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THE NONNEGATIVE P_0-MATRIX COMPLETION PROBLEM[*]

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Abstract. In this paper the nonnegative P_0-matrix completion problem is considered. It is shown that a pattern for 4×4 matrices that includes all diagonal positions has nonnegative P_0-completion if and only if its digraph is complete when it has a 4-cycle. It is also shown that any positionally symmetric pattern that includes all diagonal positions and whose graph is an n-cycle has nonnegative P_0-completion if and only if $n \neq 4$.

Key words. Matrix completion, P_0-matrix, Nonnegative P_0-matrix, L-digraph, n-cycle.

AMS subject classifications. 15A48.

1. Introduction. A *partial matrix* is a rectangular array of numbers in which some entries are specified while others are free to be chosen. A *completion* of a partial matrix is a specific choice of values for the unspecified entries. Let $\mathcal{N} = \{1, \ldots, n\}$. A *pattern* for $n \times n$ matrices is a subset of $\mathcal{N} \times \mathcal{N}$. A partial matrix *specifies a pattern* if its specified entries lie exactly in those positions listed in the pattern. For a particular class Π of matrices, we say a pattern *has* Π*-completion* if every partial Π-matrix specifying the pattern can be completed to a Π-matrix. The Π-matrix completion problem for patterns is to determine which patterns have Π-completion. For example, the positive definite completion problem asks, "Which patterns have the property that any partial positive definite matrix specifying the pattern can be completed to a positive definite matrix?" The answer to this question is given in [4] through the use of graph theoretic methods. Matrix completion problems arise in applications whenever a full set of data is not available, but it is known that the full matrix of data must have certain properties. Such applications include seismic reconstruction problems and data transmission, coding, and image enhancement problems in electrical and computer engineering.

A *positionally symmetric* pattern is a pattern with the property that (i, j) is in the pattern if and only if (j, i) is also in the pattern. An *asymmetric* pattern is a pattern with the property that if (i, j) is in the pattern, then (j, i) is not in the pattern.

For α a subset of $\mathcal{N}$, the *principal submatrix* obtained from A by deleting all rows and columns not in α is denoted by $A(\alpha)$. The *principal subpattern* $Q(\alpha)$ is obtained from the pattern Q by deleting all positions whose first or second coordinate is not in α. A *principal minor* is the determinant of a principal submatrix. A real $n \times n$ matrix is called a P_0*-matrix* if all of its principal minors are nonnegative. A P_0-

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matrix is called a $P_{0,1}$-*matrix* if all diagonal entries are nonzero. A *partial P_0-matrix* (*$P_{0,1}$-matrix*) is a partial matrix in which all fully specified principal submatrices are P_0-matrices ($P_{0,1}$-matrices). A *nonnegative P_0-matrix* (*nonnegative $P_{0,1}$-matrix*) is a P_0-matrix ($P_{0,1}$-matrix) whose entries are nonnegative. A *partial nonnegative P_0-matrix* (*partial nonnegative $P_{0,1}$-matrix*) is a partial P_0-matrix ($P_{0,1}$-matrix) whose specified entries are nonnegative.

The following properties of nonnegative P_0-matrices will be used: A nonnegative block triangular matrix, all of whose diagonal blocks are nonnegative P_0-matrices, is a nonnegative P_0-matrix. If A is a nonnegative P_0-matrix and D is a positive diagonal matrix, then DA and $D^{-1}AD$ are both nonnegative P_0-matrices. If A is a nonnegative P_0-matrix and P_π is a permutation matrix, then $P_\pi^{-1}AP_\pi$ is a nonnegative P_0-matrix.

In many situations we need to permute the entries of a partial matrix. We do this by defining a permutation of a pattern: If Q is a pattern for $n \times n$ matrices and π is a permutation of $\mathcal{N}$, then $\pi(Q) = \{(\pi(i), \pi(j)) : (i,j) \in Q\}$. For a partial matrix A specifying Q, define the partial matrix $\pi(A)$ specifying $\pi(Q)$ by $\pi(A)_{\pi(i)\pi(j)} = A_{ij}$ for $(i,j) \in Q$. Note that for a fully specified matrix A, $\pi(A) = P_\pi^{-1}AP_\pi$ with $P_\pi = [e_{\pi(1)}, \ldots, e_{\pi(n)}]^T$. In completion problems we permute the entries of a given partial matrix A to obtain $\pi(A)$, complete $\pi(A)$ to $\widehat{\pi(A)}$, and use $\pi^{-1}(\widehat{\pi(A)})$ to complete A.

Throughout the paper we will denote the entries of a partial matrix as follows: d_i denotes a specified diagonal entry, a_{ij} a specified off-diagonal entry, and x_{ij} an unspecified entry, $1 \leq i,j \leq n$. In addition, c_{ij} may be used to denote the value assigned to the unspecified entry x_{ij} during the process of completing a partial matrix.

The next result is known [6]. We include the proof here, because the explicit values chosen in this proof will be used in subsequent proofs.

LEMMA 1.1. *A pattern for 3×3 matrices that includes all diagonal positions has nonnegative P_0-completion.*

Proof. Let

$$A = \begin{bmatrix} d_1 & a_{12} & x_{13} \\ a_{21} & d_2 & a_{23} \\ a_{31} & a_{32} & d_3 \end{bmatrix}$$

be a partial nonnegative P_0-matrix and let Q be the pattern A specifies. We will look at two cases: 1) If $a_{12}a_{21} = 0$, $a_{23}a_{32} = 0$ or $a_{12}a_{23}a_{31} \geq d_1d_2d_3$, set $x_{13} = 0$ to complete A. 2) If $a_{12} > 0$, $a_{21} > 0$, $a_{23} > 0$, $a_{32} > 0$, and $a_{12}a_{23}a_{31} < d_1d_2d_3$, set $x_{13} = \frac{a_{12}a_{23}}{d_2}$ to complete A. In both cases, this completes A to a nonnegative P_0-matrix. Thus, the pattern Q that A specifies has nonnegative P_0-completion.

This implies that any pattern R for a 3×3 matrix with one unspecified off-diagonal entry has nonnegative P_0-completion, since any partial matrix specifying R can be transformed via a permutation matrix in order to specify the above pattern Q. Also, any matrix with more than one unspecified off-diagonal entry can be completed by first setting all except one unspecified off-diagonal entry to zero and then completing the resulting matrix as shown above. $\square$

A *digraph* $G = (V_G, E_G)$ is a finite set V_G of positive integers, whose members are called *vertices*, and a set E_G of ordered pairs (v, u) of distinct vertices, called *arcs*.

