

SEMIDEFINITE GEOMETRY OF THE NUMERICAL RANGE*

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Abstract. The numerical range of a matrix is studied geometrically via the cone of positive semidefinite matrices (or semidefinite cone for short). In particular, it is shown that the feasible set of a two-dimensional linear matrix inequality (LMI), an affine section of the semidefinite cone, is always dual to the numerical range of a matrix, which is therefore an affine projection of the semidefinite cone. Both primal and dual sets can also be viewed as convex hulls of explicit algebraic plane curve components. Several numerical examples illustrate this interplay between algebra, geometry and semidefinite programming duality. Finally, these techniques are used to revisit a theorem in statistics on the independence of quadratic forms in a normally distributed vector.

Key words. Numerical range, Semidefinite programming, LMI, Algebraic plane curves.

AMS subject classifications. 14H50, 14Q05, 47A12, 52A10, 90C22.

*Received by the editors March 8, 2009. Accepted for publication May 28, 2010. Handling Editor: Shmuel Friedland.

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