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NON-EXISTENCE OF 5×5 FULL RAY-NONSINGULAR MATRICES*

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Abstract. An $n \times n$ complex matrix is full ray-nonsingular if it has no zero entries and every matrix obtained by changing the magnitudes of its entries is nonsingular. It is shown that a 5×5 full ray-nonsingular matrix does not exist. This, combined with earlier results, shows that there exists an $n \times n$ full ray-nonsingular matrix if and only if $n \leq 4$.

Key words. Ray-patterns, Ray-nonsingular matrices.

AMS subject classifications. 15A48, 15A57.

1. Introduction. A complex matrix is a *ray-pattern matrix* if each of its nonzero entries has modulus 1. A ray-pattern matrix is *full* if each of its entries is nonzero. An $n \times n$ complex matrix A is *ray-nonsingular* if $A \circ X$ is nonsingular for all (entry-wise) positive matrices X, where $A \circ X$ denotes the Schur (entry-wise) product. Ray-nonsingular matrices with real entries are simply *sign-nonsingular* matrices; see [2] and its references. In [2], the authors posed the following question: for which n does there exist an $n \times n$ full ray-nonsingular matrix? It is not hard to construct examples of $n \times n$ full ray-nonsingular matrices for $n \leq 4$; see [1, 2]. In [1], the authors showed that there are no full $n \times n$ ray-nonsingular matrices for $n \geq 6$. The question of whether there are 5×5 full ray-nonsingular matrices remains open. In this paper, we show that there is no 5×5 full ray-nonsingular matrix. As a result, we have the following complete answer for the question raised in [2]:

MAIN THEOREM *There is an $n \times n$ full ray-nonsingular matrix if and only if $n \leq 4$.*

The proof of the main theorem is quite detailed. In section 2, we recall some known results and outline our strategy for the proof. The key to the proof is an understanding of 3×3 full ray-patterns that are not ray-nonsingular. These are studied in section 3. The proof of the main theorem is given in section 4.

2. Preliminary results and basic strategies of proof. We first recall some terminology from [1]. A nonzero, diagonal ray-pattern matrix is a called a *complex signing*. A complex signing is *strict* if each diagonal entry is nonzero. A $(1, -1)$-*signing* is a diagonal matrix with diagonal entries in $\{1, -1\}$. A vector v is *balanced* if zero is in the relative interior of the convex hull its entries (viewed as points of the complex plane). Furthermore, it is *strongly balanced* if its entries take on at least three distinct values. A ray-pattern vector v is *generic* if for all $i < j$, $v_i \neq \pm v_j$.

Let $\overline{A}$ denote the entry-wise conjugate of A. Consider the relation on the set of ray-patterns defined by $A \sim B$ if and only if there exist matrices P and Q, each a

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product of permutation matrices and strict complex signings, such that $B = P\hat{A}Q$ where $\hat{A} = A$, A^t or $\overline{A}$. Clearly, $\sim$ is an equivalence relation, and we have the following observation.

LEMMA 2.1. *Suppose A and B are ray-pattern matrices with $A \sim B$. Then A is ray-nonsingular if and only if B is ray-nonsingular.*

We say that the matrix A is *strongly balanceable* if $A \sim B$ for some B each of whose columns is strongly balanced. The following three lemmas from [1] will be useful in establishing the nonexistence of a 5×5 full ray-nonsingular matrix.

LEMMA 2.2. [1, Lemma 3.7] *Let A be an $n \times n$ full ray-pattern. If A has an $m \times m$ strongly balanceable submatrix with $m \geq 3$, then A is not ray-nonsingular.*

In section 3, we establish sufficient conditions for a 3×3 full ray-pattern to be strongly balanceable.

LEMMA 2.3. [1, Theorem 4.3] *Let $A = (a_{jk})$ be a 5×5 full ray-pattern. If $a_{jk} \in \{1, -1, i, -i\}$ for all j and k, then A is not ray-nonsingular.*

LEMMA 2.4. [1, Proposition 4.4] *Let A be a 5×5 full ray-pattern with first column consisting of all 1's and each remaining column generic. Then A is not ray-nonsingular.*

Note that Lemma 2.4 implies that if A is a 5×5 full ray-nonsingular matrix, then each row and column of A intersects a 2×2 submatrix of the form

$$\begin{bmatrix} x & y \\ z & \pm yz/x \end{bmatrix}.$$

We now give a basic outline of our strategy for proving the main theorem. The proof will be by contradiction. Thus, we will assume to the contrary that there is a 5 by 5 full ray-nonsingular matrix A. We then use the results of section 3 (that give sufficient conditions for a 3×3 full ray-pattern to be strongly balanceable) and Lemmas 2.1–2.4 to show that, up to $\sim$-equivalence, the leading 3×3 submatrix of A has one of the following forms:

$$\text{(a)} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & e^{i\alpha} & e^{i\beta} \end{bmatrix}, \quad \text{(b)} \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & e^{i\alpha} & e^{i\beta} \end{bmatrix}, \quad \text{(c)} \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & e^{i\beta} \\ 1 & e^{i\alpha} & -1 \end{bmatrix},$$

$$\text{(d)} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & e^{i\beta} \\ 1 & e^{i\alpha} & -1 \end{bmatrix}, \quad \text{or} \quad \text{(e)} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & e^{i\beta} \\ 1 & e^{i\alpha} & 1 \end{bmatrix}.$$

Next, for each of these cases, we use Lemma 2.2 and the results of section 3 to conclude that either

(i) all entries of A belong to $\{1, -1, i, -i\}$, or
(ii) all entries of A belong to $\{1, e^{2\pi i/3}, e^{4\pi i/3}\}$ arranged in certain patterns.

Finally, we obtain a contradiction by showing that if A satisfies (i) or (ii), then A is not ray-nonsingular.

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3. Sufficient conditions for 3×3 patterns to be strongly balanceable. One of the keys to our proof of the main theorem is Lemma 2.2 which implies that no 3×3 submatrix of a 5×5, full ray-nonsingular matrix is strongly balanceable. In this section, we give sufficient conditions for a 3×3 full ray-pattern to be strongly balanceable.

By Lemma 2.1, we may restrict our attention to ray-patterns of the form

$$B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{i\alpha_2} & e^{i\beta_2} \\ 1 & e^{i\alpha_3} & e^{i\beta_3} \end{bmatrix}. \tag{3.1}$$

As the function e^{ix}, x real, is 2π-periodic, we may assume that each of α_2, α_3, β_2 and β_3 lies in the interval $(-\pi, \pi]$. For convenience we partition $(-\pi, \pi]$ into the following sets:

$$\mathcal{P} = (0, \pi), \quad \mathcal{N} = (-\pi, 0), \quad \{0\}, \quad \{\pi\}.$$

We first determine the strict signings S for which the vector $(1, 1, 1)S$ is strongly balanced. Note that for each $\theta \in (-\pi, \pi]$, the vector $(1, 1, 1)S$ is strongly balanced if and only if the vector $(1, 1, 1)(e^{i\theta}S)$ is strongly balanced. Hence, it suffices to determine the S whose leading diagonal entry is 1.

LEMMA 3.1. *Let $S = \mathrm{diag}(1, e^{ix}, e^{iy})$ be a strict signing with $x, y \in (-\pi, \pi]$. Then $(1, 1, 1)S$ is strongly balanced if and only if $x \in \mathcal{P}$ and $-\pi < y < x - \pi$, or $x \in \mathcal{N}$ and $\pi + x < y < \pi$.*

Proof. Note that $(1, 1, 1)S$ is strongly balanced if and only if no two of 1, e^{ix} and e^{iy} are equal or opposite, and the convex hull, H, of $\{1, e^{ix}, e^{iy}\}$ contains the origin. Thus, $(1, 1, 1)S$ is not strongly balanced if $x = 0$, $x = \pi$, $y = 0$, $y = \pi$ or $x = y \pm \pi$. If $x \in \mathcal{P}$, then it is easy to verify that H contains the origin if and only if $-\pi < y < x - \pi$. If $x \in \mathcal{N}$, then it is easy to verify that H contains the origin if and only if $\pi + x < y < \pi$. The lemma now follows. ☐

The shaded regions without their boundaries given in Figure 1, represent the region of the Cartesian plane determined by the inequalities in Lemma 3.1.

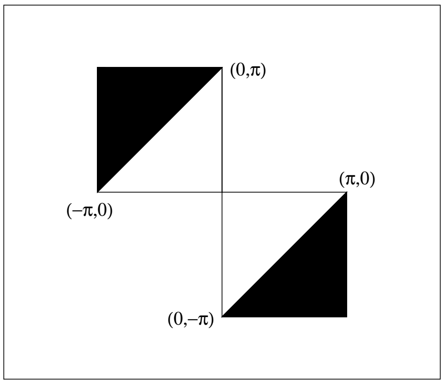

Figure 1

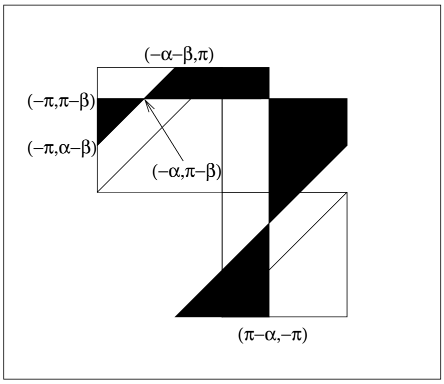

Figure 2

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Next, we investigate a general vector $z = (1, e^{i\alpha}, e^{i\beta})$, and let $R(\alpha, \beta)$ be the region of the Cartesian plane consisting of the points (x, y) such that z diag$(1, e^{ix}, e^{iy})$ is strongly balanced and $x, y \in (-\pi, \pi]$. Thus $R(0, 0)$ is the region described in Lemma 3.1, and illustrated in Figure 1. Let $D = \text{diag}(1, e^{i\alpha}, e^{i\beta})$. Note that S is a strict signing such that zS is strongly balanced if and only if DS is a strict signing such that $(1, 1, 1)DS$ is strongly balanced. It follows that $R(\alpha, \beta)$ can be obtained from $R(0, 0)$ by identifying opposite edges of the square $[-\pi, \pi] \times [-\pi, \pi]$ to form a torus, and then translating the shaded region in Figure 1 by $(-\alpha, -\beta)$. For example, $R(\alpha, \beta)$ (where $\alpha \in \mathcal{P}$ and $\beta < \alpha$) is presented in Figure 2.

Note that $R(0, 0) \cap R(\alpha, \beta)$ represents the points (x, y) in the plane such that $-\pi \le x, y < \pi$ and both rows of

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{i\alpha} & e^{i\beta} \end{bmatrix} \text{diag}(1, e^{ix}, e^{iy})$$

are strongly balanced. It is a tedious, but straightforward, to determine the regions $R(0, 0) \cap R(\alpha, \beta)$. We do this as follows. First partition the vectors of the form $z = \begin{bmatrix} 1 \; e^{i\alpha} \; e^{i\beta} \end{bmatrix}$ according to the locations and relationships between α and β as given by the 24 types described in Table 1. The sets $R(0, 0) \cap R(\alpha, \beta)$ for each of these 24 types are the shaded regions without the boundaries illustrated in the Appendix.

Table 1. Types for the vector $[1 \, e^{ia} \, e^{i\beta}]$.