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ELA
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In the arc (v, u), v is the *tail* and u is the *head*. The *indegree* $d^-(u)$ of vertex u is the number of arcs with head u. The *outdegree* $d^+(v)$ of vertex v is the number of arcs with tail v. The *order* of G is the number of vertices of G. A *subdigraph* of the digraph $G = (V_G, E_G)$ is a digraph $H = (V_H, E_H)$, where V_H is a subset of V_G and E_H is a subset of E_G (note that $(v, u) \in E_H$ requires $v, u \in V_H$, since H is a digraph). If W is a subset of V_G, the *subdigraph induced by* W, $< W >$, is the digraph (W, E_W) with E_W the set of all arcs of G between the vertices in W. A subdigraph induced by a subset of vertices is also called an *induced subdigraph*.

For a pattern Q for $n \times n$ matrices that contains all diagonal positions, the *digraph* of Q (*pattern-digraph*) is the digraph having vertex set $\mathcal{N}$ and, as arcs, the ordered pairs $(i, j) \in Q$ where $i \neq j$. A partial matrix that specifies a pattern is also referred to as specifying the digraph of the pattern. Note that the use of a permutation on a pattern or partial matrix corresponds to renumbering the vertices of the pattern-digraph that the matrix specifies. Since nonnegative P_0-matrices are closed under permutation similarity, we are free to renumber digraph vertices as convenient.

When all diagonal entries in a matrix are nonzero or all diagonal positions are present in a pattern, digraphs can be used to study matrices (nonzero digraphs) and patterns (pattern-digraphs). When diagonal positions are omitted or diagonal entries of a matrix can be zero, it is sometimes necessary to use mardigraphs (cf. [6]) or digraphs that include loops (L-digraphs, defined below, cf. [8, Definition 6.2.11] and [1, p. 53]). In this paper we study only patterns that include all diagonal positions, so we use pattern-digraphs, but we will use L-digraphs to study matrices, since it is necessary to distinguish between zero and nonzero diagonal entries, both of which occur. Note that the term "digraph" is sometimes used to describe what is here called an L-digraph. We use our teminology because we need to distinguish L-digraphs from what we call digraphs.

An L-*digraph* is a digraph that is allowed to have loops, i.e., arcs (v, v). The terms *indegree, outdegree, order, sub-L-digraph*, and *induced sub-L-digraph* are defined analogously to the corresponding terms for digraphs. For an L-digraph G, let $\mathrm{Sub}(G)$ denote the set of all sub-L-digraphs of G. The L-digraph $G = (V_G, E_G)$ is *isomorphic* to the L-digraph $H = (V_H, E_H)$ if there is a one-to-one map ϕ from V_G onto V_H and $(v, w) \in E_G$ if and only if $(\phi(v), \phi(w)) \in E_H$.

Let A be a (fully specified) $n \times n$ matrix. The *nonzero-L-digraph* of A is the L-digraph having vertex set $\mathcal{N}$ and, as arcs, the ordered pairs (i, j) where $a_{ij} \neq 0$. If G is the nonzero-L-digraph of A, then the nonzero-L-digraph of the principal submatrix $A(\alpha)$ is isomorphic to $< \alpha >$. We may abuse the notation and refer to $< \alpha >$ as the nonzero-L-digraph of $A(\alpha)$.

We use the term (L-)digraph to mean digraph or L-digraph. A *path* (respectively, *semipath*) in the (L-)digraph $G = (V_G, E_G)$ is a sequence of vertices $v_1, v_2, \ldots, v_{k-1}, v_k$ in V_G such that for $i = 1, \ldots, k-1$, the arc $(v_i, v_{i+1}) \in E_G$ (respectively, $(v_i, v_{i+1}) \in E_G$ or $(v_{i+1}, v_i) \in E_G$) and all vertices are distinct except possibly $v_1 = v_k$. Clearly, a path is a semipath, although the converse is false. The length of the (semi)path $v_1, v_2, \ldots, v_{k-1}, v_k$ is $k - 1$. A *cycle* is a path in which $v_1 = v_k$. A cycle is *even* or *odd* according as its length is even or odd. A digraph whose vertex set consists of the n vertices $v_1, \ldots v_n$, and whose arc set consists of exactly the arcs in the two cycles

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$v_1, v_2, \ldots, v_n, v_1$ and $v_n, v_{n-1}, \ldots, v_1, v_n$ is a *symmetric* n*-cycle*. Two distinct vertices v and w are *connected* if there is a semipath $v = v_1, v_2, \cdots, v_k = w$. Any vertex v is connected to itself, whether or not the loop (v, v) is in E_G. The relationship of being connected is an equivalence relation on vertices of G. A sub-(L-)digraph induced by an equivalence class defined by this relation is called a *component* of G. The (L-)digraph G is *connected* if it has only one component, i.e., if any two vertices of G are connected. An (L-)digraph is *strongly connected* if for any two vertices v and w there is a path from v to w.

An (L-)digraph that contains all possible arcs between its vertices is called *complete*. A complete sub(-L-)digraph is called a *clique*. A *cut-vertex* is a vertex that if removed along with all incidental arcs, causes a component of the (L-)digraph to be separated into more than one component. If a connected (L-)digraph does not contain any cut-vertices, the (L-)digraph is *nonseparable*. A *block* is a maximal nonseparable sub(-L-)digraph, and a (L-)digraph where all blocks are cliques is called *block-clique*.

Let S_n denote the group of permutations of $\mathcal{N}$. Let G_π denote the L-digraph of the permutation matrix P_π for some $\pi \in S_n$; G_π is called a *permutation L-digraph* (cf. [1, p. 291]).

The following two lemmas are obvious.

Lemma 1.2. *Let A be an $n \times n$ matrix and let G be its nonzero-L-digraph. Then*

$$\mathrm{Det}A = \sum_{\pi \in S_n : G_\pi \in \mathrm{Sub}(G)} (\mathrm{sgn}\ \pi) a_{1\pi(1)} \cdots a_{n\pi(n)},$$

where the sum over the empty set is zero.

