using the relation $x = Px + Qx$ and noting $\operatorname{im} Q = \ker E \subset \ker f_y(t,x,y)$ for all $t \in \mathbb{I}$, we obtain

$$
\begin{aligned}
(Px)' &= PE_1^{-1}A(Px) + PE_1^{-1}f(t, Px + Qx, (Px)'), \\
Qx &= QE_1^{-1}A(Px) + QE_1^{-1}f(t, Px + Qx, (Px)').
\end{aligned}
$$

Denoting again $u = Px$ and $v = Qx$, then the system can be rewritten as

$$u' = PE_1^{-1}Au + PE_1^{-1}f(t, u + v, u'), \tag{3.6}$$

$$v = QE_1^{-1}Au + QE_1^{-1}f(t, u + v, u'). \tag{3.7}$$

Using on the second equation of (3.7), we will try to represent v by u and u'. Put $F(t,u,u',v) = QE_1^{-1}Au + QE_1^{-1}f(t, u + v, u')$. Due to (3.5), we have

$$\|F(t,x,y,z_1) - F(t,x,y,z_2)\| \le \alpha_2(t)\|z_1 - z_2\|, \quad \text{for all } x,\ y,\ z_1,\ z_2 \in \mathbb{C}^n,\ t \in \mathbb{I}.$$

Since $\sup_{t\in\mathbb{I}} \alpha_2(t) < 1$, $F(t,x,y,z)$ defined as above is a contractive mapping with respect to variable z. Applying the contractive mapping principle, there exists a function $\psi(t,x,y)$ such that $z = \psi(t,x,y)$, i.e.,

$$\psi(t,u,u') = QE_1^{-1}Au + QE_1^{-1}f(t, u + \psi(t,u,u'), u').$$

We can see that ψ is invariant under projector Q, i.e., $Q\psi(t,u,u') = \psi(t,u,u')$. Due to (3.5), we have

$$
\begin{aligned}
&\|\psi(t,x_1,y_1) - \psi(t,x_2,y_2)\| \\
&\quad \le \|QE_1^{-1}A(x_1 - x_2)\| + \|QE_1^{-1}f(t, x_1 + \psi(t,x_1,y_1), y_1) - QE_1^{-1}f(t, x_2 + \psi(t,x_2,y_2), y_2)\|, \\
&\quad \le C_1\|x_1 - x_2\| + \alpha_2(t)\|x_1 - x_2\| + \alpha_2(t)\|\psi(t,x_1,y_1) - \psi(t,x_2,y_2)\| + \beta_2(t)\|y_1 - y_2\|,
\end{aligned}
$$

for all $x_1,\ x_2,\ y_1,\ y_2 \in \mathbb{C}^n,\ t \in \mathbb{I}$, where $C_1 = \|QE_1^{-1}A\|$.

Hence, we get

$$\|\psi(t,x_1,y_1) - \psi(t,x_2,y_2)\| \le \frac{C_1 + \alpha_2(t)}{1 - \alpha_2(t)}\|x_1 - x_2\| + \frac{\beta_2(t)}{1 - \alpha_2(t)}\|y_1 - y_2\|. \tag{3.8}$$

Replacing $v = \psi(t,u,u')$ in (3.6), we obtain

$$u' = PE_1^{-1}Au + PE_1^{-1}f(t, u + \psi(t,u,u'), u').$$

Put $K(t,u,u') = PE_1^{-1}Au + PE_1^{-1}f(t, u + \psi(t,u,u'), u')$. We will show that $K(t,x,y)$ is a contractive mapping with respect to variable y. Indeed, for $x,\ y_1,\ y_2 \in \mathbb{C}^n,\ t \in \mathbb{I}$, we have

$$
\begin{aligned}
\|K(t,x,y_1) - K(t,x,y_2)\| &= \|PE_1^{-1}f(t, x + \psi(t,x,y_1), y_1) - PE_1^{-1}f(t, x + \psi(t,x,y_2), y_2)\| \\
&\le \alpha_1(t)\|\psi(t,x,y_1) - \psi(t,x,y_2)\| + \beta_1(t)\|y_1 - y_2\| \\
&\le \frac{\alpha_1(t)\beta_2(t)}{1 - \alpha_2(t)}\|y_1 - y_2\| + \beta_1(t)\|y_1 - y_2\| \quad \text{(by (3.8))} \\
&\le \Big(\frac{\alpha_1(t)\beta_2(t)}{1 - \alpha_2(t)} + \beta_1(t)\Big)\|y_1 - y_2\| = \gamma(t)\|y_1 - y_2\|.
\end{aligned}
$$