Type	α in	β in	Conditions
1	$\mathcal{P}$	$\mathcal{P}$	$\alpha > \beta$
2	$\mathcal{P}$	$\mathcal{P}$	$\alpha < \beta$
3	$\mathcal{N}$	$\mathcal{N}$	$\alpha > \beta$
4	$\mathcal{N}$	$\mathcal{N}$	$\alpha < \beta$
5	$\mathcal{P}$	$\mathcal{N}$	$\alpha - \beta < \pi$
6	$\mathcal{P}$	$\mathcal{N}$	$\alpha - \beta > \pi$
7	$\mathcal{N}$	$\mathcal{P}$	$\beta - \alpha < \pi$
8	$\mathcal{N}$	$\mathcal{P}$	$\beta - \alpha > \pi$
9	$\mathcal{P}$	$\mathcal{P}$	$\alpha = \beta$
10	$\mathcal{N}$	$\mathcal{N}$	$\alpha = \beta$
11	$\mathcal{P}$	$\mathcal{N}$	$\alpha - \beta = \pi$
12	$\mathcal{N}$	$\mathcal{P}$	$\beta - \alpha = \pi$

Class	α in	β in
C1	$\mathcal{P}$	$\{0\}$
C2	$\mathcal{N}$	$\{0\}$
C3	$\mathcal{P}$	$\{\pi\}$
C4	$\mathcal{N}$	$\{\pi\}$
C5	$\{0\}$	$\mathcal{P}$
C6	$\{0\}$	$\mathcal{N}$
C7	$\{\pi\}$	$\mathcal{P}$
C8	$\{\pi\}$	$\mathcal{N}$
C9	$\{0\}$	$\{0\}$
C10	$\{0\}$	$\{\pi\}$
C11	$\{\pi\}$	$\{0\}$
C12	$\{\pi\}$	$\{\pi\}$

We finally turn our attention to studying the strong balanceability of the matrix B in (3.1). Note that B is strongly balanceable if and only if $R(0, 0) \cap R(\alpha_2, \beta_2) \cap R(\alpha_3, \beta_3) \neq \emptyset$, or equivalently if and only if

$$(R(0, 0) \cap R(\alpha_2, \beta_2)) \cap (R(0, 0) \cap R(\alpha_3, b_3)) \neq \emptyset.$$

If the second (or third) row has form (C9), i.e. is $[1, 1, 1]$, then trivially, this intersection corresponds to the solution set of the first and third (or second) row. Also, if the second (or third) row has form (C10)-(C12), then the solution set is empty and

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so intersection is trivially empty. Thus, we need only consider those cases when the second and third rows are of one of the first 20 types listed on Table 1 that is, for each pair of these 20 types we need to study the intersection of the pair of corresponding regions listed in the Appendix.

The results of this straight-forward but tedious study are summarized in Table 2 below. The rows and columns of Table 2 are indexed by the 20 classes other than (C9)-(C12). An entry of '1' indicates that the pair of specified regions always has nonempty intersection. For example, the fact that there is a '1' in the row indexed by 9 and column indexed by 1, implies that every 3×3 matrix whose first row is $[1, 1, 1]$, whose second row is of type (9), and whose third row is type (2), is strongly balanceable, and hence not ray-nonsingular.

For some pairs of types the regions intersect only under certain conditions on $\alpha_1, \alpha_2, \beta_1, \beta_2$. For example, consider a matrix B whose rows are $u_1 = [1\ 1\ 1]$, $u_2 = [1\ e^{i\alpha_1}\ e^{i\beta_1}]$ of type (C1) and $u_3 = [1\ e^{i\alpha_2}\ e^{i\beta_2}]$ of type (C5). From the figures in the Appendix, we see that the corresponding solution sets have empty intersection if and only if $\pi - \beta_2 \le \alpha_1$. Table 3 lists other conditions, derived from an analysis of the regions in Figure 1, for certain pairs to have empty intersection. We include only those pairs relevant to our discussion.

Table 2. Types of pairs whose solution sets always have nonempty intersection

	1	2	3	4	5	6	7	8	9	10	11	12	C1	C2	C3	C4	C5	C6	C7	C8
1	1																			
2		1																		
3		1	1																	
4	1			1																
5					1															
6		1		1	1	1														
7					1		1													
8	1		1				1	1												
9	1	1	1	1					1											
10	1	1	1	1						1										
11		1		1	1	1	1				1									
12	1		1		1		1	1				1								
C1	1			1	1		1		1			1	1							
C2	1			1	1		1			1	1			1						
C3		1	1	1	1	1				1	1				1					
C4	1	1	1				1	1	1			1				1				
C5		1	1		1		1		1		1			1		1	1			
C6		1	1		1		1			1		1	1		1			1		
C7	1		1	1			1	1		1		1		1		1			1	
C8	1	1		1	1	1			1		1		1		1					1

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Table 3. Necessary and sufficient conditions for empty intersection of the two solution sets.

Type of $[1\ e^{i\alpha_1}\ e^{i\beta_1}]$	Type of $[1\ e^{i\alpha_2}\ e^{i\beta_2}]$	Condition on $\alpha_1, \alpha_2, \beta_1, \beta_2$
1	6	$\alpha_2 \leq \alpha_1$
1	11	$\alpha_2 \leq \alpha_1$
6	9	$\alpha_1 \leq \alpha_2$
6	10	$\beta_2 \leq \beta_1$
6	C1	$\alpha_1 \leq \alpha_2$
6	C5	$\beta_1 - \beta_2 + 2\pi \leq \alpha_1$
6	C6	$\beta_2 \leq \beta_1$
9	11	$\alpha_2 \leq \alpha_1$
9	C7	$\beta_2 \leq \alpha_1 = \beta_1$
10	12	$\alpha_1 \leq \alpha_2$
10	C1	$\beta_1 + \pi = \alpha_1 + \pi \leq \alpha_2$
10	C5	$\beta_1 + \pi = \alpha_1 + \pi \leq \beta_2$
10	C8	$\beta_1 = \alpha_1 \leq \beta_2$
11	C6	$\beta_2 \leq \beta_1 = \alpha_1 - \pi$
12	C5	$\alpha_1 + \pi = \beta_1 \leq \beta_2$
C1	C5	$\pi \leq \alpha_1 + \beta_2$
C3	C5	$\pi \leq \alpha_1 + \beta_2$
C5	C7	$\beta_2 \leq \beta_1$
C5	C8	$\beta_2 + \pi \leq \beta_1$
C6	C8	$\beta_1 \leq \beta_2$

We conclude this section by illustrating how to use Tables 2 and 3 to determine information about the *columns* of certain matrices. This will allow the reader to get a feel for how these arguments work while also providing information needed later.

Example 3.2. Let B be a 3×3 matrix which is not strongly balanceable and whose rows are

$$u_1 = [\ 1 \quad 1 \quad 1\ ], \quad u_2 = [\ 1 \quad 1 \quad e^{i\beta_1}\ ], \quad u_3 = [\ 1 \quad -1 \quad e^{i\beta_2}\ ],$$

where $\{e^{i\beta_1}, e^{i\beta_2}\} \cap \{\pm 1\} = \emptyset$.

If we assume that $\beta_1 \in \mathcal{P}$, then u_2 has type (C5) and u_3 has type (C7) or (C8). By Table 3, if u_3 has type (C7) then $\beta_2 \leq \beta_1$, if u_3 has type (C8), then $\beta_2 + \pi \leq \beta_1$. If we are interested in the vector $v = [\ 1 \quad e^{i\beta_1} \quad e^{i\beta_2}\ ]$, then v has one of the following types: (1), (6), (9) or (11).

If $\beta_1 \in \mathcal{N}$, then we may apply the above reasoning to $\bar{B}$, and thereby conclude that v has one of the following types: (4), (8), (10), (12).

Therefore, v has type (1), (4), (6), (8), (9), (10), (11) or (12).

A similar analysis gives the following.

Example 3.3. Let B be a 3×3 matrix which is not strongly balanceable and whose rows are

$$u_1 = [\ 1 \quad 1 \quad 1\ ], \quad u_2 = [\ 1 \quad -1 \quad e^{i\beta_1}\ ], \quad u_3 = [\ 1 \quad 1 \quad e^{i\beta_2}\ ],$$

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where $\{e^{i\beta_1}, e^{i\beta_2}\} \cap \{\pm 1\} = \emptyset$. Then $[1\ e^{i\beta_1}\ e^{i\beta_2}]$ has one of the following types: (2), (3), (6) or (8)–(12).

4. Proof of main theorem. Throughout the remainder of the paper we will let A denote a 5×5 full ray nonsingular matrix. We say that A is in standard form if each entry in row and column 1 is equal to 1. By Lemma 2.1, there is no loss of generality in assuming that A is in standard form. We first show that A has a 3×3 submatrix that is $\sim$-equivalent to one of several special forms.

PROPOSITION 4.1. *If A is a 5×5 full ray-nonsingular matrix, then A has a 3×3 submatrix that is $\sim -$equivalent to a matrix of one of the following forms:*

$$\text{(a)}\ \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & e^{i\alpha} & e^{i\beta} \end{bmatrix}, \qquad \text{(b)}\ \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & e^{i\alpha} & e^{i\beta} \end{bmatrix}, \qquad \text{(c)}\ \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & e^{i\beta} \\ 1 & e^{i\alpha} & -1 \end{bmatrix},$$

$$\text{(d)}\ \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & e^{i\beta} \\ 1 & e^{i\alpha} & -1 \end{bmatrix}, \qquad \textit{or} \qquad \text{(e)}\ \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & e^{i\beta} \\ 1 & e^{i\alpha} & 1 \end{bmatrix}.$$

Proof. Let A be a 5×5 full ray-nonsingular matrix in standard form. By Lemma 2.4, each row and column of A intersects a 2×2 submatrix of the form

$$\begin{bmatrix} x & y \\ z & \pm\frac{yz}{x} \end{bmatrix}.$$

By Lemma 2.1, we may assume that the 2×2 submatrix intersecting the first row is $A[\{1,2\},\{1,2\}]$, and that $a_{jk} = 1$ whenever $j = 1$ or $k = 1$. Then $a_{22} = \pm 1$. Let $a_{jk} = e^{ix_{jk}}$ and $u_j = [1, e^{ix_{j2}}, e^{ix_{j3}}]$ for $j, k = 1, 2, 3, 4, 5$.

We claim that one of the following conditions holds:

$$\begin{aligned} &e^{ix_{23}} = \pm 1 \text{ or } e^{ix_{j2}} = \pm 1 \text{ for some } j \in \{3,4,5\}, \\ &e^{ix_{j3}} = \pm 1 \text{ for some } j \in \{3,4,5\}, \\ &e^{ix_{j2}} = \pm e^{ix_{j3}} \text{ for some } j \in \{3,4,5\}. \end{aligned} \tag{4.1}$$

Suppose to the contrary that none of these conditions hold. Then u_3, u_4 and u_5 do not have types (C1)–(C12) nor (9)–(12). Also, u_2 can only have type (C5)–(C8). In fact, since $A \sim \overline{A}$, we may assume without loss of generality that $x_{23} \in \mathcal{P}$, and therefore u_2 has either type (C5) or (C7).

First consider the case that u_2 has type (C5). Because the matrix with rows u_1, u_2 and u_j is not strongly balanceable, Table 2 implies that each u_j $(j = 3, 4, 5)$ has type

(1), (4), (6) or (8).

Note that if two vectors, say u_j and u_k, have the same type, then the matrix with rows u_1, u_j and u_k is strongly balanceable by Table 2. Also, from Table 2, any collection

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of three distinct rows of types (1), (4), (6) or (8) contains two rows whose solutions sets intersect, and we have the contradiction that A contains a strongly balanceable 3×3 submatrix. Thus, u_2 does not have type (C5).

Next consider the remaining case that u_2 has type (C7). Because the matrix with rows u_1, u_2 and u_j is not strongly balanceable, Table 2 implies that each u_j $(j = 3, 4, 5)$ has type

$$(2),\ (5) \text{ or } (6).$$

As no type can be repeated, we can assume that u_3 has type (2), u_4 has type (5) and u_5 has type (6). But then the matrix with rows u_1, u_3 and u_5 is strongly balanceable by Table 2. Thus, u_2 does not have type (C7) and we have a contradiction.

Therefore we have shown that at least one of the three conditions in (4.1) holds. If $e^{ix_{23}} = \pm 1$ or $e^{ix_{j2}} = \pm 1$ for some $j \in \{3, 4, 5\}$, then A has a 3×3 submatrix equivalent to a matrix of form (a) or (b). If for some $j \in \{3, 4, 5\}$ we have $e^{ix_{j3}} = \pm 1$, then A has a 3×3 submatrix equivalent to a matrix of form (c), (d) or (e). Suppose, for some $j \in \{3, 4, 5\}$, that $e^{ix_{j2}} = \pm e^{ix_{j3}}$. Then

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & \pm 1 & e^{ix_{23}} \\ 1 & e^{ix_{j2}} & \pm e^{ix_{j2}} \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ \pm 1 & 1 & \pm e^{ix_{23}} \\ e^{-ix_{j2}} & 1 & \pm 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 1 & \pm 1 & \pm e^{ix_{23}} \\ 1 & e^{-ix_{j2}} & \pm 1 \end{bmatrix}.$$

In other words, A has a 3×3 submatrix equivalent to a matrix of form (c), (d) or (e). □

In the following subsections we show the presence of each type of 3×3 submatrix in Proposition 4.1 leads to a contradiction.

