The Electronic Journal of Linear Algebra.
A publication of the International Linear Algebra Society.
Volume 1, pp. 1-17, June 1996.
ISSN 1081-3810. http://gauss.technion.ac.il/iic/ela

ELA

# NUMERICAL RANGES OF AN OPERATOR ON AN INDEFINITE INNER PRODUCT SPACE*

CHI-KWONG LI†, NAM-KIU TSING‡ AND FRANK UHLIG§

*Dedicated to Chandler Davis*

**Abstract.** For $n \times n$ complex matrices $A$ and an $n \times n$ Hermitian matrix $S$, we consider the $S$-*numerical range* of $A$ and the *positive* $S$-*numerical range* of $A$ defined by

$$W_S(A) = \left\{ \frac{\langle Av, v\rangle_S}{\langle v, v\rangle_S} : v \in \mathbb{C}^n, \langle v, v\rangle_S \neq 0 \right\}$$

and

$$W_S^+(A) = \{\langle Av, v\rangle_S : v \in \mathbb{C}^n, \langle v, v\rangle_S = 1\},$$

respectively, where $\langle u, v\rangle_S = v^*Su$. These sets generalize the classical numerical range, and they are closely related to the joint numerical range of three Hermitian forms and the cone generated by it. Using some theory of the joint numerical range we can give a detailed description of $W_S(A)$ and $W_S^+(A)$ for arbitrary Hermitian matrices $S$. In particular, it is shown that $W_S^+(A)$ is always convex and $W_S(A)$ is always $p$-convex for all $S$. Similar results are obtained for the sets

$$V_S(A) = \left\{ \frac{\langle Av, v\rangle}{\langle Sv, v\rangle} : v \in \mathbb{C}^n, \langle Sv, v\rangle \neq 0 \right\}, \quad V_S^+(A) = \{\langle Av, v\rangle : v \in \mathbb{C}^n, \langle Sv, v\rangle = 1\},$$

where $\langle u, v\rangle = v^*u$. Furthermore, we characterize those linear operators preserving $W_S(A)$, $W_S^+(A)$, $V_S(A)$, or $V_S^+(A)$. Possible generalizations of our results, including their extensions to bounded linear operators on an infinite dimensional Hilbert or Krein space, are discussed.

**Key words.** Field of values, numerical range, generalized numerical range, Krein space, convexity, linear preserver, indefinite inner product space.

**AMS(MOS) subject classification.** 15A60, 15A63, 15A45, 46C20, 52A30

* Received by the editors on 19 December 1995. Final manuscript accepted by Daniel Hershkowitz on 29 May 1996.

† Department of Mathematics, College of William and Mary, Williamsburg, VA 23187, USA (ckli@cs.wm.edu)

‡ Department of Mathematics, The University of Hong Kong, Hong Kong (nktsing@hkuxa.hku.hk)

§ Department of Mathematics, Auburn University, Auburn, AL 36849-5310, USA (uhligfd@mail.auburn.edu)