Since $\sup_{t\in\mathbb{I}}\gamma(t) < 1$, it follows that $K(t,x,y)$ is a contraction with respect to variable y. Applying the contractive mapping principle, there exists a function $g(t,x)$ such that $y = g(t,x)$, i.e.,

$$g(t,u) = PE_1^{-1}Au + PE_1^{-1}f(t,u+\psi(t,u,g(t,u)),g(t,u)).$$

Obviously, g is invariant under projector P, i.e., $Pg(t,u) = g(t,u)$. By (3.4) and (3.8), for $u_1,\, u_2 \in \mathbb{C}^n,\ t\in\mathbb{I}$, we have

$$\begin{aligned}
&\|g(t,u_1)-g(t,u_2)\|\\
&\quad\le \|PE_1^{-1}A\|\|u_1-u_2\| + \alpha_1(t)\|u_1-u_2\| + \alpha_1(t)\|\psi(t,u_1,g(t,u_1))-\psi(t,u_2,g(t,u_2))\|\\
&\qquad + \beta_1(t)\|g(t,u_1)-g(t,u_2)\|\\
&\quad\le (C_2+\alpha_1(t))\|u_1-u_2\| + \frac{\alpha_1(t)(C_1+\alpha_2(t))}{1-\alpha_2(t)}\|u_1-u_2\|\\
&\qquad + \frac{\alpha_1(t)\beta_2(t)}{1-\alpha_2(t)}\|g(t,u_1)-g(t,u_2)\| + \beta_1(t)\|g(t,u_1)-g(t,u_2)\| \quad (\text{put } C_2 = \|PE_1^{-1}A\|)\\
&\quad\le \left(C_2 + \frac{(C_1+1)\alpha_1(t)}{1-\alpha_2(t)}\right)\|u_1-u_2\| + \gamma(t)\|g(t,u_1)-g(t,u_2)\|.
\end{aligned}$$

Thus, we have

$$(1-\gamma(t))\|g(t,u_1)-g(t,u_2)\| \le \left(C_2 + \frac{(C_1+1)\alpha_1(t)}{1-\alpha_2(t)}\right)\|u_1-u_2\|,$$

i.e.,

$$\|g(t,u_1)-g(t,u_2)\| \le \frac{1}{1-\gamma(t)}\left(C_2 + \frac{(C_1+1)\alpha_1(t)}{1-\alpha_2(t)}\right)\|u_1-u_2\|. \tag{3.9}$$

Put $\widetilde{g}(t,u) = g(t,u) - PE_1^{-1}Au$. Then $\widetilde{g}$ is also invariant under projector P, i.e., $P\widetilde{g}(t,u) = \widetilde{g}(t,u)$. We have

$$\begin{aligned}
\|\widetilde{g}(t,u_1)-\widetilde{g}(t,u_2)\| &\le \alpha_1(t)\|u_1-u_2\| + \alpha_1(t)\|\psi(t,u_1,g(t,u_1))-\psi(t,u_2,g(t,u_2))\|\\
&\quad + \beta_1(t)\|g(t,u_1)-g(t,u_2)\|\\
&\le \alpha_1(t)\|u_1-u_2\| + \frac{\alpha_1(t)(C_1+\alpha_2(t))}{1-\alpha_2(t)}\|u_1-u_2\|\\
&\quad + \frac{\alpha_1(t)\beta_2(t)}{1-\alpha_2(t)}\|g(t,u_1)-g(t,u_2)\| + \beta_1(t)\|g(t,u_1)-g(t,u_2)\|\\
&\le \frac{(C_1+1)\alpha_1(t)}{1-\alpha_2(t)}\|u_1-u_2\| + \frac{\gamma(t)}{1-\gamma(t)}\left(C_2 + \frac{(C_1+1)\alpha_1(t)}{1-\alpha_2(t)}\right)\|u_1-u_2\|\\
&\le \left[\frac{(C_1+1)\alpha_1(t)}{(1-\alpha_2(t))(1-\gamma(t))} + \frac{C_2\gamma(t)}{1-\gamma(t)}\right]\|u_1-u_2\|.
\end{aligned}$$