Lemma 1.3. *An L-digraph H is a permutation L-digraph if and only if $V_H = \mathcal{N}$ and each component of H is a single cycle.*

Corollary 1.4. *A nonnegative matrix whose nonzero L-digraph contains no even cycles is a P_0-matrix.*

Proof. For every $\pi \in S_n$ such that G_π is a sub-L-digraph of G, G_π is composed of disjoint cycles, which by hypothesis must all be of odd length. Thus π is the product of odd cycles and so sgn $\pi = 1$. Thus Det$A$ is the sum of positive terms, or zero if there are no permutation L-digraphs in G. Since any cycle in an induced sub-L-digraph of G is also a cycle of G, every induced sub-L-digraph $< \alpha >$ inherits the property of containing no even cycles, and hence Det$A(\alpha)$ is nonnegative. □

References [3, 6, 7] contain some results on nonnegative P- and nonnegative P_0-matrix completion problems. Of particular interest is Lemma 3.5 in [3]: Any partial positive P-matrix, the graph of whose specified entries is an n-cycle can be completed to a positive P-matrix. (For this result all diagonal entries are assumed specified, and "n-cycle" means what we call here a "symmetric n-cycle" because all patterns discussed in [3] are positionally symmetric.) In [6, Theorem 8.4], it is noted that the same method of proof applies to partial nonnegative $P_{0,1}$-matrices. In contrast, it is shown in [2] that the analogous statement for P_0-matrices is true if and only if $n \neq 4$. That is, a pattern for $n \times n$ matrices that includes all diagonal positions whose pattern-digraph is a symmetric n-cycle has P_0-completion if and only if $n \neq 4$. Note that the cases $n = 2$ and 3 are trivial, since the pattern includes all positions. The interesting

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ELA
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cases are $n = 4$ (no completion) and $n \geq 5$ (completion). In Section 3 it is shown that the same situation holds for the nonnegative P_0-matrix completion problem, that is, a pattern for $n \times n$ matrices that includes all diagonal positions whose pattern digraph is a symmetric n-cycle has nonnegative P_0-completion if and only if $n \neq 4$. Section 2 contains a classification of patterns for 4×4 matrices that include all diagonal positions as either having nonnegative P_0-completion or not having nonnegative P_0-completion, as [2] provided the analogous classification for P_0-completion. However, this classification has a more elegant description (cf. Theorem 2.6). In [2], it was established that asymmetric patterns have P_0-completion. This is not the case for nonnegative P_0-completion, as Lemma 2.1 makes clear.

2. Classification of Patterns for 4×4 Matrices. In this section we will classify all patterns for 4×4 matrices as either having nonnegative P_0-completion, or not. The patterns discussed here are assumed to include all diagonal positions. The digraphs are numbered following [5]; q is the number of edges, n is the diagram number. The classification is broken up into a series of lemmas.

LEMMA 2.1. *If the digraph of a pattern Q contains a 4-cycle Γ and the subdigraph induced by Γ is not a clique, then Q does not have nonnegative P_0-completion.*

Proof. Without loss of generality, assume that Γ is $1, 2, 3, 4, 1$. Let A be a partial nonnegative P_0-matrix specifying Q, with $a_{12} = 1$, $a_{23} = 1$, $a_{34} = 1$, $a_{41} = 1$, and all other specified entries equal to zero. Then every fully specified principal submatrix of A is triangular, so A is a partial nonnegative P_0-matrix. Suppose that $\widehat{A}$ is a nonnegative P_0-matrix that completes A, and let $\widehat{a}_{ij}$ denote the ij-entry of $\widehat{A}$, whether specified in A or chosen for $\widehat{A}$. Since all of the diagonal entries of $\widehat{A}$ are zero, $\mathrm{Det}\widehat{A}(\{i, j\}) = -\widehat{a}_{ij}\widehat{a}_{ji}$. But $\widehat{a}_{ij} \geq 0$, $\widehat{a}_{ji} \geq 0$, and $\mathrm{Det}\widehat{A}(\{i, j\}) \geq 0$. Therefore, $\widehat{a}_{21} = 0$, $\widehat{a}_{32} = 0$, $\widehat{a}_{43} = 0$, and $\widehat{a}_{14} = 0$. In addition, $\widehat{a}_{13}\widehat{a}_{31} = 0$. Then $\mathrm{Det}\widehat{A} = -1 + \widehat{a}_{13}\widehat{a}_{31}\widehat{a}_{24}\widehat{a}_{42} = -1$, a contradiction. Thus, A cannot be completed to a nonnegative P_0-matrix. Therefore, $< \Gamma >$ does not have nonnegative P_0-completion, so by [6, Lemma 3.1], neither does the pattern Q. □

LEMMA 2.2. *The patterns for the digraphs listed below have nonnegative P_0-completion.*

$q = 0$	
$q = 1$	
$q = 2$	$n = 1 - 5$
$q = 3$	$n = 1 - 13$
$q = 4$	$n = 1 - 15, 17 - 27$
$q = 5$	$n = 1 - 6, 8 - 31, 33, 34, 36 - 38$
$q = 6$	$n = 1 - 3, 5, 6, 8 - 21, 23 - 27, 29, 32, 35, 36, 38 - 41, 43, 44, 46 - 48$
$q = 7$	$n = 1, 3 - 6, 9, 11, 14, 16, 19, 22, 24, 26, 28, 29, 31, 34, 36, 37$
$q = 8$	$n = 1, 10, 12, 18, 21, 27$
$q = 9$	$n = 8, 11$
$q = 12$	

Proof. For each of these digraphs, the pattern of every nonseparable strongly connected induced subdigraph has nonnegative P_0-completion, thus by [6, Theorem

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5.8] the patterns of these digraphs have nonnegative P_0-completion. □

LEMMA 2.3. *The patterns for the digraphs listed below have nonnegative P_0-completion.*

$$\begin{array}{ll} q = 5 & n = 35 \\ q = 6 & n = 28, 30, 31 \\ q = 7 & n = 7 \end{array}$$

Proof. Let A be a partial nonnegative P_0-matrix specifying any of the patterns of the above digraphs. Then set all unspecified entries of the partial matrix A to zero. It is straightforward to verify by computation that this completion results in a nonnegative P_0-matrix. □

LEMMA 2.4. *The patterns for the digraphs $q = 8$, $n = 14$; $q = 8$, $n = 15$; $q = 7$, $n = 15$; $q = 7$, $n = 17$; $q = 7$, $n = 21$; and $q = 7$, $n = 23$ all have nonnegative P_0-completion.*

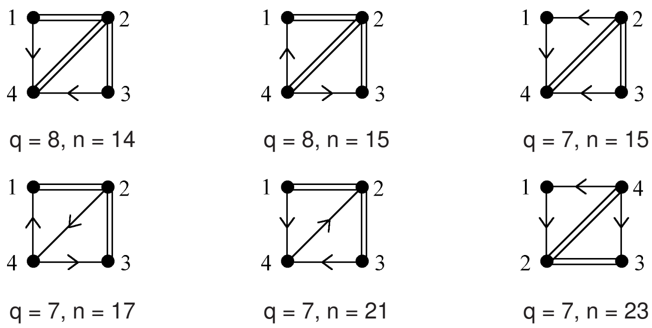

FIG. 2.1. *Digraphs having P_0-completion.*

Proof. Let

$$A = \begin{bmatrix} d_1 & a_{12} & x_{13} & a_{14} \\ a_{21} & d_2 & a_{23} & a_{24} \\ x_{31} & a_{32} & d_3 & a_{34} \\ x_{41} & a_{42} & x_{43} & d_4 \end{bmatrix}$$

be a partial nonnegative P_0-matrix specifying the pattern of the digraph $q = 8$, $n = 14$ with the vertices labeled as in Figure 2.1.