4.1. Form (a).

In this section, we show that a full 5 by 5 full ray-nonsingular matrix does not have a 3×3 submatrix with form (a).

PROPOSITION 4.2. *Let A be a 5×5 full ray pattern whose leading submatrix has form* (a). *Then A is not ray-nonsingular.*

Proof. Assume to the contrary that A is ray-nonsingular. Without loss of generality we may assume that A is in standard form.

Let u_1, u_2, u_3, u_4, u_5 be the rows of $A[\{1, 2, 3, 4, 5\}, \{1, 2, 3\}]$. Thus, $u_1 = u_2 = [\ 1 \quad 1 \quad 1\ ]$. Let

$$u_3 = [\ 1 \quad e^{i\alpha_3} \quad e^{i\beta_3}\ ], \quad u_4 = [\ 1 \quad e^{i\alpha_4} \quad e^{i\beta_4}\ ], \quad u_5 = [\ 1 \quad e^{i\alpha_5} \quad e^{i\beta_5}\ ].$$

Note that since $u_1 = u_2 = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$, u_i $(i \in \{3, 4, 5\})$ is not of the type (1)–(12) nor (C1)–(C9); otherwise the matrix with rows u_1, u_2, u_i is strongly balanceable. So u_i $(i \in \{3, 4, 5\})$ has type (C10), (C11) or (C12). If $u_3 = u_4 = u_5$, then the matrix with rows u_3, u_4, u_5 is strongly balanceable. Thus, u_3, u_4, u_5 are not all equal. Thus, by $\sim$-equivalence, we may assume that A is one of the following two matrices.

$$B_1 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & a_{24} & a_{25} \\ 1 & 1 & -1 & a_{34} & a_{35} \\ 1 & 1 & -1 & a_{44} & a_{45} \\ 1 & -1 & 1 & a_{54} & a_{55} \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & a_{24} & a_{25} \\ 1 & 1 & -1 & a_{34} & a_{35} \\ 1 & -1 & -1 & a_{44} & a_{45} \\ 1 & -1 & 1 & a_{54} & a_{55} \end{bmatrix}.$$

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In both cases, A has the 3×3 submatrices

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & a_{2j} \\ 1 & 1 & a_{3j} \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & a_{2j} \\ 1 & 1 & a_{5j} \end{bmatrix}$$

for $j = 4, 5$. Since the transpose of neither of these matrices is strongly balanceable, $a_{ij} = \pm 1$ for $i = 2, 3, 5$ and $j = 4, 5$. If $A = B_1$, then A has the 3×3 submatrices

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & a_{2j} \\ 1 & 1 & a_{4j} \end{bmatrix},$$

and so $a_{4j} = \pm 1$ for $j = 4, 5$. If $A = B_2$, then A has the 3×3 submatrices

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & a_{2j} \\ -1 & -1 & a_{4j} \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & a_{2j} \\ 1 & 1 & -a_{4j} \end{bmatrix}.$$

Thus, $a_{4j} = \pm 1$ for $j = 4, 5$. And therefore, $a_{kj} = \pm 1$ for all k, j, and by Lemma 2.3, A is not ray-nonsingular–a contradiction.

Therefore, no 3×3 submatrix of A is equivalent to (a). ◻

4.2. Form (b). In this section, we show that the existence of a submatrix of A with form (b) implies that all entries of A are in $\{\pm 1\}$, and thereby contradict Lemma 2.3.

PROPOSITION 4.3. *Let A be a 5×5 full ray pattern whose leading 3×3 principle submatrix has form* (b). *Then A is not ray-nonsingular.*

Proof. Suppose to the contrary that A is ray-nonsingular. Without loss of generality we may assume that A is in standard form.

Let $u_1, \ldots, u_5$ be the rows of $A[\{1, 2, 3, 4, 5\}, \{1, 2, 3\}]$. Thus $u_1 = [1\ 1\ 1]$ and $u_2 = [1\ \ -1\ 1]$. Let

$$u_3 = [\ 1 \quad e^{ix_3} \quad e^{iy_3}\ ], \quad u_4 = [\ 1 \quad e^{ix_4} \quad e^{iy_4}\ ], \quad u_5 = [\ 1 \quad e^{ix_5} \quad e^{iy_5}\ ].$$

By Proposition 4.2, no e^{iy_j} $(j = 3, 4, 5)$ is equal to 1, and not all e^{iy_j} $(j = 3, 4, 5)$ equal -1. Thus, there exists a $j \in \{3, 4, 5\}$, such that $e^{iy_j} \neq \pm 1$ and, since $A \sim \overline{A}$, we may assume without loss of generality that $y_3 \in \mathcal{P}$.

Since for each $j \in \{3, 4, 5\}$ the matrix with rows u_1, u_2, u_j is not strongly balanceable and since no submatrix of A is $\sim$-equivalent to a matrix of form (a), Example 3.3 implies that each u_j $(j = 3, 4, 5)$ has one of the following types:

$$(2), (3), (6), (8)\text{–}(12), (C3)\text{–}(C8), (C10) \text{ or } (C12).$$

In particular, we see that u_3 has one of the following types:

$$(2), (8), (9), (12), (C5) \text{ or } (C7).$$

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Next, let $v_1, \ldots, v_5$ be the rows of the matrix obtained from

$$A[\{1,2,\ldots,5\},\{1,2,3\}]$$

by multiplying its second column by -1. Then

$v_1 = [1\ {-1}\ 1]$, $v_2 = [1\ 1\ 1]$, $v_3 = [1\ {-e^{ix_3}}\ e^{iy_3}]$, $v_4 = [1\ {-e^{ix_4}}\ e^{iy_4}]$, $v_5 = [1\ {-e^{ix_5}}\ e^{iy_5}]$.

Note that u_3 has type (2), (9), (C5) if and only if v_3 has type (8), (12), (C7), respectively. Thus, we may assume without loss of generality that u_3 has type

(2), (9) or (C5).

Now, we consider several subcases.

Case A: Either u_4 or u_5 has type (C10) or (C12).
We may assume that u_5 has type (C10) or (C12); otherwise we permute the fourth and fifth rows of A.

Subcase A.i: u_5 has type (C10); i.e. $u_5 = \begin{bmatrix} 1 & 1 & -1 \end{bmatrix}$.
Recall that u_3 has type (2), (9) or (C5) while u_4 has one of the types: (2), (3), (6), (8), (9)–(12), (C3)–(C8), (C10) or (C12).

Consider the matrix

$$\begin{bmatrix} u_1 \\ u_5 \\ u_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & e^{ix_3} & e^{iy_3} \end{bmatrix}$$

Proposition 4.2 implies that u_3 does not have type (C5). It follows from Example 3.2 that u_3 does not have type (2). Hence u_3 has type (9).

Since the matrix

$$\begin{bmatrix} u_1 \\ u_3 \\ u_4 \end{bmatrix}$$

is not strongly balanceable, Table 2 implies that u_4 is not of type (2), (3), (9), (C4), (C5), or (C8). Since this matrix is not equivalent to a matrix of form (a), u_4 is not of type (10) or (C12).

Since

$$\begin{bmatrix} u_1 \\ u_4 \\ u_5 \end{bmatrix}$$

is not $\sim$-equivalent to a matrix of form (a), u_4 does not have type (C6) or (C10).

Note that

$$\begin{bmatrix} u_2 \\ u_5 \\ u_4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & e^{ix_4} \\ 1 & -1 & e^{iy_4} \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -e^{ix_4} \\ 1 & -1 & e^{iy_4} \end{bmatrix}.$$

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From Table 2, we see that if $e^{ix_j}, e^{iy_j} \neq \pm 1$, then the sign of the imaginary parts of $-e^{ix_j}$ and e^{iy_j} do not agree. In other words, $x_j \in \mathcal{P}$ implies $y_j \in \mathcal{P}$ and $x_j \in \mathcal{N}$ implies $y_j \in \mathcal{N}$. Thus, u_4 does not have type (6), (8), (11) or (12).

If u_4 has type (C3), then v_3 has type (12) and v_4 has type (C4), and so (by Table 2) the matrix with rows v_2, v_3, v_4 is strongly balanceable. Thus, u_4 does not have type (C3).

If u_4 has type (C7), then by Table 2 the matrix with rows u_5, u_3, u_4 is strongly balanceable because it is equivalent to

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & e^{ix_3} & e^{ix_3} \\ 1 & -1 & e^{iy_4} \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{ix_3} & -e^{ix_3} \\ 1 & -1 & -e^{iy_4} \end{bmatrix}$$

which has rows of type (11) and (C8).

Hence, for each possible type of u_4 we obtain a contradiction. Therefore, subcase A.i, does not occur.

Subcase A.ii: u_5 has type (C12); i.e. $u_5 = \begin{bmatrix} 1 & -1 & -1 \end{bmatrix}$.
Recall that u_3 has type (2), (9) or (C5) while u_4 has one of the following types: (2), (3), (6), (8)–(12), (C3)–(C8), (C10) or (C12). If u_3 has type (2) or (9), then

$$\begin{bmatrix} u_1 \\ u_3 \\ u_5 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{ix_3} & e^{iy_3} \\ 1 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{ix_3} & -1 \\ 1 & e^{iy_3} & -1 \end{bmatrix}$$

is strongly balanceable (by Table 2) because the last two rows both have type (C3). Hence u_3 has type (C5).

Note that if u_4 has type (C10) then we are back to Case A.i., and if it has type (C12) then we have a 3×3 submatrix of form (a) and we contradict Proposition 4.2. Table 2 applied to the matrix with rows u_1, u_3, u_4 implies that u_4 does not have type (2), (3), (9), (11), (C4) or (C5).

The type of u_4 is not (6) or (8); otherwise $\begin{bmatrix} 1 & -e^{ix_4} & -e^{iy_4} \end{bmatrix}$ has either type (5) or (7),

$$\begin{bmatrix} u_5 \\ u_2 \\ u_4 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 \\ 1 & -1 & 1 \\ 1 & e^{ix_4} & e^{iy_4} \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -e^{ix_4} & -e^{iy_4} \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -e^{ix_4} \\ 1 & -1 & -e^{iy_4} \end{bmatrix},$$

and Example 3.2 leads to the contradiction that the last matrix is strongly balanceable by Table 2.

The type of u_4 is not (10); otherwise

$$\begin{bmatrix} u_1 \\ u_4 \\ u_5 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{ix_4} & e^{ix_4} \\ 1 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{ix_4} & -1 \\ 1 & e^{iy_4} & -1 \end{bmatrix},$$

which has second and third row of type (C4), and the matrix is strongly balanceable by Table 2.

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The type of u_4 is not (12); otherwise

$$\begin{bmatrix} u_5 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & e^{iy_3} \\ 1 & -e^{iy_4} & e^{iy_4} \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -e^{iy_3} \\ 1 & e^{iy_4} & -e^{iy_4} \end{bmatrix},$$

where $y_3, y_4 \in \mathcal{P}$. This matrix has second row of type (C8) and third row of type (11), and hence is strongly balanceable by Table 2.

The type of u_4 is not (C3); otherwise

$$\begin{bmatrix} u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & e^{iy_3} \\ 1 & e^{ix_4} & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & e^{iy_3} \\ 1 & -e^{ix_4} & -1 \end{bmatrix},$$

where $y_3, x_4 \in \mathcal{P}$. But then the second row has type (C7) and the third row has type (C4), and the matrix is strongly balanceable by Table 2.

If u_4 has type (C6), then the matrix with rows u_1, u_3, u_4 is equivalent to

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & e^{iy_3} \\ 1 & 1 & e^{iy_4} \end{bmatrix}$$

and we contradict Proposition 4.2. Likewise, if u_4 has type (C7) or (C8), then

$$\begin{bmatrix} u_2 \\ u_4 \\ u_5 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -1 & e^{iy_4} \\ 1 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & e^{iy_4} \\ 1 & 1 & -1 \end{bmatrix}$$

and we contradict Proposition 4.2.

Therefore, we conclude that Subcase A.ii does not occur. Moreover, Case A does not occur.