Thus, we get

$$\|\widetilde{g}(t,u_1)-\widetilde{g}(t,u_2)\| \le \widetilde{\gamma}(t)\|u_1-u_2\|, \tag{3.10}$$

where

$$\widetilde{\gamma}(t) = \frac{(C_1+1)\alpha_1(t)}{(1-\alpha_2(t))(1-\gamma(t))} + \frac{C_2\gamma(t)}{1-\gamma(t)}. \tag{3.11}$$

On the other hand, $\alpha_1(t)$ is bounded, $\sup_{t\in[0,\infty)}\alpha_2(t) < 1$ and $\sup_{t\in[0,\infty)}\gamma(t) < 1$, then there exists a positive constant L such that $\|\widetilde{g}(t,u_1)-\widetilde{g}(t,u_2)\| \le L\|u_1-u_2\|$, for all $u_1,u_2\in\mathbb{C}^n$, i.e., $\widetilde{g}(t,u)$ is Lipschitz continuous with respect to u.

We conclude that system (1.1) can be reduced to the decoupled form

$$u' = PE_1^{-1}Au + \widetilde{g}(t,u), \tag{3.12}$$
$$v = \psi(t,u,u'). \tag{3.13}$$

and initial condition $P(x(t_1) - x^1) = 0$ is equivalent to $u(t_1) = Px^1$. Since $\widetilde{g}(t,u)$ is a Lipschitz continuous function for u, the IVP for equation (3.12) with initial condition $u(t_1) = Px^1$ has a unique solution $u(t)$ defined on $[t_1, t_1 + a)$ for some positive number a and this solution satisfies $Pu(t) = u(t)$. Then, we obtain $v(t)$ from (3.13). Hence, the unique solution $x(t)$ is defined by $x(t) = u(t) + v(t)$ for all $t \in [t_1, t_1 + a)$. The proof is complete. □

We present a simple example that illustrates the feasibility of the conditions given in Theorem 3.1.

Example 3.2. We consider the equation

$$Ex' = Ax + f(t,x,x') \tag{3.14}$$

with

$$E = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 3 & 0 \\ 1 & 2 & 0 \end{bmatrix}, \qquad A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}.$$

The nonlinear part $f = (f_1, f_2, f_3)^\top$ will be specified later. Let us choose

$$Q = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

It is seen that Q is a projection onto $\ker E$ and the matrix pencil $\{E, A\}$ is of index-1. Furthermore, we have

$$PE_1^{-1} = \begin{bmatrix} 4 & -2 & -1 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad QE_1^{-1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix}.$$

Let us define f such that

$$PE_1^{-1}f(t,x,y) = \begin{bmatrix} a_1(t)\sin(x_2) + b_1(t)\cos(y_1) \\ a_2(t)\sin(x_3) + a_3(t)\cos(x_2) \\ 0 \end{bmatrix}$$

and

$$QE_1^{-1}f(t,x,y) = \begin{bmatrix} 0 \\ 0 \\ -a_4(t)\cos(x_3) - b_2(t)\cos(y_2) \end{bmatrix}.$$

Choose $a_i(t) = \frac{\delta_i}{(1+t)^2}$, $(i = 1,2,3,4)$, $b_1(t) = \frac{\varepsilon_1}{(1+t)^2}$, and $b_2(t) = \frac{1-\delta_4}{(1+t)^2}$, where δ_i and ε_1 are positive constants such that $\delta_4 < 1$.