We will consider two cases: 1) $a_{12}a_{21} = 0$ or $a_{23}a_{32} = 0$ or $a_{24}a_{42} = 0$, and 2) a_{12}, a_{21}, a_{23}, a_{32}, a_{24}, and a_{42} are all nonzero. Notice that the submatrices $A(\{1, 2\})$, $A(\{2, 3\})$, and $A(\{2, 4\})$ are fully specified, and thus their determinants are given to be nonnegative in all cases.

Case 1: $a_{12}a_{21} = 0$ or $a_{23}a_{32} = 0$ or $a_{24}a_{42} = 0$. We prove the case $a_{12}a_{21} = 0$; the other two cases are similar. Consider the two subcases: i) $a_{12} = 0$ and ii) $a_{21} = 0$. For both subcases set $x_{31} = 0$, $x_{41} = 0$, and $x_{13} = 0$. Next, complete the submatrix $A(\{2, 3, 4\})$ to a nonnegative P_0-matrix by using Lemma 1.1. This will determine x_{43}, and we will say $x_{43} = c_{43}$. Thus A is completed to $\widehat{A}$.

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TABLE 2.1

Principal Submatrix	Determinant
$\widehat{A}(\{1,2\})$	d_1d_2
$\widehat{A}(\{2,3\})$	$d_2d_3 - a_{23}a_{32}$
$\widehat{A}(\{2,4\})$	$d_2d_4 - a_{24}a_{42}$
$\widehat{A}(\{3,4\})$	$d_3d_4 - a_{34}c_{43}$
$\widehat{A}(\{2,3,4\})$	$d_2d_3d_4 + a_{23}a_{34}a_{42} + a_{24}a_{32}c_{43} - a_{34}c_{43}d_2 - a_{24}a_{42}d_3 - a_{23}a_{32}d_4$
$\widehat{A}(\{1,3\})$	d_1d_3
$\widehat{A}(\{1,4\})$	d_1d_4
$\widehat{A}(\{1,2,3\})$	$d_1d_2d_3 - a_{23}a_{32}d_1$
$\widehat{A}(\{1,2,4\})$	$d_1d_2d_4 + a_{14}a_{21}a_{42} - a_{24}a_{42}d_1$
$\widehat{A}(\{1,3,4\})$	$d_1d_3d_4 - a_{34}c_{43}d_1$
$\widehat{A}$	$d_1d_2d_3d_4 - a_{14}a_{21}a_{32}c_{43} + a_{23}a_{34}a_{42}d_1 + a_{24}a_{32}c_{43}d_1 - a_{34}c_{43}d_1d_2 + a_{14}a_{21}a_{42}d_3 - a_{24}a_{42}d_1d_3 - a_{23}a_{32}d_1d_4$

For subcase i), the principal minors of $\widehat{A}$ are shown in Table 2.1.
The first three principal minors are known to be nonnegative because A is a partial nonnegative P_0-matrix. $\mathrm{Det}\widehat{A}(\{3,4\})$ and $\mathrm{Det}\widehat{A}(\{2,3,4\})$ are nonnegative due to the selection of c_{43}. $\mathrm{Det}\widehat{A}(\{1,2,3\}) = d_1 \cdot \mathrm{Det}\widehat{A}(\{2,3\}) \geq 0$, $\mathrm{Det}\widehat{A}(\{1,2,4\}) = a_{14}a_{21}a_{42} + d_1 \cdot \mathrm{Det}\widehat{A}(\{2,4\}) \geq 0$, and $\mathrm{Det}\widehat{A}(\{1,3,4\}) = d_1 \cdot \mathrm{Det}\widehat{A}(\{3,4\}) \geq 0$.
Recall that the completion of $A(\{2,3,4\})$ sets c_{43} to be either 0 or $\frac{a_{23}a_{42}}{d_2}$.
If $c_{43} = 0$, then $a_{24}a_{42} = 0$, $a_{23}a_{32} = 0$, or $a_{23}a_{34}a_{42} \geq d_2d_3d_4$, and $\mathrm{Det}\widehat{A} = d_1d_2d_3d_4 + a_{23}a_{34}a_{42}d_1 + a_{14}a_{21}a_{42}d_3 - a_{24}a_{42}d_1d_3 - a_{23}a_{32}d_1d_4$. If $a_{24}a_{42} = 0$, then $\mathrm{Det}\widehat{A} = a_{23}a_{34}a_{42}d_1 + a_{14}a_{21}a_{42}d_3 + d_1d_4 \cdot \mathrm{Det}\widehat{A}(\{2,3\}) \geq 0$. If $a_{23}a_{32} = 0$, then $\mathrm{Det}\widehat{A} = a_{23}a_{34}a_{42}d_1 + a_{14}a_{21}a_{42}d_3 + d_1d_3 \cdot \mathrm{Det}\widehat{A}(\{2,4\}) \geq 0$. If $a_{23}a_{34}a_{42} \geq d_2d_3d_4$, then $\mathrm{Det}\widehat{A} \geq d_1d_2d_3d_4 + d_1d_2d_3d_4 + a_{14}a_{21}a_{42}d_3 - a_{24}a_{42}d_1d_3 - a_{23}a_{32}d_1d_4 = a_{14}a_{21}a_{42}d_3 + d_1d_3 \cdot \mathrm{Det}\widehat{A}(\{2,4\}) + d_1d_4 \cdot \mathrm{Det}\widehat{A}(\{2,3\}) \geq 0$.
If $c_{43} = \frac{a_{23}a_{42}}{d_2}$, then

$$\begin{aligned}\mathrm{Det}\widehat{A} &= d_1d_2d_3d_4 - \frac{a_{14}a_{21}a_{23}a_{32}a_{42}}{d_2} + \frac{a_{23}a_{24}a_{32}a_{42}d_1}{d_2} \\ &\quad + a_{14}a_{21}a_{42}d_3 - a_{24}a_{42}d_1d_3 - a_{23}a_{32}d_1d_4 \\ &= \frac{(d_2d_3 - a_{23}a_{32})(a_{14}a_{21}a_{42} + d_1d_2d_4 - a_{24}a_{42}d_1)}{d_2} \\ &= \frac{\mathrm{Det}A(\{2,3\}) \cdot (a_{14}a_{21}a_{42} + d_1 \cdot \mathrm{Det}A(\{2,4\}))}{d_2} \geq 0.\end{aligned}$$