Case B: u_4 and u_5 have neither type (C10) nor (C12).
Recall from the beginning of the proof that u_3 has type (2), (9) or (C5). Also, for $j = 4, 5$, u_j has type (2), (3), (6), (8) – (12), (C3), (C4), (C6), (C7) or (C8).

Subcase B.i: $\{e^{ix_3}, e^{ix_4}, e^{ix_5}\} \cap \{\pm 1\} = \emptyset$.
Since $\{e^{ix_3}, e^{ix_4}, e^{ix_5}\} \cap \{\pm 1\} = \emptyset$, u_3 does not have type (C5), and u_j $(j = 4, 5)$, does not have type (C6), (C7), or (C8).

First suppose u_3 has the type (2). Table 2 applied to the matrices with rows u_1, u_3, u_4, and rows u_1, u_3, u_5 implies that u_j $(j = 4, 5)$ has type (8) or (12). Since the matrix with rows u_1, u_4, u_5 is not strongly balanceable, by Table 2, u_4 and u_5 do not have the same type. We may assume that u_4 has type (8) and u_5 has type (12), but then, by Table 2, the matrix with rows u_1, u_4, u_5 is strongly balanceable. Hence u_3 does not have type (2).

Next suppose u_3 has type (9). By Table 2, applied to the matrix with rows u_1, u_3, u_j, the vector u_j $(j = 4, 5)$ has type (6), (8), (10), (11), (12) or (C3). Note that v_3 has type (12), and Table 2 applied to the matrix with rows v_1, v_3, v_j $(j = 4, 5))$

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implies that v_j has type (2), (4), (6), (9), (10), (11), (C2), (C3), (C5), (C8); that is, u_j has type (8), (5), (3), (12), (11), (10), (C1), (C4), (C7) or (C6). Upon comparison of the two list of possibilities for the type of u_j, we conclude that u_j $(j = 4, 5)$ has type (8), (10), (11) or (12).

If u_j has type (8), then v_j has type (2) and we are back to case handled in the second paragraph of this subcase. Hence u_j does not have type (8).

If u_j has type (10), then the matrix with rows u_1, u_3, u_j is $\sim$-equivalent to a matrix of type (a), contrary to Proposition 4.2. Hence u_j does not have type (10).

Now u_j must have the type (11) or (12). By Table 2, u_4 and u_5 do not have the same type. We may assume that u_4 has type (11) and u_5 has type (12). But then

$$\begin{bmatrix} u_2 \\ u_4 \\ u_5 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -e^{ix_4} & e^{ix_4} \\ 1 & -e^{ix_5} & e^{ix_5} \end{bmatrix} \sim \begin{bmatrix} 1 & e^{-ix_4} & e^{-ix_5} \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix},$$

and we contradict Proposition 4.2.

Thus, we conclude that Subcase B.i does not occur.

Subcase B.ii. $\{e^{ix_3}, e^{ix_4}, e^{ix_5}\} \cap \{\pm 1\} \neq \emptyset$**.**
We know that u_3 has type (2), (9) or (C5). We claim that without loss of generality we may assume that u_3 has type (C5).

To see this, suppose u_3 has type (2) or (9). Since $\{e^{ix_3}, e^{ix_4}, e^{ix_5}\} \cap \{\pm 1\} \neq \emptyset$ and since neither u_4 nor u_5 have type (C10) nor (C12), there exists $j \in \{4, 5\}$ such that $e^{ix_j} = \pm 1$ while $e^{iy_j} \neq \pm 1$. Now, interchange rows 3 and j and if $e^{ix_j} = -1$, multiply the second column by -1 and interchange the first two rows. We may assume $y_3 \in \mathcal{P}$ since $A \sim \overline{A}$. Note that this new third row has type (C5). Therefore, we may assume that u_3 has type (C5).

Recall that u_j $(j = 4, 5)$ has type (2), (3), (6), (8)–(12), (C3), (C4), (C6), (C7) or (C8). By Table 2 applied to the matrix with rows u_1, u_3 and u_j, the vector u_j does not have type (2), (3), (9), (11), or (C4). By considering the matrix with rows u_1, u_3, u_j, we see that by Proposition 4.2, $e^{ix_j} \neq 1$ for $j = 4, 5$; thus, u_j does not have type (C6). Hence u_j can only have one of the following types: (6), (8), (10), (12), (C3), (C7) or (C8).

But v_3 has type (C7) and Table 2 applied to the matrix with rows v_2, v_3 and v_j, implies that has one of the types (8), (4), (3), (12), (10), (C2), (C4), (C7), (C8), or (C6). By comparing the two lists of possibilities for the type of u_j, we conclude that u_j $(j = 4, 5)$ has type (8), (10), (12), (C7) or (C8).

Let $\{j, k\} = \{4, 5\}$. Suppose u_j has type (8). Then u_k does not have type (8), (12) or (C7) else, by Table 2, the matrix with rows u_1, u_j, u_k is strongly balanceable. Note that v_j has type (2). If u_k has type (10) or (C8), then v_k has type (11) or (C6) and so the matrix with rows v_1, v_j and v_k is strongly balanceable. Therefore, u_j does not have type (8).

Next suppose u_j has type (C7). But then the matrix with rows u_1, u_j and u_k is strongly balanceable for u_k of any type but (C8). However, if u_k has type (C8), then

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the matrix formed by rows u_2, u_j, u_k is equivalent to

$$\begin{bmatrix} u_2 \\ u_j \\ u_k \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -1 & e^{iy_j} \\ 1 & -1 & e^{iy_k} \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & e^{iy_j} & e^{iy_k} \end{bmatrix},$$

which contradicts Proposition 4.2. Thus, u_j $(j = 4, 5)$ does not have type $(C7)$.

Therefore, u_j and u_k will have one of the following types: (10), (12) or (C8). They will not both have the same type, else the matrix with rows u_1, u_j and u_k is strongly balanceable. We now examine the restrictions on the entries in each of the possible combinations of types by considering the following subcases.

Subcase B.ii.a: u_4 and u_5 have types (C8) and (10) respectively.
In other words, there exist $\alpha, \beta, \gamma \in \mathcal{P}$ such that

$$u_3 = [\ 1 \quad 1 \quad e^{i\alpha}\ ], \qquad u_4 = [\ 1 \quad -1 \quad e^{i(\beta-\pi)}\ ], \qquad u_5 = [\ 1 \quad e^{i(\gamma-\pi)} \quad e^{i(\gamma-\pi)}\ ].$$

By Table 3, we find conditions on these angles such that there are no 3×3 submatrices that are strongly balanceable. Because the matrix with rows u_1, u_4, u_5 is not strongly balanceable,

$$\gamma \leq \beta. \tag{4.2}$$

Because the matrix with rows v_2, v_4, v_5 is not strongly balanceable,

$$\gamma \geq \beta. \tag{4.3}$$

Equations (4.2) and (4.3) imply $\gamma = \beta$. Also, because the matrix with rows u_1, u_3, u_4 is not strongly balanceable,

$$\gamma = \beta \leq \alpha.$$

Suppose $\gamma = \beta < \alpha$. For $j = 3, 4, 5$, let $\hat{u}_j$ be such that

$$\begin{bmatrix} u_3 \\ u_4 \\ u_5 \end{bmatrix} = \begin{bmatrix} 1 & 1 & e^{i\alpha} \\ 1 & -1 & e^{i(\beta-\pi)} \\ 1 & e^{i(\beta-\pi)} & e^{i(\beta-\pi)} \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & e^{i(\beta-\alpha-\pi)} \\ 1 & e^{i(\beta-\pi)} & e^{i(\beta-\alpha-\pi)} \end{bmatrix} = \begin{bmatrix} \hat{u}_3 \\ \hat{u}_4 \\ \hat{u}_5 \end{bmatrix}.$$

Since $0 < \beta < \beta + (\pi - \alpha) < \alpha + (\pi - \alpha) = \pi$ and $e^{i(\beta+\pi-\alpha)} = e^{i(\beta-\alpha-\pi)}$, $\hat{u}_4$ has type (C7) and $\hat{u}_5$ has type (8), and hence, by Table 2, the matrix with rows $\hat{u}_3, \hat{u}_4, \hat{u}_5$ is strongly balanceable. Therefore

$$\gamma = \beta = \alpha.$$

Subcase B.ii.b. u_4 and u_5 have types (C8) and (12) respectively.
In other words, there exist $\alpha, \beta, \gamma \in \mathcal{P}$ such that

$$u_3 = [\ 1 \quad 1 \quad e^{i\alpha}\ ], \qquad u_4 = [\ 1 \quad -1 \quad e^{i(\beta-\pi)}\ ], \qquad u_5 = [\ 1 \quad -e^{i\gamma} \quad e^{i\gamma}\ ].$$

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The matrix formed by rows $u_1, \ldots, u_5$ is equivalent, by complex conjugation, $(1,-1)$-signings and row permutation, to

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & e^{i\alpha} \\ 1 & -1 & e^{i(\beta-\pi)} \\ 1 & -e^{i\gamma} & e^{i\gamma} \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & -1 & e^{-i\alpha} \\ 1 & 1 & e^{-i(\beta-\pi)} \\ 1 & e^{-i\gamma} & e^{-i\gamma} \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & e^{i(\pi-\beta)} \\ 1 & -1 & e^{-i\alpha} \\ 1 & e^{-i\gamma} & e^{-i\gamma} \end{bmatrix}.$$

Note that the third, fourth and fifth rows of this matrix have types (C5), (C8) and (10) respectively. Thus, by Case B.ii.a., $\alpha = \beta = \gamma$.

Subcase B.ii.c. u_4 **and** u_5 **have types (10) and (12) respectively.**
Therefore, there exists $\alpha, \beta, \gamma \in \mathcal{P}$ such that

$$u_3 = [\ 1 \quad 1 \quad e^{i\alpha}\ ], \qquad u_4 = [\ 1 \quad e^{i(\beta-\pi)} \quad e^{i\beta}\ ], \qquad u_5 = [\ 1 \quad e^{i(\gamma-\pi)} \quad e^{i(\gamma-\pi)}\ ].$$

Once again, we use Table 3 to find necessary conditions on these angles for there to be no 3×3 submatrices that are strongly balanceable. Because the matrix with rows u_1, u_3, u_4 is not strongly balanceable,

$$\beta \le \alpha.$$

Because the matrix with rows v_1, v_3, v_4 is not strongly balanceable,

$$\beta \ge \alpha.$$

Therefore, $\beta = \alpha$. Also, because the matrix with rows u_1, u_4, u_5 is not strongly balanceable,

$$\gamma \le \beta = \alpha.$$

Suppose $\gamma < \beta = \alpha$. For $j = 3, 4, 5$, let $\hat{u}_j$ be such that

$$\begin{bmatrix} u_3 \\ u_4 \\ u_5 \end{bmatrix} = \begin{bmatrix} 1 & 1 & e^{i\alpha} \\ 1 & e^{i(\alpha-\pi)} & e^{i\alpha} \\ 1 & e^{i(\gamma-\pi)} & e^{i(\gamma-\pi)} \end{bmatrix} \sim \begin{bmatrix} 1 & e^{i(\pi-\gamma)} & e^{i(\alpha+\pi-\gamma)} \\ 1 & e^{i(\alpha-\gamma)} & e^{i(\alpha+\pi-\gamma)} \\ 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} \hat{u}_3 \\ \hat{u}_4 \\ \hat{u}_5 \end{bmatrix}.$$

Because $\gamma < \alpha$, thus $0 < (\alpha-\gamma) < \pi - \gamma < \pi$. Also, $e^{i(\alpha-\gamma+\pi)} = e^{i(\alpha-\gamma-\pi)}$. So $\hat{u}_4$ has type (11) and $\hat{u}_3$ has type (6), and therefore the matrix with rows $\hat{u}_5, \hat{u}_3, \hat{u}_4$ is strongly balanceable. Hence

$$\gamma = \beta = \alpha.$$

We now summarize the implications of the analysis in subcases B.ii.a, B.ii.b, and B.ii.c. Let $\alpha = y_3$. Then $\alpha \in \mathcal{P}$ and $A[\{1,2,3\},\{1,2,3\}]$ has the form

$$A_1 = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & e^{i\alpha} \end{bmatrix}.$$

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Let

$$b_1 = \begin{bmatrix} 1 & -1 & -e^{i\alpha} \end{bmatrix},$$

$$b_2 = \begin{bmatrix} 1 & -e^{i\alpha} & -e^{i\alpha} \end{bmatrix}, \quad b_3 = \begin{bmatrix} 1 & -e^{i\alpha} & e^{i\alpha} \end{bmatrix}.$$

Because $A_1^t = A_1$, the constraints found in Subcases B.ii.a–B.ii.c imply that A has the form

$$A = \begin{bmatrix} A_1 & v_4^t & v_5^t \\ u_4 & e^{iz_{44}} & e^{iz_{45}} \\ u_5 & e^{iz_{54}} & e^{iz_{55}} \end{bmatrix},$$

where $u_4, u_5, v_4, v_5 \in \{b_1, b_2, b_3\}$, $u_4 \neq u_5$ and $v_4 \neq v_5$.