Using the maximum norm, for any $t \geq 0$ and any $x, \bar{x}, y, \bar{y} \in \mathbb{R}^3$, we have

$$\begin{aligned} \|PE_1^{-1}f(t,x,y) - PE_1^{-1}f(t,\bar{x},\bar{y})\| \leq &\max\{a_1(t)|\sin(x_2) - \sin(\bar{x}_2)| + b_1(t)|\cos(y_1) - \cos(\bar{y}_1)|, \\ &a_2(t)|\sin(x_3) - \sin(\bar{x}_3)| + a_3(t)|\cos(x_2) - \cos(\bar{x}_2)|\} \\ \leq &(a_1(t) + a_2(t) + a_3(t))\|x - \bar{x}\| + b_1(t)\|y - \bar{y}\| \end{aligned}$$

and

$$\|QE_1^{-1}f(t,x,y) - QE_1^{-1}f(t,\bar{x},\bar{y})\| \le a_4(t)|\cos(x_3) - \cos(\bar{x}_3)| + b_2(t)|\cos(y_2) - \cos(\bar{y}_2)|$$
$$\le a_4(t)\|x - \bar{x}\| + b_2(t)\|y - \bar{y}\|.$$

We put

$$\alpha_1(t) = \frac{\delta_1 + \delta_2 + \delta_3}{(1+t)^2}, \qquad \beta_1(t) = \frac{\varepsilon_1}{(1+t)^2}, \qquad \alpha_2(t) = \frac{\delta_4}{(1+t)^2}, \qquad \beta_2 = \frac{1-\delta_4}{(1+t)^2}.$$

It is trivial to see that, for t in $[0,\infty)$, the following estimates hold:

$$0 < \alpha_1(t) \le \delta_1 + \delta_2 + \delta_3, \qquad 0 < \alpha_2(t) \le \delta_4,$$
$$0 < \beta_1(t) \le \varepsilon_1, \qquad 0 < \beta_2(t) \le 1 - \delta_4,$$
$$0 < \gamma(t) = \frac{\alpha_1(t)\beta_2(t)}{1 - \alpha_2(t)} + \beta_1(t) < \delta_1 + \delta_2 + \delta_3 + \varepsilon_1.$$

Therefore, if $\delta_4 < 1$ and $\delta_1 + \delta_2 + \delta_3 + \varepsilon_1 < 1$ simultaneously hold, then all the conditions in Theorem 3.1 are satisfied. We conclude that for any $t_1 \in [0,\infty)$ and $x^1 \in \mathbb{R}^3$, the IVP for equation (3.14) with initial condition $P(x(t_1) - x^1) = 0$ has a unique solution defined in $[t_1, t_1 + a)$ with some positive number a.

4 Asymptotic behavior of solutions for quasilinear DAEs

In this section, we extend the results in Section 2.2 to quasilinear DAEs of the form (1.1).

Theorem 4.1. *Let pencil $\{E, A\}$ be regular of index-1 and let f satisfy all the conditions in Theorem 3.1 with*

$$\int_0^\infty \alpha_1(t)dt < \infty, \qquad \int_0^\infty \beta_1(t)dt < \infty, \tag{4.1}$$

and $f(t,0,0) \equiv 0$. Let all the solutions of the linear DAE (2.1) be bounded for $t \ge 0$, i.e., there exists a positive constant M such that the fundamental matrix $X(t)$ of (2.1) satisfies $\|X(t)\| \le M$ for all $t \ge 0$. Then, there exists positive constant M_1 such that any solution $x = x(t)$ of the system (1.1) satisfies $\|x(t)\| \le M_1\|x(0)\|$ for all $t \ge 0$.

Proof. In space $\operatorname{im} P$, let us consider again an orthogonal basis $u_1, u_2, \ldots, u_d$. We denote $U = (u_1, u_2, \ldots, u_d)$ which is a $n \times d$-size matrix and $U^T U = I_d$. Using the change of variables $u(t) = Uw(t)$, it is easy to see that from the equation (3.12) we obtain the EUODE

$$w'(t) = U^T P E_1^{-1} A U w(t) + U^T \tilde{g}(t, Uw(t)). \tag{4.2}$$

Due to Lemma 2.3, the spectra of pencil $\{E, A\}$ and of $N = U^T P E_1^{-1} A U$ coincide. Put $\bar{g}(t, w(t)) = U^T \tilde{g}(t, Uw(t))$. Clearly, the equation (4.2) is an ODE for $w(t)$. From (3.10), together with (4.1), $\alpha_1(t)$ and $\beta_2(t)$ are bounded, $\sup_{t\in[0,\infty)} \alpha_2(t) < 1$ and $\sup_{t\in[0,\infty)} \gamma(t) < 1$, $\tilde{g}(t,0) = 0$, we have

