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ON THE MAXIMUM POSITIVE SEMI-DEFINITE NULLITY AND THE CYCLE MATROID OF GRAPHS*

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Abstract. Let $G = (V, E)$ be a graph with $V = \{1, 2, \ldots, n\}$, in which we allow parallel edges but no loops, and let $\mathcal{S}_+(G)$ be the set of all positive semi-definite $n \times n$ matrices $A = [a_{i,j}]$ with $a_{i,j} = 0$ if $i \neq j$ and i and j are non-adjacent, $a_{i,j} \neq 0$ if $i \neq j$ and i and j are connected by exactly one edge, and $a_{i,j} \in \mathbb{R}$ if $i = j$ or i and j are connected by parallel edges. The maximum positive semi-definite nullity of G, denoted by $M_+(G)$, is the maximum nullity attained by any matrix $A \in \mathcal{S}_+(G)$. A k-separation of G is a pair of subgraphs (G_1, G_2) such that $V(G_1) \cup V(G_2) = V$, $E(G_1) \cup E(G_2) = E$, $E(G_1) \cap E(G_2) = \emptyset$ and $|V(G_1) \cap V(G_2)| = k$. When G has a k-separation (G_1, G_2) with $k \leq 2$, we give a formula for the maximum positive semi-definite nullity of G in terms of G_1, G_2, and in case of $k = 2$, also two other specified graphs. For a graph G, let c_G denote the number of components in G. As a corollary of the result on k-separations with $k \leq 2$, we obtain that $M_+(G) - c_G = M_+(G') - c_{G'}$ for graphs G and G' that have isomorphic cycle matroids.

Key words. Positive semi-definite matrices, Nullity, Graphs, Separation, Matroids.

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