We will use this to show that each entry of A is in $\{1, -1, i, -i\}$. We do this by analyzing the 4×4 submatrices

$$\begin{bmatrix} A_1 & v_j^t \\ u_k & e^{iz_{kj}} \end{bmatrix},$$

and show that $e^{iz_{kj}} \in \{\pm 1, \pm e^{i\alpha}\}$ and that $e^{i\alpha} = i$.

First suppose $v_j = b_1$. Note that

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & e^{i\alpha} & -e^{i\alpha} \\ 1 & r & s & t \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ 1 & 1 & e^{i\alpha} & -e^{i\alpha} \\ \overline{r} & 1 & \overline{r}s & \overline{r}t \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -e^{i\alpha} & e^{i\alpha} \\ 1 & \overline{r} & \overline{r}t & \overline{r}s \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -e^{-i\alpha} & e^{-i\alpha} \\ 1 & r & r\overline{t} & r\overline{s} \end{bmatrix}.$$

Therefore, $r, r\overline{t} \in \{\pm 1, \pm e^{-i\alpha}\}$. Let $u_k = [1, r, s]$ and $e^{iz_{kj}} = t$. Note that if $u_k \in \{b_2, b_3\}$, then $r = -e^{i\alpha}$. But this implies $-e^{i\alpha} = e^{-i\alpha}$, i.e. $e^{i\alpha} = i$. And thus $t \in \{\pm 1, \pm i\}$. If $u_k = b_1$, then $r = -1$ and $t \in \{\pm 1, \pm e^{i\alpha}\}$.

Next suppose $v_j = b_2$. Note that

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -e^{i\alpha} \\ 1 & 1 & e^{i\alpha} & -e^{i\alpha} \\ 1 & r & s & t \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & e^{-i\alpha} & -e^{-i\alpha} \\ 1 & -1 & e^{-i\alpha} & 1 \\ 1 & 1 & 1 & 1 \\ 1 & r & se^{-i\alpha} & -te^{-i\alpha} \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & e^{-i\alpha} \\ 1 & 1 & -e^{-i\alpha} & e^{-i\alpha} \\ 1 & r & -te^{-i\alpha} & se^{-i\alpha} \end{bmatrix}.$$

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Therefore, $r, -te^{-i\alpha} \in \{\pm 1, \pm e^{-i\alpha}\}$. Let $u_k = [1, r, s]$ and $e^{iz_{kj}} = t$. Note that if $u_k \in \{b_2, b_3\}$, then $r = -e^{i\alpha}$. But then $-e^{i\alpha} = e^{-i\alpha}$, i.e. $e^{i\alpha} = i$. And thus $t \in \{\pm 1, \pm i\}$. If $u_k = b_1$, then $r = -1$ and $t \in \{\pm 1, \pm e^{i\alpha}\}$.

Finally suppose $v_j = b_3$. Note that

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -e^{i\alpha} \\ 1 & 1 & e^{i\alpha} & e^{i\alpha} \\ 1 & r & s & t \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & e^{i\alpha} \\ 1 & 1 & e^{i\alpha} & e^{i\alpha} \\ \overline{r} & 1 & s\overline{r} & t\overline{r} \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & e^{-i\alpha} & e^{-i\alpha}1 \\ -1 & 1 & -e^{-i\alpha} & 1 \\ 1 & 1 & 1 & 1 \\ \overline{r} & 1 & s\overline{r}e^{-i\alpha} & t\overline{r}e^{-i\alpha} \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -e^{-i\alpha} \\ 1 & 1 & e^{-i\alpha} & e^{-i\alpha} \\ 1 & \overline{r} & t\overline{r}e^{-i\alpha} & s\overline{r}e^{-i\alpha} \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -e^{i\alpha} \\ 1 & 1 & e^{i\alpha} & e^{i\alpha} \\ 1 & r & r\overline{t}e^{i\alpha} & r\overline{s}e^{i\alpha} \end{bmatrix}.$$

So $r, r\overline{t}e^{i\alpha} \in \{\pm 1, \pm e^{i\alpha}\}$, i.e. $t \in \{\pm r, \pm re^{i\alpha}\}$. Let $u_k = [1, r, s]$. If $u_k = b_1$ then $r = -1$ and so $e^{iz_{kj}} = t \in \{\pm 1, \pm e^{i\alpha}\}$. If $u_k \in \{b_2, b_3\}$, or in other words, $r = -e^{i\alpha}$, then $t \in \{\pm e^{i\alpha}, \pm e^{i2\alpha}\}$.

Note that there are 3 choices for the two vectors v_j, therefore, at least one, say v_4 is in $\{b_1, b_2\}$. Similarly, there are two vectors u_k, thus at least one of them, say u_4 is in $\{b_2, b_3\}$. Therefore, $e^{i\alpha} = i$, and so $e^{i2\alpha} = -1$ and $e^{iz_{kj}} \in \{\pm 1, \pm i\}$ for all $j, k = 4, 5$. By Lemma 2.3, A is not ray-nonsingular. □

4.3. Form (c). In this section we show that A does not have a submatrix of form (c).

PROPOSITION 4.4. *Let A be a 5×5 full ray pattern whose leading 3×3 principal submatrix has form* (c). *Then A is not ray-nonsingular.*

Proof. Suppose to the contrary that A is ray-nonsingular. Without loss of generality we may assume that A is in standard form. By Lemma 2.2, no 3×3 submatrix of A is strongly balanceable.

Let

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & e^{i\alpha} \\ 1 & e^{i\beta} & -1 \\ 1 & e^{ix_4} & e^{iy_4} \\ 1 & e^{ix_5} & e^{iy_5} \end{bmatrix}.$$

Propositions 4.2 and 4.3 imply that $e^{i\alpha}, e^{i\beta}, e_{ix_j}, e_{iy_j} \notin \{\pm 1\}$ for $j = 4, 5$. Furthermore, since $A \sim \overline{A}$, we may assume that $\alpha \in \mathcal{P}$. Therefore, u_2 has type (C7). Since the matrix with rows u_1, u_2 and u_3 is not strongly balanceable, by Table 2, u_3 has type (C3), i.e. $\beta \in \mathcal{P}$. Table 2 applied to the matrix with rows u_1, u_3 and u_j, and the matrix with rows u_1, u_2, u_j implies that for $j = 4, 5$, u_j has type (9). But this means that rows u_4 and u_5 both have type (9) and therefore the matrix with rows u_1, u_4 and u_5 is strongly balanceable, which is the desired contradiction. □

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4.4. Form (d). In this section we show that A does not have a submatrix of form (d).

PROPOSITION 4.5. *Let A be a 5×5 full ray pattern whose leading 3×3 principal submatrix has form* (d). *Then A is not ray-nonsingular.*

Proof. Suppose to the contrary that A is ray-nonsingular. Without loss of generality we may assume that A is in standard form. By Lemma 2.2, no 3×3 submatrix of A is strongly balanceable.

Let

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & e^{i\alpha} \\ 1 & e^{i\beta} & -1 \\ 1 & e^{ix_4} & e^{iy_4} \\ 1 & e^{ix_5} & e^{iy_5} \end{bmatrix}.$$

By Propositions 4.2 and 4.3, $e^{i\alpha}, e^{i\beta}, e^{ix_j}, e^{iy_j} \notin \{\pm 1\}$ for $j = 4, 5$. Furthermore, we may assume that $\alpha \in \mathcal{P}$ as $A \sim \overline{A}$. Therefore, u_2 has type (C5). By Table 2, applied to the matrix with rows u_1, u_2 and u_3, u_3 has type (C3), i.e. $\beta \in \mathcal{P}$. As in the proof of Proposition 4.4, u_3 has type (C3), and so for $j = 4, 5$, u_j has type (1), (8), or (12). Since the matrix with rows u_1, u_2 and u_j is also not strongly balanceable, u_j can only have type (1), (8) or (12). By Table 2, we see that the pairwise intersections of the solution sets are non-empty; thus, the matrix with rows u_1, u_4 and u_5 is strongly balanceable–a contradiction. $\square$

4.5. Form (e). In this section we show that A does not have any 3×3 submatrix that is $\sim$-equivalent to a matrix of form (e).

PROPOSITION 4.6. *Let A be a 5×5 full ray-pattern whose leading principle submatrix has the form* (e). *Then A is not ray-nonsingular.*

Proof. Suppose to the contrary that A is ray-nonsingular. Without loss of generality we may assume that A is in standard form. By Lemma 2.2, no 3×3 submatrix of A is strongly balanceble, and by Propositions 4.2-4.5, no 3×3 submatrix of A is equivalent to a matrix of form (a), (b), (c), or (d). We will show that this implies that each entry of A lies in $\{1, e^{\pm i2\pi/3}\}$ and that A has a 4×4 strongly balanceable submatrix (which contradicts Lemma 2.2).

Suppose $u_1, \ldots, u_5$ are the five rows of $[a_{ij}]_{1 \le i \le 5, 1 \le j \le 3}$. Then

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & e^{i\beta} \\ 1 & e^{i\alpha} & 1 \end{bmatrix}.$$

Let

$$u_4 = [\ 1 \quad e^{ix_4} \quad e^{iy_4}\ ], \quad u_5 = [\ 1 \quad e^{ix_5} \quad e^{iy_5}\ ].$$

Note that $e^{i\alpha}, e^{i\beta}, e^{ix_j}, e^{iy_j} \neq \pm 1$, for $j = 4, 5$; otherwise we contradict Proposition 4.2 or 4.3. Furthermore, we may assume that $\alpha \in \mathcal{P}$, otherwise replace A with $\overline{A}$.

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Therefore, u_3 has type (C1). Because the matrix with rows u_1, u_2, u_3 is not strongly balanceable, by Table 2, u_2 has type (C5), i.e. $\beta \in \mathcal{P}$. We also know that

$$\pi \leq \alpha + \beta \tag{4.4}$$

by Table 3. Because the matrix with rows u_1, u_2, u_j, for $j = 4, 5$, is not strongly balanceable, u_j has types (1), (4), (6), (8), (10) or (12). Because the matrix with rows u_1, u_3, u_j is also not strongly balanceable, u_j has one of the following types:

(6), (8) or (10).

We now consider the three cases where u_j has type (6), (8) and (10) and examine the matrices with rows u_l, u_k, u_j where $l, k \in \{1, 2, 3\}$, to find bounds on x_j and y_j dependent on α and β. These bounds are found by using Table 3 for the given matrices.

Case A: u_j has type (6), i.e., $x_j \in \mathcal{P}$, $y_j \in \mathcal{N}$ and $x_j - y_j > \pi$.
Table 3 applied to u_1, u_2, u_j gives

$$0 < y_j - \beta + 2\pi \leq x_j. \tag{4.5}$$

Table 3 applied to u_1, u_3, u_j gives

$$x_j \leq \alpha. \tag{4.6}$$

Note that

$$\begin{bmatrix} u_2 \\ u_3 \\ u_j \end{bmatrix} = \begin{bmatrix} 1 & 1 & e^{i\beta} \\ 1 & e^{i\alpha} & 1 \\ 1 & e^{ix_j} & e^{iy_j} \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{i\alpha} & e^{-i\beta} \\ 1 & e^{ix_j} & e^{i(y_j-\beta)} \end{bmatrix}.$$

The second row of the second matrix has type (6) or (11) because $\alpha + \beta \geq \pi$ by (4.4). Because $e^{i(y_j-\beta)} = e^{i(y_j-\beta+2\pi)}$ and (4.5) holds, the third row has either type (1) or (9). By Table 3,

$$\alpha \leq x_j. \tag{4.7}$$

Equations (4.6) and (4.7) imply

$$\alpha = x_j. \tag{4.8}$$

Also, equation (4.5) implies

$$\alpha + \beta \geq y_j + 2\pi > \pi. \tag{4.9}$$

Case B: u_j has type (8), i.e., $x_j \in \mathcal{N}$, $y_j \in \mathcal{P}$ and $y_j - x_j > \pi$.
Note that the following matrices are equivalent.