$$\|\tilde{g}(t,u)\| \le \tilde{\gamma}(t)\|u\|,$$

where $\tilde{\gamma}(t)$ defined in (3.11) satisfies $\int_0^\infty \tilde{\gamma}(t)dt < \infty$. Without loss of generality let us use the Euclidean norm, due to the definition of U and the properties that $u(t) = Pu(t)$, $\tilde{g}(t,u) = P\tilde{g}(t,u)$, we obtain $\|u(t)\| = \|w(t)\|$ and

$$\|\bar{g}(t, w(t))\| = \|U^T \tilde{g}(t, Uw(t))\| = \|\tilde{g}(t, Uw(t))\| \le \tilde{\gamma}(t)\, \|Uw(t)\| = \tilde{\gamma}(t)\, \|w(t)\|.$$

Thus, we get

$$\|\overline{g}(t, w(t))\| \leq \widetilde{\gamma}(t)\|w(t)\|, \quad \text{for all } t \geq 0. \tag{4.3}$$

Due to the properties of $\overline{g}$ and $\widetilde{\gamma}$, by Proposition 2.5, one concludes that if the equation (4.2) has a solution $w(t)$ defined for all $t \in [0, \infty)$, there exists a constant $\widetilde{M}$ such that

$$\|w(t)\| < \widetilde{M}\|w(0)\|.$$

Then, $u(t) = Uw(t)$ is a solution of the equation (3.12) such that

$$\|u(t)\| = \|w(t)\| < \widetilde{M}\|w(0)\| = \widetilde{M}\|u(0)\|.$$

The equation (1.1) has the solution of the form $x(t) = u(t) + v(t)$. Therefore, we obtain $x(t) = u(t) + \psi(t, u(t), g(t, u(t)))$. Note that, since $f(t, 0, 0) \equiv 0$, it is not difficult to show that $\psi(t, 0, 0) \equiv 0$, $g(t, 0) \equiv 0$ and $\widetilde{g}(t, 0) \equiv 0$. Then, we have

$$\begin{aligned}
\|x(t)\| &\leq \|u(t)\| + \|\psi(t, u(t), g(t, u(t)))\| \\
&\leq \|u(t)\| + \frac{(C_1 + \alpha_2(t))}{1 - \alpha_2(t)}\|u(t)\| + \frac{\beta_2(t)}{1 - \alpha_2(t)}\|g(t, u(t))\| \quad \text{(by (3.8))} \\
&\leq \frac{C_1 + 1}{1 - \alpha_2(t)}\|u(t)\| + \frac{\beta_2(t)}{(1 - \alpha_2(t))(1 - \gamma(t))}\left(C_2 + \frac{(C_1 + 1)\alpha_1(t)}{1 - \alpha_2(t)}\right)\|u(t)\| \quad \text{(by (3.9))} \\
&\leq \frac{1}{1 - \alpha_2(t)}\left[C_1 + 1 + \frac{\beta_2(t)}{1 - \gamma(t)}\left(C_2 + \frac{(C_1 + 1)\alpha_1(t)}{1 - \alpha_2(t)}\right)\right]\|u(t)\|.
\end{aligned}$$

In addition, $\alpha_1(t)$ and $\beta_2(t)$ are bounded, $\sup_{t\in[0,\infty)} \alpha_2(t) < 1$ and $\sup_{t\in[0,\infty)} \gamma(t) < 1$, there exists a constant $\bar{K}$ such that

$$\|x(t)\| \leq \bar{K}\|u(t)\|. \tag{4.4}$$

Therefore, we obtain

$$\|x(t)\| \leq \bar{K}\|u(t)\| \leq \bar{K}\widetilde{M}\|u(0)\| \leq \bar{K}\widetilde{M}\|Px(0)\| \leq \bar{K}\widetilde{M}\|P\|\|x(0)\|.$$