For subcase ii), $\widehat{A}$ is a nonnegative block upper triangular matrix with diagonal blocks $[d_1]$ and $A(\{2,3,4\})$, which are nonnegative P_0-matrices. Therefore, $\widehat{A}$ is a nonnegative P_0-matrix.

Case 2: $a_{12} > 0$, $a_{21} > 0$, $a_{23} > 0$, $a_{32} > 0$, $a_{24} > 0$, and $a_{42} > 0$. These assumptions imply that $d_1 > 0$, $d_2 > 0$, $d_3 > 0$, and $d_4 > 0$. By left multiplication

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of A by a positive diagonal matrix, we may assume without loss of generality that $d_1 = d_2 = d_3 = d_4 = 1$. By use of a diagonal similarity we may also assume without loss of generality that $a_{21} = a_{32} = a_{42} = 1$. Thus,

$$A = \begin{bmatrix} 1 & a_{12} & x_{13} & a_{14} \\ 1 & 1 & a_{23} & a_{24} \\ x_{31} & 1 & 1 & a_{34} \\ x_{41} & 1 & x_{43} & 1 \end{bmatrix}.$$

The submatrices in Table 2.2 are fully specified and thus their determinants are nonnegative. We will look at three subcases:
i) $a_{14} \geq 1$ and $a_{34} \geq 1$, ii) $a_{14} < 1$, and iii) $a_{34} < 1$.

Table 2.2

Principal Submatrix	Determinant
$A(\{1,2\})$	$1 - a_{12}$
$A(\{2,3\})$	$1 - a_{23}$
$A(\{2,4\})$	$1 - a_{24}$

For subcase i), set $x_{31} = 0$, $x_{41} = 0$, $x_{43} = 0$ and $x_{13} = a_{23}$. The principal minors of $\widehat{A}$ are shown in Table 2.3.

Table 2.3

Principal Submatrix	Determinant
$\widehat{A}(\{1,3\})$	1
$\widehat{A}(\{1,4\})$	1
$\widehat{A}(\{3,4\})$	1
$\widehat{A}(\{1,2,3\})$	$1 - a_{12}$
$\widehat{A}(\{1,2,4\})$	$1 - a_{12} + a_{14} - a_{24}$
$\widehat{A}(\{1,3,4\})$	1
$\widehat{A}(\{2,3,4\})$	$1 - a_{23} - a_{24} + a_{23}a_{34}$
$\widehat{A}$	$1 - a_{12} + a_{14} - a_{24}$

$\mathrm{Det}\widehat{A}(\{1,2,3\}) = 1 - a_{12} = \mathrm{Det}\widehat{A}(\{1,2\}) \geq 0$. Since $a_{14} \geq 1$, $\mathrm{Det}\widehat{A}(\{1,2,4\}) \geq 1 - a_{12} + 1 - a_{24} = \mathrm{Det}\widehat{A}(\{1,2\}) + \mathrm{Det}\widehat{A}(\{2,4\}) \geq 0$. $\mathrm{Det}\widehat{A}(\{2,3,4\}) = \mathrm{Det}A(\{2,4\}) + a_{23}(a_{34} - 1) \geq 0$, since $a_{34} \geq 1$. And

$$\mathrm{Det}\widehat{A} = \mathrm{Det}\widehat{A}(\{1,2,4\}) \geq 0.$$

For subcase ii), specify the unspecified entries in the following order. First, set $x_{41} = 1$ and $x_{31} = 1$. Then, complete the submatrix $A(\{2,3,4\})$ to a nonnegative P_0-matrix

Electronic Journal of Linear Algebra ISSN 1081-3810
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Volume 10, pp. 46-59, March 2003

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by using Lemma 1.1. This will determine x_{43}, which we will call c_{43}. Next, complete $A(\{1,3,4\})$ to a nonnegative P_0-matrix by Lemma 1.1, and say $x_{13} = c_{13}$. The resulting matrix is

$$\widehat{A} = \begin{bmatrix} 1 & a_{12} & c_{13} & a_{14} \\ 1 & 1 & a_{23} & a_{24} \\ 1 & 1 & 1 & a_{34} \\ 1 & 1 & c_{43} & 1 \end{bmatrix}.$$

The principal minors of $\widehat{A}$ are shown in Table 2.4.

Table 2.4

Principal Submatrix	Determinant
$\widehat{A}(\{3,4\})$	$1 - a_{34}c_{43}$
$\widehat{A}(\{2,3,4\})$	$1 - a_{23} - a_{24} + a_{23}a_{34} + a_{24}c_{43} - a_{34}c_{43}$
$\widehat{A}(\{1,3\})$	$1 - c_{13}$
$\widehat{A}(\{1,3,4\})$	$1 - a_{14} - c_{13} + a_{34}c_{13} + a_{14}c_{43} - a_{34}c_{43}$
$\widehat{A}(\{1,4\})$	$1 - a_{14}$
$\widehat{A}(\{1,2,3\})$	$1 - a_{12} - a_{23} + a_{12}a_{23}$
$\widehat{A}(\{1,2,4\})$	$1 - a_{12} - a_{24} + a_{12}a_{24}$
$\widehat{A}$	$1 - a_{12} - a_{23} + a_{12}a_{23} - a_{24} + a_{12}a_{24} + a_{23}a_{34} - a_{12}a_{23}a_{34} + a_{24}c_{43} - a_{12}a_{24}c_{43} - a_{34}c_{43} + a_{12}a_{34}c_{43}$