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & e^{i\beta} \\ 1 & e^{i\alpha} & 1 \\ 1 & e^{ix_j} & e^{iy_j} \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & e^{i\alpha} \\ 1 & e^{i\beta} & 1 \\ 1 & e^{iy_j} & e^{ix_j} \end{bmatrix}.$$

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Using the same argument in Case A, we have

$$y_j = \beta \text{ and} \tag{4.10}$$

$$\alpha + \beta \geq x_j + 2\pi > \pi. \tag{4.11}$$

Case C: u_j has type (10), i.e. $x_j = y_j \in \mathcal{N}$.
Because the matrix with rows u_1, u_2, u_j is not strongly balanceable,

$$\beta \geq x_j + \pi. \tag{4.12}$$

Also, because the matrix with rows u_1, u_3, u_j is not strongly balanceable,

$$\alpha \geq x_j + \pi. \tag{4.13}$$

We now use the above information to further determine the structure of $u_1, \ldots, u_5$. We have the following three cases.

Case A′. Assume u_4 and u_5 have types (6) and (8) respectively.
Then (4.8)–(4.11) imply that $x_4 = \alpha$, $y_5 = \beta$ and $\alpha + \beta \geq \gamma + 2\pi > \pi$ for $\gamma \in \{y_4, x_5\}$.

Suppose that $\alpha + \beta > y_4 + 2\pi$. Then the following matrices are equivalent.

$$\begin{bmatrix} u_2 \\ u_4 \\ u_5 \end{bmatrix} = \begin{bmatrix} 1 & 1 & e^{i\beta} \\ 1 & e^{i\alpha} & e^{iy_4} \\ 1 & e^{ix_5} & e^{i\beta} \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{i\alpha} & e^{i(y_4-\beta)} \\ 1 & e^{ix_5} & 1 \end{bmatrix}.$$

But the second row has type (1) and the third row has type (C2) and, by Table 2, the matrix is strongly balanceable. Therefore,

$$\alpha + \beta = y_4 + 2\pi. \tag{4.14}$$

Similarly, using the matrix with rows u_3, u_4, u_5, we can show that

$$\alpha + \beta = x_5 + 2\pi. \tag{4.15}$$

Therefore,

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & e^{i\beta} \\ 1 & e^{i\alpha} & 1 \\ 1 & e^{i\alpha} & e^{i(\alpha+\beta)} \\ 1 & e^{i(\alpha+\beta)} & e^{i\beta} \end{bmatrix}.$$

Case B′. Assume u_4 and u_5 have types (6) and (10) respectively.
Then by (4.8), (4.9), (4.12) and (4.13) we have

$$x_4 = \alpha, \qquad \alpha + \beta \geq y_4 + 2\pi > \pi,$$

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$$x_5 = y_5 \in \mathcal{N} \qquad \text{and } x_5 + \pi \le \alpha, \beta.$$

Table 3 applied to u_1, u_4, u_5 implies

$$y_4 \ge x_5. \tag{4.16}$$

Note that

$$\begin{bmatrix} u_2 \\ u_4 \\ u_5 \end{bmatrix} = \begin{bmatrix} 1 & 1 & e^{i\beta} \\ 1 & e^{i\alpha} & e^{iy_4} \\ 1 & e^{ix_5} & e^{ix_5} \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{i\alpha} & e^{i(y_4-\beta)} \\ 1 & e^{ix_5} & e^{i(x_5-\beta)} \end{bmatrix}.$$

Label the second row $\hat{u}_4$ and the third row $\hat{u}_5$. By equation (4.9), $\hat{u}_4$ has type (1) when $y_4 + 2\pi < \alpha + \beta$, and has type (9) when $y_4 + 2\pi = \alpha + \beta$. Also, by (4.12) and $x_5 - \alpha > x_5 - \pi$, we see that $\hat{u}_5$ has type (8) when $\beta > x_5 + \pi$, and has type (C4) when $\beta = x_5 + \pi$. Referring to Table 2, we see that $\hat{u}_4$ must have type (9), i.e. $y_4 + 2\pi = \alpha + \beta$, and $\hat{u}_5$ must have type (8), i.e. $\beta > x_5 + \pi$ because this matrix is not strongly balanceable.

Similarly, we note that the matrix

$$\begin{bmatrix} u_3 \\ u_4 \\ u_5 \end{bmatrix} = \begin{bmatrix} 1 & e^{i\alpha} & 1 \\ 1 & e^{i\alpha} & e^{iy_4} \\ 1 & e^{ix_5} & e^{ix_5} \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & e^{iy_4} \\ 1 & e^{i(x_5-\alpha)} & e^{ix_5} \end{bmatrix}.$$

Again, label the second row $\hat{u}_4$ and the third row $\hat{u}_5$. Note that $\hat{u}_4$ has type (C6) and $\hat{u}_5$ has type (6) or (C8) because $x_5 - \alpha > x_5 - \pi$ and (4.13). But this matrix is not strongly balanceable and so by Table 3,

$$x_5 \ge y_4. \tag{4.17}$$

Equations (4.16), (4.17) and the fact that $y_4 + 2\pi = \alpha + \beta$ imply

$$x_5 = y_4 = \alpha + \beta - 2\pi. \tag{4.18}$$

So

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & e^{i\beta} \\ 1 & e^{i\alpha} & 1 \\ 1 & e^{i\alpha} & e^{i(\alpha+\beta)} \\ 1 & e^{i(\alpha+\beta)} & e^{i(\alpha+\beta)} \end{bmatrix}.$$

Case C′. Assume u_4 and u_5 have types (8) and (10) respectively.
Then

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & e^{i\beta} \\ 1 & e^{i\alpha} & 1 \\ 1 & e^{ix_4} & e^{i\beta} \\ 1 & e^{ix_5} & e^{ix_5} \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & e^{i\alpha} \\ 1 & e^{i\beta} & 1 \\ 1 & e^{i\beta} & e^{ix_4} \\ 1 & e^{ix_5} & e^{ix_5} \end{bmatrix}.$$

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Using the argument in Case B′, we see that

$$x_5 = x_4 = \alpha + \beta - 2\pi$$

and

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & e^{i\beta} \\ 1 & e^{i\alpha} & 1 \\ 1 & e^{i(\alpha+\beta)} & e^{i\beta} \\ 1 & e^{i(\alpha+\beta)} & e^{i(\alpha+\beta)} \end{bmatrix}.$$

We now describe the implications of the analysis in Cases A–C, and A'–C'. Let

$$c_1 = \begin{bmatrix} 1 & e^{i\alpha} & e^{i(\alpha+\beta)} \end{bmatrix},\ c_2 = \begin{bmatrix} 1 & e^{i(\alpha+\beta)} & e^{i\beta} \end{bmatrix},\ c_3 = \begin{bmatrix} 1 & e^{i(\alpha+\beta)} & e^{i(\alpha+\beta)} \end{bmatrix},$$

$$c_4 = \begin{bmatrix} 1 & e^{i\beta} & e^{i(\alpha+\beta)} \end{bmatrix},\quad c_5 = \begin{bmatrix} 1 & e^{i(\alpha+\beta)} & e^{i\alpha} \end{bmatrix}.$$

Using both A by A^t, we see that if A_1 is the 3×3 leading principal submatrix of A, i.e., with rows u_1, u_2, u_3, then

$$A = \begin{bmatrix} A_1 & v_4^t & v_5^t \\ u_4 & e^{iz_{44}} & e^{iz_{45}} \\ u_5 & e^{iz_{54}} & e^{iz_{55}} \end{bmatrix}$$

where $u_4, u_5 \in \{c_1, c_2, c_3\}$ and $v_4, v_5 \in \{c_3, c_4, c_5\}$. We consider the possible 4×4 submatrices for the different values of u_j and v_k and determine the possible values of $e^{iz_{kj}}$.

First suppose $u_j = c_3$. Let $v_k = [1, e^{i\gamma}, e^{i\delta}]$ and $z_{kj} = \lambda$. We consider the following submatrix of A^t:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & e^{i\alpha} & e^{i(\alpha+\beta)} \\ 1 & e^{i\beta} & 1 & e^{i(\alpha+\beta)} \\ 1 & e^{i\gamma} & e^{i\delta} & e^{i\lambda} \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & e^{-i\alpha} & e^{-i(\alpha+\beta)} \\ 1 & 1 & 1 & 1 \\ 1 & e^{i\beta} & e^{-i\alpha} & 1 \\ 1 & e^{i\gamma} & e^{i(\delta-\alpha)} & e^{i(\lambda-\alpha-\beta)} \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & e^{-i(\alpha+\beta)} & e^{-i\alpha} \\ 1 & e^{i\beta} & 1 & e^{-i\alpha} \\ 1 & e^{i\gamma} & e^{i(\lambda-\alpha-\beta)} & e^{i(\delta-\alpha)} \end{bmatrix}.$$

Applying the arguments in Cases A, B, C, A′, B′, C′ to the right most matrix with (α, β) replaced by $(\beta, -(\alpha+\beta))$, we conclude that

$$\begin{bmatrix} 1 \\ e^{i\gamma} \\ e^{i(\lambda-\alpha-\beta)} \end{bmatrix} \in \left\{ \begin{bmatrix} 1 \\ e^{i\beta} \\ e^{-i\alpha} \end{bmatrix}, \begin{bmatrix} 1 \\ e^{-i\alpha} \\ e^{-i(\alpha+\beta)} \end{bmatrix}, \begin{bmatrix} 1 \\ e^{-i\alpha} \\ e^{-i\alpha} \end{bmatrix} \right\}.$$

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Thus, $e^{i\lambda} \in \{1, e^{i\beta}\}$ and $e^{i\gamma} \in \{e^{i\beta}, e^{i(-\alpha)}\}$. Because $-\alpha, (\alpha+\beta) \in \mathcal{N}$ and $\beta \in \mathcal{P}$, if $v_k = c_3$ or c_5, then $e^{i\gamma} = e^{i(\alpha+\beta)} = e^{-i\alpha}$, i.e. $e^{i(2\alpha+\beta)} = 1$.