Thus, by setting $M_1 = \bar{K}\widetilde{M}\|P\|$, we get

$$\|x(t)\| < M_1\|x(0)\|, \quad \text{for all } t \in [0, \infty).$$

The proof of Theorem 4.1 is complete. □

The boundedness of the solutions of DAE (2.1) implies that

$$Z(t) = e^{tU^T PE_1^{-1}AU} = Z_1(t) + Z_2(t), \tag{4.5}$$

where $Z_1(t)$ contains elements which are sums of exponential terms $e^{i\lambda_j t}$ for real λ_j and

$$\|Z_1(t)\| \leq K_1, \quad -\infty < t < \infty, \tag{4.6}$$

$$\|Z_2(t)\| \leq K_2 e^{-\sigma t}, \quad 0 \leq t < \infty \tag{4.7}$$

for some $\sigma > 0$, where K_1 and K_2 are constants. From (4.5) and (2.10) we have

$$Y(t) = UZ(t) = UZ_1(t) + UZ_2(t). \tag{4.8}$$

Thus, the fundamental matrix $X(t)$ of equation (2.1) can be decomposed as

$$X(t) = X_1(t) + X_2(t), \tag{4.9}$$

where

$$X_1(t) = (I + QE_1^{-1}A)UZ_1(t),\ X_2(t) = (I + QE_1^{-1}A)UZ_2(t).$$

Therefore, the estimates

$$\|X_1(t)\| \le \overline{K}_1, \quad -\infty < t < \infty, \tag{4.10}$$

$$\|X_2(t)\| \le \overline{K}_2 e^{-\sigma t}, \quad 0 \le t < \infty \tag{4.11}$$

hold for some positive constants $\sigma > 0$, $\overline{K}_1$ and $\overline{K}_2$.

Theorem 4.2. *Let the assumptions of Theorem 4.1 be satisfied. Moreover, let $\alpha_2(t) \to 0$ and $\beta_2(t) \to 0$ as $t \to \infty$ hold. Then, for any solution x of (1.1), there is a constant vector $p \in \mathbb{R}^d$ such that*

$$x(t) - X_1(t)p \to 0 \quad \text{as } t \to \infty.$$

Proof. Given a solution x, let us define u, v and w as above. By Proposition 2.6, there exists a constant vector $p \in \mathbb{R}^d$ such that

$$\lim_{t\to\infty}[w(t) - Z_1(t)p] = 0$$

i.e.,

$$\lim_{t\to\infty}[u(t) - UZ_1(t)p] = 0,$$

where $u(t)$ is a solution of (3.12).

On the other hand, the solution of (1.1) of the form $x(t) = u(t) + \psi(t, u(t), g(t, u(t)))$. Thus, we obtain

$$\begin{aligned} x(t) - X_1(t)p &= u(t) + \psi(t, u(t), g(t, u(t))) - (I + QE_1^{-1}A)UZ_1(t)p \\ &= u(t) + QE_1^{-1}Au(t) + QE_1^{-1}f(t, u(t) + \psi(t, u(t), g(t, u(t))), g(t, u(t))) \\ &\quad - (I + QE_1^{-1}A)UU_1(t)p \\ &= (I+QE_1^{-1}A)[u(t)-UZ_1(t)p] + QE_1^{-1}f(t, u(t)+\psi(t, u(t), g(t, u(t))), g(t, u(t))). \end{aligned}$$

Therefore, we have the following inequality

$$\begin{aligned} \|x(t) - X_1(t)p\| \le{}& \|(I + QE_1^{-1}A)(u(t) - UZ_1(t)p)\| \\ &+ \|QE_1^{-1}f(t, u(t) + \psi(t, u(t), g(t, u(t))), g(t, u(t)))\|. \end{aligned}$$