By the choice of c_{43}, $\mathrm{Det}\widehat{A}(\{3,4\})$ and $\mathrm{Det}\widehat{A}(\{2,3,4\})$ are nonnegative. And by the choice of c_{13}, $\mathrm{Det}\widehat{A}(\{1,3\})$ and $\mathrm{Det}\widehat{A}(\{1,3,4\})$ are nonnegative. Since $a_{14} < 1$, $\mathrm{Det}\widehat{A}(\{1,4\}) \geq 0$. Also, $\mathrm{Det}\widehat{A}(\{1,2,3\}) = \mathrm{Det}\widehat{A}(\{1,2\}) \cdot \mathrm{Det}\widehat{A}(\{2,3\}) \geq 0$ and $\mathrm{Det}\widehat{A}(\{1,2,4\}) = \mathrm{Det}\widehat{A}(\{1,2\}) \cdot \mathrm{Det}\widehat{A}(\{2,4\}) \geq 0$. Lastly,

$$\mathrm{Det}\widehat{A} = \mathrm{Det}\widehat{A}(\{1,2\}) \cdot \mathrm{Det}\widehat{A}(\{2,3,4\}) \geq 0.$$

Subcase iii) is similar to subcase ii). Therefore, the pattern for $q = 8$, $n = 14$ has nonnegative P_0-completion.

A partial nonnegative P_0-matrix B specifying the pattern of the digraph $q = 7$, $n = 15$ or $q = 7$, $n = 21$ with the vertices labelled as in Figure 2.1 can be extended to a partial nonnegative P_0-matrix specifying $q = 8$, $n = 14$ by setting the unspecified entry x_{12} or x_{24}, respectively, of B to zero. Then the resulting matrix can be completed to a nonnegative P_0-matrix as above. Also, notice that a matrix specifying the pattern of $q = 8$, $n = 15$ is the transpose of a matrix specifying $q = 8$, $n = 14$. Therefore, any partial nonnegative P_0-matrix specifying $q = 8$, $n = 15$ can be completed to a nonnegative P_0-matrix by taking its transpose, completing it as above, and taking its transpose again. In addition, a partial nonnegative P_0-matrix C specifying $q = 7$, $n = 17$ or $q = 7$, $n = 23$ with the vertices labeled as in Figure 2.1 can be

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 10, pp. 46-59, March 2003

extended to a partial nonnegative P_0-matrix specifying $q = 8$, $n = 15$ by setting the unspecified entry x_{42} or x_{21}, respectively, of C to zero. Then C can be completed to a nonnegative P_0-matrix as above. So the patterns for $q = 7$, $n = 15$; $q = 7$, $n = 21$; $q = 8$, $n = 15$; $q = 7$, $n = 17$; and $q = 7$, $n = 23$ have nonnegative P_0-completion. □

LEMMA 2.5. *The pattern for the digraph $q = 8$, $n = 6$ (Figure 2.2) has nonnegative P_0-completion.*

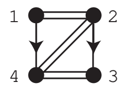

FIG. 2.2. *The digraph $q = 8, n = 6$*

The proof is omitted because it is similar to the proof of Lemma 2.4.

THEOREM 2.6. *A pattern for 4×4 matrices, that includes all diagonal positions, has nonnegative P_0-completion if and only if its digraph is complete when it has a 4-cycle, that is, if and only if its digraph is not one of the following.*

$q = 4$	$n = 16$
$q = 5$	$n = 7, 32$
$q = 6$	$n = 4, 7, 22, 33, 34, 37, 42, 45$
$q = 7$	$n = 2, 8, 10, 12, 13, 18, 20, 25, 27, 30, 32, 33, 35, 38$
$q = 8$	$n = 2 - 5, 7 - 9, 11, 13, 16, 17, 19, 20, 22 - 26$
$q = 9$	$n = 1 - 7, 9, 10, 12, 13$
$q = 10$	$n = 1 - 5$
$q = 11$	

Proof. Each of the listed digraphs contains a 4-cycle whose induced subdigraph is not a clique. By Lemma 2.1, the patterns of these digraphs do not have nonnegative P_0-completion. Lemmas 2.2, 2.3, 2.4, and 2.5 demonstrate that the patterns for all remaining digraphs have nonnegative P_0-completion. □

One question emerging from Theorem 2.6 is whether either the theorem or its obvious generalization is true for larger digraphs. That is, whether

1. a pattern has nonnegative P_0-completion if and only if its digraph has the property that the induced subdigraph of any 4-cycle is a clique, or
2. a pattern has nonnegative P_0-completion if and only if its digraph has the property that the induced subdigraph of any even cycle is a clique.

Neither of these statements is true. Example 2.7 contains a counterexample to item 1. That is, we give a partial nonnegative P_0-matrix that cannot be completed to a partial nonnegative P_0-matrix and that specifies a digraph that does not contain any 4-cycles. Furthermore, Theorem 3.2 in the next section shows that any partial nonnegative P_0-matrix that specifies a symmetric 6-cycle can be completed to a nonnegative P_0-matrix, and thus contradicts item 2.

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Volume 10, pp. 46-59, March 2003

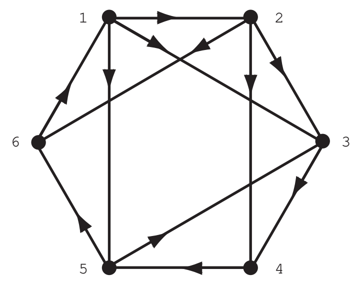

FIG. 2.3. *Digraph not containing any* 4*-cycles and not having* P_0*-completion.*

EXAMPLE 2.7. The partial nonnegative P_0-matrix

$$A = \begin{bmatrix} 0 & 1 & 0.01 & x_{14} & 0.01 & x_{16} \\ x_{21} & 0 & 1 & 0.01 & x_{25} & 0.01 \\ x_{31} & x_{32} & 0 & 1 & x_{35} & x_{36} \\ x_{41} & x_{42} & x_{43} & 0 & 1 & x_{46} \\ x_{51} & x_{52} & 0.01 & x_{54} & 0 & 1 \\ 1 & x_{62} & x_{63} & x_{64} & x_{65} & 0 \end{bmatrix}$$

specifies the digraph D, shown in Figure 2.3. Note that D does not contain any 4-cycles. But A cannot be completed to a nonnegative P_0-matrix: Examination of the 2×2 principal minors shows that if a completion $\widehat{A}$ of A is a nonnegative P_0-matrix $x_{21}, x_{32}, x_{43}, x_{54}, x_{65}, x_{16}, x_{31}, x_{51}, x_{42}, x_{62}$, and x_{35} must be zero. With these choices, $\mathrm{Det}\widehat{A}(\{1,3,5,6\}) = -0.0001x_{36}$ and $\mathrm{Det}\widehat{A}(\{1,2,3,4\}) = -x_{41}$, so x_{36} and x_{41} must be zero. Then, $\mathrm{Det}\widehat{A}(\{1,2,4,6\}) = -0.01x_{46}$ and $\mathrm{Det}\widehat{A}(\{3,4,5,6\}) = -x_{63}$, so $x_{46} = x_{63} = 0$. With these choices, $\mathrm{Det}\widehat{A} = -0.9999 - 0.0001x_{52}$, so it is impossible for $\widehat{A}$ to be a nonnegative P_0-matrix.