Next suppose $u_j = c_2$. Let $v_k = [1, e^{i\gamma}, e^{i\delta}]$ and $z_{kj} = \lambda$. We consider the following submatrix of A^t:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & e^{i\alpha} & e^{i(\alpha+\beta)} \\ 1 & e^{i\beta} & 1 & e^{i\beta} \\ 1 & e^{i\gamma} & e^{i\delta} & e^{i\lambda} \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ e^{-i\alpha} & e^{-i\alpha} & 1 & e^{i\beta} \\ 1 & e^{i\beta} & 1 & e^{i\beta} \\ e^{-i\delta} & e^{i(\gamma-\delta)} & 1 & e^{i(\lambda-\delta)} \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & e^{-i\beta} & 1 & e^{-i\beta} \\ e^{-i\alpha} & e^{-i(\beta+\alpha)} & 1 & 1 \\ 1 & 1 & 1 & 1 \\ e^{-i\delta} & e^{i(\gamma-\delta-\beta)} & 1 & e^{i(\lambda-\delta-\beta)} \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & e^{-i\beta} & e^{-i\beta} \\ 1 & e^{-i\alpha} & 1 & e^{-i(\alpha+\beta)} \\ 1 & e^{-i\delta} & e^{i(\lambda-\delta-\beta)} & e^{i(\gamma-\delta-\beta)} \end{bmatrix}.$$

Applying the arguments in Cases A, B, C, A′, B′, C′ to the right most matrix with (α, β) replaced by $(-\alpha, -\beta)$, we conclude that

$$\begin{bmatrix} 1 \\ e^{-i\delta} \\ e^{i(\lambda-\delta-\beta)} \end{bmatrix} \in \left\{ \begin{bmatrix} 1 \\ e^{-i\alpha} \\ e^{-i(\alpha+\beta)} \end{bmatrix}, \begin{bmatrix} 1 \\ e^{-i(\alpha+\beta)} \\ e^{-i\beta} \end{bmatrix}, \begin{bmatrix} 1 \\ e^{-i(\alpha+\beta)} \\ e^{-i(\alpha+\beta)} \end{bmatrix} \right\}.$$

Thus, $e^{i\lambda} \in \{e^{i\delta}, e^{i(\delta-\alpha)}\}$. Also,

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & e^{i\alpha} & e^{i(\alpha+\beta)} \\ 1 & e^{i\beta} & 1 & e^{i\beta} \\ 1 & e^{i\gamma} & e^{i\delta} & e^{i\lambda} \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & e^{i\alpha} & e^{i(\alpha+\beta)} \\ e^{-i\beta} & 1 & e^{-i\beta} & 1 \\ e^{-i\gamma} & 1 & e^{i(\delta-\gamma)} & e^{i(\lambda-\gamma)} \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & e^{i(\alpha+\beta)} & e^{i\alpha} \\ 1 & e^{-i\beta} & 1 & e^{-i\beta} \\ 1 & e^{-i\gamma} & e^{i(\lambda-\gamma)} & e^{i(\delta-\gamma)} \end{bmatrix}.$$

Applying the arguments in A, B, C, A′, B′, C′ to the right most matrix with (α, β) replaced by $(-\beta, \alpha+\beta)$, we conclude that

$$\begin{bmatrix} 1 \\ e^{-i\gamma} \\ e^{i(\lambda-\gamma)} \end{bmatrix} \in \left\{ \begin{bmatrix} 1 \\ e^{-i\beta} \\ e^{i\alpha} \end{bmatrix}, \begin{bmatrix} 1 \\ e^{i\alpha} \\ e^{i(\alpha+\beta)} \end{bmatrix}, \begin{bmatrix} 1 \\ e^{i\alpha} \\ e^{i\alpha} \end{bmatrix} \right\}.$$

Thus, $e^{i\lambda} \in \{e^{i(\alpha+\beta+\gamma)}, e^{i(\alpha+\gamma)}\}$ and $e^{i\gamma} \in \{e^{i(-\alpha)}, e^{i\beta}\}$. Since $-\alpha \in \mathcal{N}$ and $\beta \in \mathcal{P}$, if $v_k = c_3$ or c_5, then $e^{i\gamma} = e^{i(\alpha+\beta)} = e^{-i\alpha}$. In other words, $e^{i(2\alpha+\beta)} = 1$. Furthermore, if $v_k = c_3$, then $e^{i\delta} = e^{i(\alpha+\beta)}$ and therefore, $e^{i\lambda} \in \{e^{i(\alpha+\beta)}, e^{i\beta}\} \bigcap \{e^{i\beta}, 1\}$. So $e^{i\lambda} = e^{i\beta}$. If $v_k = c_5$, then $e^{i\delta} = e^{i\alpha}$; therefore, $e^{i\lambda} \in \{e^{i\alpha}, 1\} \bigcap \{e^{i\beta}, 1\}$. So

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either $e^{i\lambda} = 1$ or $e^{i\lambda} = e^{i\alpha} = e^{i\beta}$. If $v_k = c_4$, then $e^{i\delta} = e^{i(\alpha+\beta)}$ and $e^{i\gamma} = e^{i\beta}$. Hence, $e^{i\lambda} \in \{e^{i(\alpha+\beta)}, e^{i\beta}\} \bigcap \{e^{i(2\beta+\alpha)}, e^{i(\alpha+\beta)}\}$. Recall that $\alpha, \beta \in \mathcal{P} = (0, \pi)$. Thus, $e^{i\lambda} = e^{i(\alpha+\beta)}$.

Finally suppose $u_j = c_1$. Let $v_k = [1, e^{i\gamma}, e^{i\delta}]$ and $z_{kj} = \lambda$. We consider the following submatrix of A^t:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & e^{i\alpha} & e^{i\alpha} \\ 1 & e^{i\beta} & 1 & e^{i(\alpha+\beta)} \\ 1 & e^{i\gamma} & e^{i\delta} & e^{i\lambda} \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & e^{i\beta} & e^{i(\alpha+\beta)} \\ 1 & e^{i\alpha} & 1 & e^{i\alpha} \\ 1 & e^{i\delta} & e^{i\gamma} & e^{i\lambda} \end{bmatrix}.$$

Interchanging the roles of (α, γ) and (β, δ), we see that this is similar to the case when $u_j = c_2$. In other words,

$$e^{i\lambda} \in \{e^{i\gamma}, e^{i(\gamma-\beta)}\} \bigcap \{e^{i(\alpha+\beta+\delta)}, e^{i(\beta+\gamma)}\} \text{ and } e^{i\delta} \in \{e^{i(-\beta)}, e^{i\alpha}\}.$$

If $v_k = c_3$ or c_4, then $e^{i\delta} = e^{i(\alpha+\beta)}$ and so $e^{i(\alpha+\beta)} = e^{-i\beta}$, i.e. $e^{i(2\beta+\alpha)} = 1$. Furthermore, if $v_k = c_3$, then $e^{i\lambda} = e^{i\alpha}$. If $v_k = c_4$, then either $e^{i\lambda} = 1$ or $e^{i\lambda} = e^{i\alpha} = e^{i\beta}$. If $v_k = c_5$, then $e^{i\lambda} = e^{i(\alpha+\beta)}$.

We now turn our attention to the possible forms of v_j. First suppose that $v_j = c_5$. Let $u_k = [1, e^{i\gamma}, e^{i\delta}]$ and $z_{jk} = \lambda$. Interchanging α and β, and using the transpose of A, we see that this is similar to the case when $u_j = c_2$. Thus, $e^{i\lambda} \in \{e^{i\delta}, e^{i(\delta-\beta)}\} \bigcap \{e^{i(\alpha+\beta+\gamma)}, e^{i(\beta+\gamma)}\}$ and $e^{i\gamma} \in \{e^{i\alpha}, e^{-i\beta}\}$. Therefore, if $u_k = c_2$ or c_3, then $e^{i\gamma} = e^{i(\alpha+\beta)} = e^{-i\beta}$, i.e. $e^{i(\alpha+2\beta)} = 1$. Furthermore, if $u_k = c_3$, then $e^{i\lambda} = e^{i\alpha}$. If $u_k = c_2$, then either $e^{i\lambda} = 1$ or $e^{i\lambda} = e^{i\beta} = e^{i\alpha}$. And if $u_k = c_1$, then $e^{i\lambda} = e^{i(\alpha+\beta)}$.

Next suppose that $v_j = c_4$. Let $u_k = [1, e^{i\gamma}, e^{i\delta}]$ and $z_{jk} = \lambda$. Interchanging α and β, and using the transpose of A, we see that this is similar to the case when $u_j = c_1$. Hence, $e^{i\lambda} \in \{e^{i\gamma}, e^{i(\gamma-\alpha)}\} \bigcap \{e^{i(\alpha+\beta+\delta)}, e^{i(\alpha+\delta)}\}$ and $e^{i\delta} \in \{e^{i\beta}, e^{-i\alpha}\}$. Therefore, if $u_k = c_1$ or c_3, then $e^{i\delta} = e^{i(\alpha+\beta)} = e^{-i\alpha}$, i.e. $e^{i(2\alpha+\beta)} = 1$. Furthermore, if $u_k = c_3$, then $e^{i\lambda} = e^{i\beta}$. And if $u_k = c_1$, then either $e^{i\lambda} = 1$ or $e^{i\lambda} = e^{i\beta} = e^{i\alpha}$. If $u_k = c_2$, then $e^{i\lambda} = e^{i(\alpha+\beta)}$.

Finally suppose $v_j = c_3$. Let $u_k = [1, e^{i\gamma}, e^{i\delta}]$ and $z_{jk} = \lambda$. Interchanging α and β, and using the transpose of A, we see that this is similar to the case when $u_j = c_3$. Thus, $e^{i\lambda} \in \{1, e^{i\alpha}\}$ and $e^{i\gamma} \in \{e^{i\alpha}, e^{-i\beta}\}$. So, if $u_k = c_2$ or c_3, then $e^{i\gamma} = e^{i(\alpha+\beta)} = e^{-i\beta}$. In other words, $e^{i(\alpha+2\beta)} = 1$.

Note that $\{u_4, u_5\} \bigcap \{c_2, c_3\} \neq \emptyset$ and also $\{v_4, v_5\} \bigcap \{c_3, c_5\} \neq \emptyset$. Therefore, $e^{-i\beta} = e^{i(\alpha+\beta)} = e^{-i\alpha}$ and so $\alpha = \beta$ and $e^{i(3\alpha)} = 1$. Let $\omega = e^{i\alpha}$ so that $w^2 = e^{i(\alpha+\beta)}$.

We can always assume that if $c_3 \in \{u_4, u_5, v_4, v_5\}$, then $u_5 = c_3$ (since $A \sim A^t$ and $A \sim PAQ$ where P, Q are permutation matrices). Also, if $u_j = c_1$, then interchange the second and third row and column to get $u_j = c_2$. Thus, we may assume that the pair of pairs $((u_4, u_5), (v_4, v_5))$ is one of the following: $((c_2, c_3), (c_5, c_3))$, $((c_2, c_3), (c_4, c_3))$, $((c_2, c_3), (c_5, c_4))$, $((c_2, c_1), (c_5, c_4))$. Hence, A is one of the follow-

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ing matrices:

$$B_1 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & \omega & \omega^2 & \omega^2 \\ 1 & \omega & 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega & x_1 & \omega \\ 1 & \omega^2 & \omega^2 & \omega & y_1 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & \omega & \omega & \omega^2 \\ 1 & \omega & 1 & \omega^2 & \omega^2 \\ 1 & \omega^2 & \omega & \omega^2 & \omega \\ 1 & \omega^2 & \omega^2 & \omega & x_2 \end{bmatrix},$$

$$B_3 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & \omega & \omega^2 & \omega \\ 1 & \omega & 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega & x_3 & \omega^2 \\ 1 & \omega^2 & \omega^2 & \omega & \omega \end{bmatrix}, \qquad B_4 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & \omega & \omega^2 & \omega \\ 1 & \omega & 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega & x_4 & \omega^2 \\ 1 & \omega & \omega^2 & \omega^2 & y_4 \end{bmatrix},$$

with $x_i, y_i \in \{1, \omega\}$ for $i = 1, \ldots, 4$. However, if x_1, y_1, x_2, x_3 or $y_4 = \omega$, then we contradict Proposition 4.2 because A has the submatrix

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega^2 & \omega \end{bmatrix} \sim \begin{bmatrix} 1 & \omega & \omega^2 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Also, if $x_4 = \omega$, then we contradict Proposition 4.2 because A has the submatrix

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega & \omega^2 \end{bmatrix} \sim \begin{bmatrix} 1 & \omega^2 & \omega \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Thus, $x_i, y_i = 1$ for all i. But, for each of the four matrices B_1, B_2, B_3, B_4, there exists a 4×4 strongly balanceable submatrix (since each row contains each of the entries $1, \omega, \omega^2$). To find these submatrices, in each case remove the first row. For B_1 and B_4, remove the first column. For B_2 and B_3, remove the third and second columns, respectively. Thus A is not ray-nonsingular. ☐

Propositions 4.1-4.6 imply our main result:
Theorem 4.7. *There does not exist a* 5×5 *full ray-nonsingular matrix.*

Combined with the results of [1, 2], we have the the main theorem:

Main Theorem *There is an* $n \times n$ *full ray-nonsingular matrix if and only if* $n \leq 4$.

REFERENCES

[1] G.Y. Lee, J.J. McDonald, B.L. Shader, and M.J. Tsatsomeros. Extremal properties of ray-nonsingular matrices. *Discrete Math.*, 216:221–233, 2000.
[2] J.J. McDonald, D.D. Olesky, M.J. Tsatsomeros, and P. van den Driessche. Ray patterns of matrices and nonsingularity. *Linear Algebra Appl.*, 267:359–373, 1997.