Moreover, we have the following estimate

$$\begin{aligned} &\|QE_1^{-1}f(t, u(t) + \psi(t, u(t), g(t, u(t))), g(t, u(t)))\| \\ &\quad \le \alpha_2(t)\|u(t) + \psi(t, u(t), g(t, u(t)))\| + \beta_2(t)\|g(t, u(t))\| \\ &\quad \le \alpha_2(t)\left(\|u(t)\| + \frac{C_1 + \alpha_2(t)}{1 - \alpha_2(t)}\|u(t)\|\right) + \frac{\alpha_2(t)\beta_2(t)}{1 - \alpha_2(t)}\|g(t, u(t))\| + \beta_2(t)\|g(t, u(t))\| \\ &\quad \le \frac{(C_1 + 1)\alpha_2(t)}{1 - \alpha_2(t)}\|u(t)\| + \frac{\beta_2(t)}{1 - \alpha_2(t)}\|g(t, u(t))\| \\ &\quad \le \frac{(C_1 + 1)\alpha_2(t)}{1 - \alpha_2(t)}\|u(t)\| + \frac{\beta_2(t)}{1 - \alpha_2(t)}\frac{1}{1 - \gamma(t)}\left(C_2 + \frac{(C_1 + 1)\alpha_1(t)}{1 - \alpha_2(t)}\right)\|u(t)\| \\ &\quad \le \left[\frac{(C_1 + 1)\alpha_2(t)}{1 - \alpha_2(t)} + \frac{\beta_2(t)}{1 - \alpha_2(t)}\frac{1}{1 - \gamma(t)}\left(C_2 + \frac{(C_1 + 1)\alpha_1(t)}{1 - \alpha_2(t)}\right)\right]\|u(t)\| \\ &\quad \le K(t)\|u(t)\|, \end{aligned}$$

where

$$K(t) = \frac{(C_1+1)\alpha_2(t)}{1-\alpha_2(t)} + \frac{\beta_2(t)}{1-\alpha_2(t)}\frac{1}{1-\gamma(t)}\left(C_2 + \frac{(C_1+1)\alpha_1(t)}{1-\alpha_2(t)}\right).$$

Since $\lim_{t\to\infty}\alpha_2(t) = 0, \lim_{t\to\infty}\beta_2(t) = 0$, we have $\lim_{t\to\infty}K(t) = 0$. Thus, we obtain

$$\|x(t) - X_1(t)p\| \le \|I + QE_1^{-1}A\|\|u(t) - UZ_1(t)p\| + K(t)\|u(t)\| \to 0$$

as $t \to \infty$, because $u(t)$ is bounded for $t \ge 0$. The proof of Theorem 4.2 is complete. □

Theorem 4.3. *Let the assumptions in Theorem 3.1 be satisfied and let*

$$\int_t^{t+1}\alpha_1(s)ds \to 0, \qquad \int_t^{t+1}\beta_1(s)ds \to 0 \quad \text{as } t \to \infty. \tag{4.12}$$

Furthermore $f(t,0,0) = 0$. Suppose that x is a bounded solution of (1.1). Then, either

i) the limit

$$\lim_{t\to\infty}\frac{\ln\|x(t)\|}{t}$$

exists and is equal to the real part of one of the eigenvalues of the pencil matrix $\{E, A\}$, or

ii) $x(t) = 0$ for all large t.

Proof. Since $f(t,0,0) = 0$, equation (1.1) has the trivial solution. We consider again the EUODE (4.2), where $\|U^T\widetilde{g}(t,Uw(t))\| \le \widetilde{\gamma}(t)\|w(t)\|$ and $\int_t^{t+1}\widetilde{\gamma}(s)ds \to 0$ as $t \to \infty$. By Theorem 2.7, the solution $w(t)$ of (4.2) satisfies either the limit $\lim_{t\to\infty}\frac{\ln\|w(t)\|}{t}$ exists and is equal to the real part of one of the eigenvalues of the matrix $N = U^TPE_1^{-1}AU$, or $w(t) = 0$ for all large t. Therefore, the solution $u(t)$ of (3.12) which satisfies either the limit $\lim_{t\to\infty}\frac{\ln\|u(t)\|}{t}$ exists and is equal to the real part of one of the finite eigenvalues of the matrix pencil $\{E, A\}$ or $u(t) = 0$ for all large t.

On the other hand, from $x(t) = u(t) + \psi(t,u(t),g(t,u(t)))$ it follows that the following estimate holds:

$$\begin{aligned}
\|x(t)\| &\le \|u(t)\| + \|\psi(t,u(t),g(t,u(t)))\| \\
&\le \|u(t)\| + \frac{C_1+\alpha_2(t)}{1-\alpha_2(t)}\|u(t)\| + \frac{\beta_2(t)}{1-\alpha_2(t)}\|g(t,u(t)))\| \\
&\le \frac{C_1+1}{1-\alpha_2(t)}\|u(t)\| + \frac{1}{1-\gamma(t)}\frac{\beta_2(t)}{1-\alpha_2(t)}\left(C_2 + \frac{(C_1+1)\alpha_1(t)}{1-\alpha_2(t)}\right)\|u(t)\| \\
&\le \frac{1}{1-\alpha_2(t)}\left[1 + C_1 + \frac{\beta_2(t)}{1-\gamma(t)}\left(C_2 + \frac{(C_1+1)\alpha_1(t)}{1-\alpha_2(t)}\right)\right]\|u(t)\|.
\end{aligned}$$