3. Symmetric n-cycle. If a positionally symmetric pattern has nonnegative P_0-completion, then each principal subpattern associated with a component of the digraph either includes all diagonal positions or omits all diagonal positions [7]. Any pattern that omits all diagonal positions has nonnegative P_0-completion [6, Theorem 4.7]. Thus, to determine which positionally symmetric patterns have nonnegative P_0-completion, we need to discuss only patterns that include all diagonal positions.

In this section we prove that a pattern which includes all diagonal positions and whose digraph is a symmetric n-cycle has nonnegative P_0-completion if and only if $n \neq 4$.

LEMMA 3.1. *Let G be the digraph of the symmetric* 5*-cycle* $1, 2, 3, 4, 5, 1$. *Any partial nonnegative P_0-matrix specifying G that has at least one diagonal entry equal to zero can be completed to a nonnegative P_0-matrix.*

Proof. Let A be a partial nonnegative P_0-matrix specifying G. By use of a permutation similarity, we may assume that d_1 is zero. By examination of $\mathrm{Det}A(\{1,2\})$ and $\mathrm{Det}A(\{1,5\})$, either $a_{21} = 0$ or $a_{12} = 0$, and either $a_{51} = 0$ or $a_{15} = 0$. There are now two cases.

Electronic Journal of Linear Algebra ISSN 1081-3810
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Volume 10, pp. 46-59, March 2003

ELA
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Case 1: $a_{15} = a_{12} = 0$ or $a_{21} = a_{51} = 0$. The digraph of the pattern specified by $A(\{2,3,4,5\})$ is block-clique (see Figure 3.1 with $n = 5$), so $A(\{2,3,4,5\})$ can be completed to a nonnegative P_0-matrix [3, Theorem 4.1]. Set the remaining unspecified entries to zero. This completes A to a block-triangular matrix whose diagonal blocks are nonnegative P_0-matrices.

Case 2: $a_{15} \neq 0$ and $a_{21} \neq 0$, or $a_{51} \neq 0$ and $a_{12} \neq 0$. Without loss of generality assume $a_{15} \neq 0$ and $a_{21} \neq 0$ and by use of a diagonal similarity, $a_{15} = a_{21} = 1$ (and necessarily $a_{51} = a_{12} = 0$). Then A can be completed to a nonnegative P_0-matrix by setting

$$x_{24} = \begin{cases} \frac{d_2 d_3 d_4}{a_{43}a_{32}} & \text{if } a_{43}a_{32} \neq 0 \\ 0 & \text{if } a_{43}a_{32} = 0 \end{cases} \quad \text{and} \quad x_{35} = \begin{cases} \frac{d_3 d_4 d_5}{a_{54}a_{43}} & \text{if } a_{54}a_{43} \neq 0 \\ 0 & \text{if } a_{54}a_{43} = 0, \end{cases}$$

and all other unspecified entries equal to zero. The fact that this process yields a P_0-matrix can be verified by computing all the principal minors, most of which are clearly nonnegative. Observe

$$\mathrm{Det}A(\{2,3,4\}) = d_2d_3d_4 + x_{24}a_{43}a_{32} - d_4a_{23}a_{32} - d_2a_{34}a_{43}.$$

If $a_{43} = 0$, $\mathrm{Det}A(\{2,3,4\}) = d_2d_3d_4 - d_4a_{23}a_{32} = d_4 \cdot \mathrm{Det}A(\{2,3\}) \geq 0$. If $a_{32} = 0$, $\mathrm{Det}A(\{2,3,4\}) = d_2d_3d_4 - d_2a_{34}a_{43} = d_2 \cdot \mathrm{Det}A(\{3,4\}) \geq 0$. If $a_{43}a_{32} \neq 0$, then $\mathrm{Det}A(\{2,3,4\}) = 2d_2d_3d_4 - d_4a_{23}a_{32} - d_2a_{34}a_{43} = \mathrm{Det}A(\{3,4\}) + \mathrm{Det}A(\{2,3\}) \geq 0$. The computation of $\mathrm{Det}A(\{3,4,5\})$ is similar to that of $\mathrm{Det}A(\{2,3,4\})$.

$$\begin{aligned}\mathrm{Det}A(\{2,3,4,5\}) = a_{23}a_{32}a_{45}a_{54} - a_{45}a_{54}d_2d_3 - a_{34}a_{43}d_2d_5 - a_{23}a_{32}d_4d_5 + \\ d_2d_3d_4d_5 + a_{32}a_{43}d_5x_{24} + a_{43}a_{54}d_2x_{35}.\end{aligned}$$

If $a_{43} = 0$, we have $d_2d_5a_{34}a_{43} = 0, x_{24} = 0, x_{35} = 0$, and $\mathrm{Det}A(\{2,3,4,5\}) = \mathrm{Det}A(\{2,3\}) \cdot \mathrm{Det}A(\{4,5\})$. If $a_{32} = 0$ and $a_{54} = 0$, then $d_4d_5a_{23}a_{32} = 0$ and $d_2d_3a_{45}a_{54} = 0, x_{24} = 0, x_{35} = 0$, and $\mathrm{Det}A(\{2,3,4,5\}) = d_2d_5 \cdot \mathrm{Det}A(\{3,4\})$. If $a_{43} \neq 0$ and $a_{32} \neq 0$, then $x_{24} = \frac{d_2d_3d_4}{a_{43}a_{32}}$ and $\mathrm{Det}A(\{2,3,4,5\}) = a_{23}a_{32}a_{45}a_{54} - a_{45}a_{54}d_2d_3 - a_{34}a_{43}d_2d_5 - a_{23}a_{32}d_4d_5 + 2d_2d_3d_4d_5 + a_{43}a_{54}d_2x_{35} = \mathrm{Det}A(\{2,3\}) \cdot \mathrm{Det}A(\{4,5\}) + d_2d_5 \cdot \mathrm{Det}A(\{3,4\}) + a_{43}a_{54}d_2x_{35} \geq 0$. The case $a_{43} \neq 0$ and $a_{54} \neq 0$ is similar to the case $a_{43} \neq 0$ and $a_{32} \neq 0$. ∎

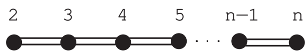

FIG. 3.1. *Block-clique*

THEOREM 3.2. *A pattern that includes all diagonal positions and whose digraph is a symmetric n-cycle has nonnegative P_0-completion for $n \geq 5$.*

Proof. The proof is by induction on n. Let A be a partial nonnegative P_0-matrix specifying a 5-cycle. If all the diagonal entries are positive, then A is a partial nonnegative $P_{0,1}$-matrix and can be completed to a nonnegative $P_{0,1}$-matrix by [6, Theorem 8.4]. If at least one diagonal entry is zero, apply Lemma 3.1.