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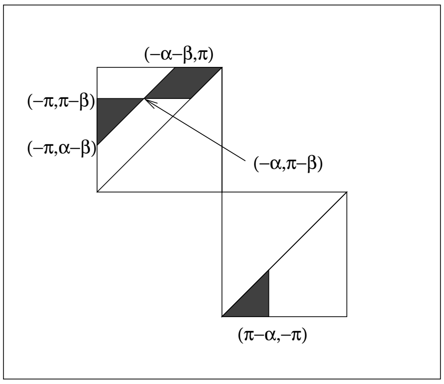
Appendix

Graphical representations of $R(0,0) \cap R(\alpha,\beta)$.



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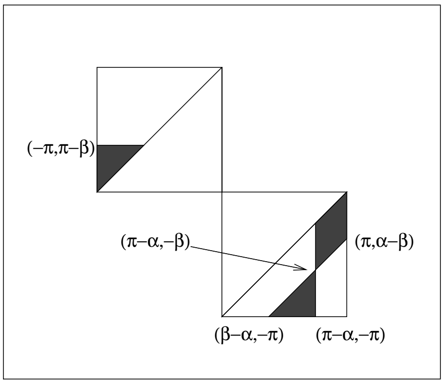

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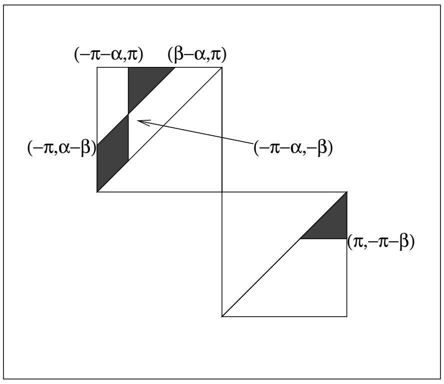

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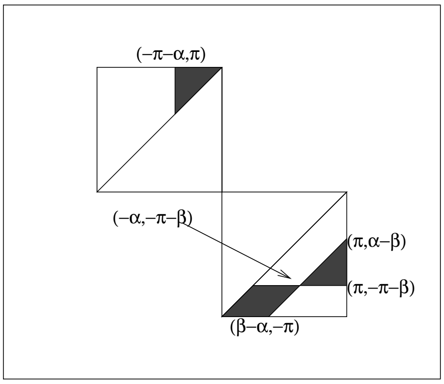

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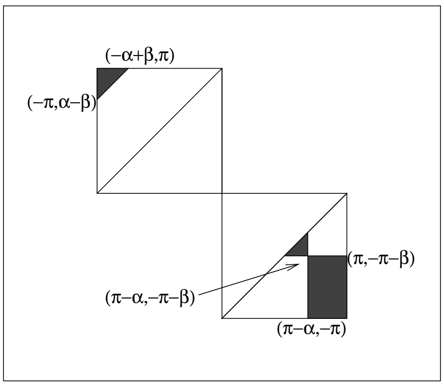

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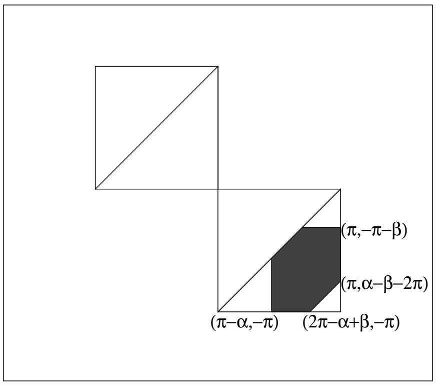

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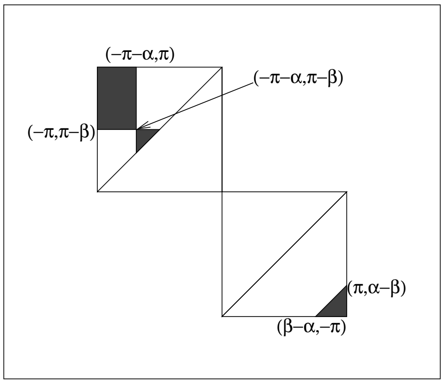

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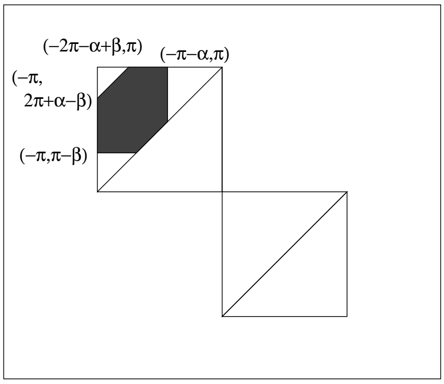

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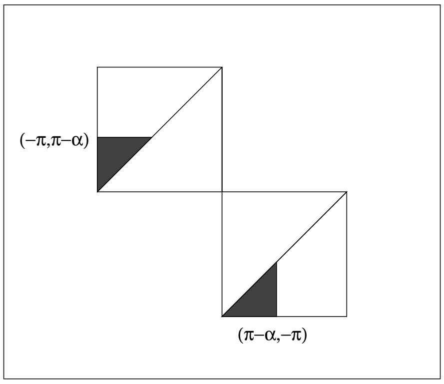

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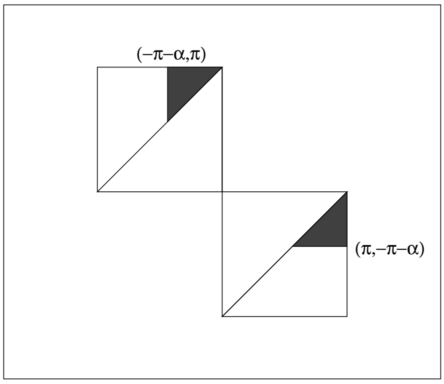

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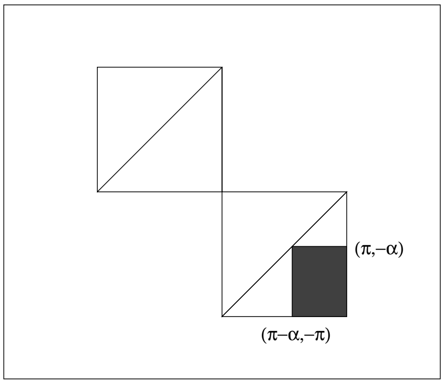

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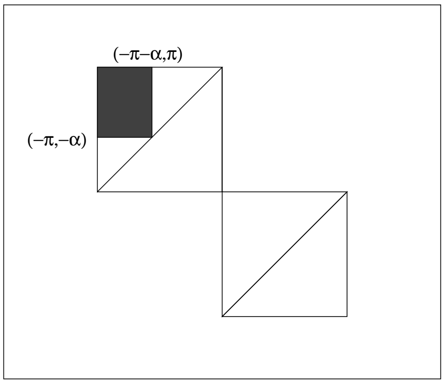

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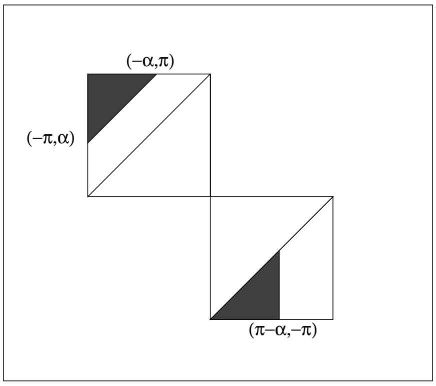

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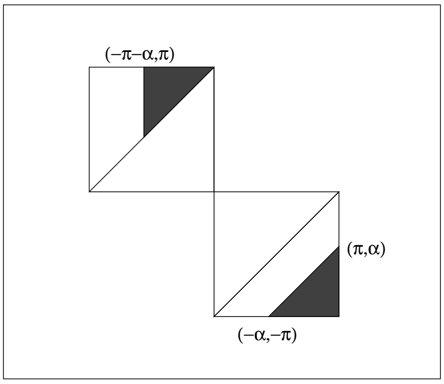

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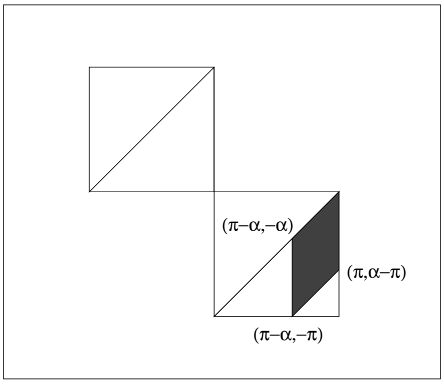

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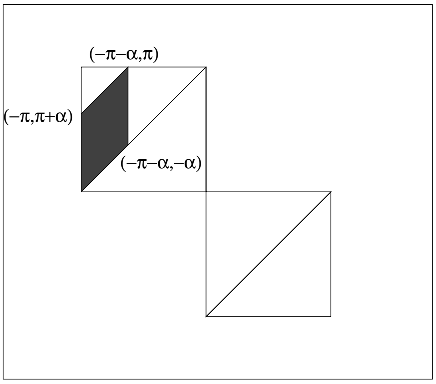

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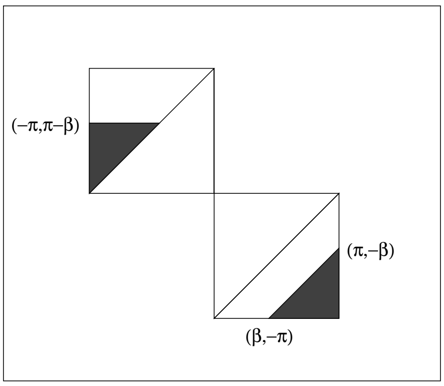

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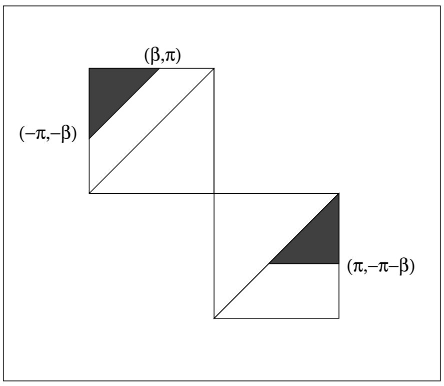

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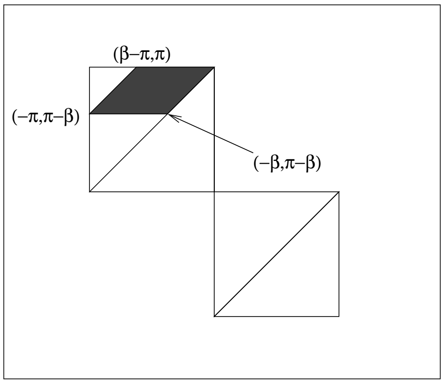

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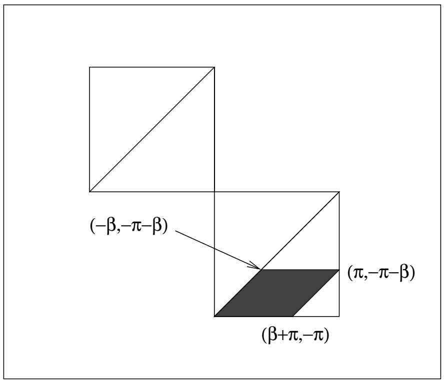

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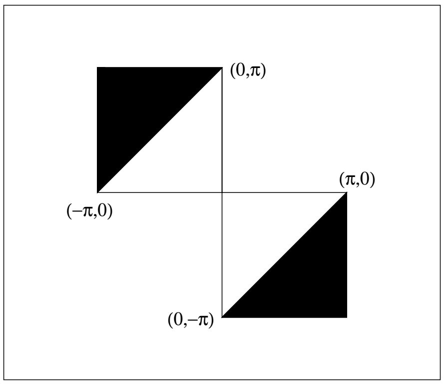

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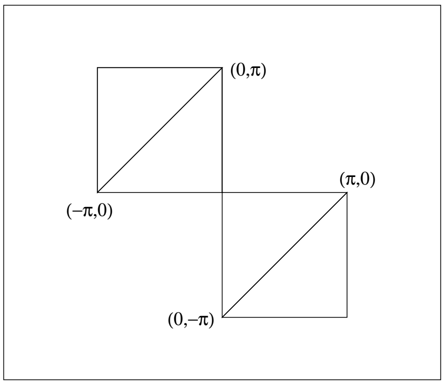

Forms (C10-12)