Put $\sup_{t\in[0,\infty)}\alpha_i = a_i,\ \sup\limits_{t\in[0,\infty)}\beta_i = b_i,\ i = 1,2$. We obtain the following inequality

$$\|x(t)\| \le \widetilde{K}\|u(t)\|, \quad t \ge 0, \tag{4.13}$$

where

$$\widetilde{K} = \frac{1}{1-a_2}\left[C_1 + 1 + \frac{b_2}{1 - \frac{a_1b_2}{1-a_2} - b_1}\left(C_2 + \frac{(C_1+1)a_1}{1-a_2}\right)\right].$$

Obviously, if $u(t) = 0$ for all large t then $x(t) = 0$ for all $t \ge 0$, too. Otherwise, from the inequality (4.13) it follows that

$$\lim_{t\to\infty}\frac{\ln\|x(t)\|}{t} \le \lim_{t\to\infty}\frac{\ln\|u(t)\|}{t}. \tag{4.14}$$

Moreover, we have

$$\|u(t)\| = \|Px(t)\| \le \|P\|\|x(t)\|, \qquad \forall t \ge 0.$$

Therefore, we obtain

$$\lim_{t\to\infty} \frac{\ln\|u(t)\|}{t} \le \lim_{t\to\infty} \frac{\ln\|x(t)\|}{t}. \tag{4.15}$$

Thus, from (4.14) and (4.15), we have

$$\lim_{t\to\infty} \frac{\ln\|x(t)\|}{t} = \lim_{t\to\infty} \frac{\ln\|u(t)\|}{t}.$$

By Theorem 2.7 and Lemma 2.3, we conclude that the limit $\lim_{t\to\infty} \frac{\ln\|x(t)\|}{t}$ exists and it is equal to the real part of one of the eigenvalues of the matrix pencil $\{E, A\}$. The proof of Theorem 4.3 is complete. □

Remark 4.4. As a special case, if the nonlinear term f does not involve the derivative term x', i.e., DAE (1.1) becomes

$$Ex'(t) = Ax(t) + f(t, x), \qquad t \in \mathbb{I}, \tag{4.16}$$

then the situation is simpler and the assumptions in Theorem 3.1 can be significantly simplified. Namely, instead of (3.4) and (3.5), we assume

$$\begin{aligned} \|PE_1^{-1}f(t,x) - PE_1^{-1}f(t,\bar{x})\| &\le \alpha_1(t)\|x - \bar{x}\|, \\ \|QE_1^{-1}f(t,x) - QE_1^{-1}f(t,\bar{x})\| &\le \alpha_2(t)\|x - \bar{x}\| \end{aligned} \tag{4.17}$$

for all $t \ge 0$ and $x, \bar{x} \in \mathbb{C}^n$, $\alpha_i(t)$ are non-negative bounded functions ($i = 1, 2$) such that $\sup_{t\in[0,\infty)} \alpha_2(t) < 1$. Furthermore, all the results in Section 4 can be stated analogously under appropriately reduced assumptions for $f(t, x)$, too.

5 Discussion

In this paper we have studied the asymptotic behavior of solutions for quasilinear DAEs, where the linear part is a DAE of index one and the nonlinearity is assumed to be small in some sense. As the main results, we have shown that any non-vanishing, bounded solution has the strict Lyapunov exponent which coincides with one of the Lyapunov exponents of the linear system. Since the coefficients of the linear system are constant, one might use alternatively the more simple Kronecker–Weierstraß decomposition or the Singular Value Decomposition as in [12] for decoupling. However, these tools will not work for time-varying systems, in general. Here we prefer using the projector-based approach because as a future problem, we want to use this approach to extend the results to quasilinear DAEs whose linear part is time-varying. The derivative should be properly stated as in [10]. This problem expects more technical difficulties since the Lyapunov spectrum of a linear time-varying system may be unstable under infinitesimally small perturbations.

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