Assume true for $n-1$. Let A be an $n \times n$ partial nonnegative P_0-matrix specifying the pattern whose digraph is the symmetric n-cycle $1, 2, \ldots, n, 1$. By multiplication

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Volume 10, pp. 46-59, March 2003

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by a positive diagonal matrix we may assume that each diagonal entry of A is either 1 or 0. (Note that subscript numbering is modulo n.) We now consider two cases.

Case 1: There exists an index i such that $d_i = d_{i+1} = 1$ and at least one of $a_{i,i+1}$ and $a_{i+1,i}$ is nonzero. Renumber so that $d_1 = d_2 = 1$ and $a_{12} \neq 0$. Then we can use the completion of an appropriate $(n-1) \times (n-1)$ principal submatrix specifying an $(n-1)$-cycle to complete A to a nonnegative P_0-matrix as in [3, Lemma 3.5] (see also [6, Theorem 8.4] and [2]). This case uses the induction hypothesis.

Case 2: The matrix does not satisfy the conditions of Case 1 and there exists an index i such that $d_i = d_{i+1} = 1$. Necessarily $a_{i,i+1} = a_{i+1,i} = 0$. Renumber so that $d_1 = d_2 = 1$ (and $a_{12} = a_{21} = 0$). At least one of a_{n1} and a_{1n} must be zero, because: if $d_n = 0$, then $\mathrm{Det}A(\{1,n\}) = -a_{n1}a_{1n}$; if $d_n = 1$, then a_{n1} and a_{1n} must both be zero, since Case 1 does not apply. The digraph of the pattern specified by $A(\{2,\ldots,n\})$ is block-clique (see Figure 3.1), so $A(\{2,\ldots,n\})$ can be completed to a nonnegative P_0-matrix. Set the remaining unspecified entries to zero, thus obtaining a nonnegative block-triangular matrix $\widehat{A}$ with diagonal blocks, $[d_1] = [1]$ and the completion of $A(\{2,\ldots,n\})$, both of which are P_0-matrices. So $\widehat{A}$ is a nonnegative P_0-matrix.

Case 3: There does not exist an index i such that $d_i = d_{i+1} = 1$. Then for each i, $\mathrm{Det}A(\{i,i+1\}) = -a_{i,i+1}a_{i+1,i}$, so $a_{i,i+1} = 0$ or $a_{i+1,i} = 0$. There are now two subcases.

Subcase i): Whenever n is odd, or if $a_{k,k+1} = 0$ for some $k \leq n$ and $a_{j+1,j} = 0$, for some $j \leq n$, we can set all unspecified entries to zero to get $\widehat{A}$. The nonzero L-digraph of $\widehat{A}$ does not contain any 2-cycles, since for all i, $a_{i,i+1} = 0$ or $a_{i+1,i} = 0$. If at least one $a_{k,k+1} = 0$ and at least one $a_{j+1,j} = 0$, there is no cycle of length greater than 1. Thus in either case the nonzero L-digraph of $\widehat{A}$ does not contain any even cycle, and so by Corollary 1.4, $\widehat{A}$ is a nonnegative P_0-matrix.

Subcase ii): Let n be even and, for all $i = 1,2,\ldots,n, a_{i,i+1} \neq 0$ or for all $i = 1,\ldots,n, a_{i+1,i} \neq 0$. Without loss of generality $a_{i,i+1} \neq 0$ for all i. Complete A to $\widehat{A}$ by choosing $x_{2n} = a_{23}$ and $x_{n-1,3} = a_{n-1,n}$, and set all other unspecified entries to zero. The nonzero L-digraph $\widehat{G}$ of $\widehat{A}$ contains the n-cycle $1,2,\ldots,n,1$; the 3-cycle $1,2,n,1$; the $(n-3)$-cycle $3,4,\ldots,n-2,n-1,3$ and possibly some loops (see Figure 3.2). Thus there are exactly two permutation L-digraphs in $\widehat{G}$, one having arc set the n-cycle and one having arc set the 3-cycle and the $(n-3)$-cycle. The two permutations have opposite signs and the products of the entries of $\widehat{A}$ corresponding to these two permutation L-digraphs are equal, so $\mathrm{Det}A = 0$ by Lemma 1.2. The nonzero L-digraph of any principal submatrix which is not the whole matrix cannot contain the n-cycle and thus has no even cycles. Therefore $\widehat{A}$ is a nonnegative P_0-matrix. □

Corollary 3.3. *A pattern that includes all diagonal positions and whose digraph is a symmetric n-cycle has nonnegative P_0-completion for $n \neq 4$.*

Proof. The cases $n = 2$ and $n = 3$ are trivial because the pattern includes all positions. The case $n = 4$ was established in Theorem 2.6, and Theorem 3.2 covers the case $n \geq 5$. □

The pattern that includes all diagonal positions and whose digraph is an n-cycle

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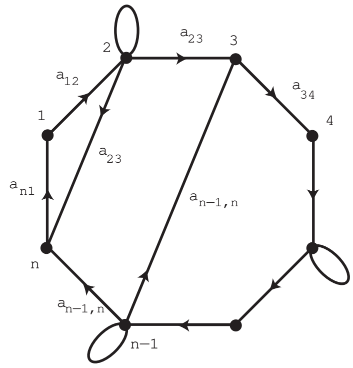

FIG. 3.2. *Nonzero L-digraph of the completion for subcase ii)*

has nonnegative P_0-completion if and only if $n \neq 4$, because each entry in the partial matrix corresponding to the reverse of each arc in the cycle can be set to zero to obtain a partial nonnegative P_0-matrix specifying a symmetric n-cycle.

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