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ON REDUCING AND DEFLATING SUBSPACES OF MATRICES*

V. MONOV† AND M. TSATSOMEROS‡

Abstract. A multilinear approach based on Grassmann representatives and matrix compounds is presented for the identification of reducing pairs of subspaces that are common to two or more matrices. Similar methods are employed to characterize the deflating pairs of subspaces for a regular matrix pencil $A + sB$, namely, pairs of subspaces $(\mathcal{L}, \mathcal{M})$ such that $A\mathcal{L} \subseteq \mathcal{M}$ and $B\mathcal{L} \subseteq \mathcal{M}$.

Key words. Reducing subspace, Deflating subspace, Invariant subspace, Compound matrix, Grassmann space, Decomposable vector, Matrix pencil.

AMS subject classifications. 15A75, 47A15, 15A69, 15A22.

1. Introduction. The notions of invariant, reducing and deflating subspaces are well known in linear algebra and matrix theory. Invariant and reducing subspaces play a key role in studying the spectral properties and canonical forms of matrices and have a number of important applications [3]. The concept of a deflating subspace is of particular importance in matrix pencil theory and in solving matrix algebraic equations arising in optimization and control theory [1]. The existence of a non-trivial common invariant subspace for two matrices is considered in [2] and [9] by employing some basic tools of multilinear algebra. Under certain assumptions, a procedure to check whether such a subspace exists is proposed in [2], and a general necessary and sufficient condition is obtained in [9]. In this paper, the approach of [9] is extended to characterize and study the existence of non-trivial reducing and deflating subspaces for two matrices. In particular, Section 3 contains necessary and sufficient conditions for the existence of a reducing subspace of dimension k $(1 \le k < n)$ for a pair of matrices. The main result in Section 4 is a characterization of the deflating subspaces for a regular matrix pencil that also yields some reducibility conditions. The results and their usage are illustrated in Section 5.

2. Preliminaries. Let $\langle .\,, . \rangle$ be the usual inner product on $\mathbb{C}^n$, i.e., $\langle x, y \rangle = x^* y$ $(x, y \in \mathbb{C}^n)$, where $*$ denotes complex conjugate transposition. Recall first that the *sum* of two subspaces $\mathcal{L}$ and $\mathcal{M}$ of $\mathbb{C}^n$ is defined as $\mathcal{L} + \mathcal{M} = \{z \in \mathbb{C}^n : z = x + y,\ x \in \mathcal{L},\ y \in \mathcal{M}\}$. The sum is said to be *direct* if $\mathcal{L} \cap \mathcal{M} = \{0\}$ in which case it is denoted by $\mathcal{L} \dot{+} \mathcal{M}$. The subspaces $\mathcal{L}$ and $\mathcal{M}$ are *complementary* (direct complements) if $\mathcal{L} \cap \mathcal{M} = \{0\}$ and $\mathcal{L} \dot{+} \mathcal{M} = \mathbb{C}^n$. Subspaces $\mathcal{L}$ and $\mathcal{M}$ are *orthogonal* if $\langle x, y \rangle = 0$ for every $x \in \mathcal{L}$ and $y \in \mathcal{M}$; they are *orthogonal complements* if, in addition, they are complementary. In the latter case we write $\mathcal{L} = \mathcal{M}^{\perp}$ and $\mathcal{M} = \mathcal{L}^{\perp}$.

For any $A \in \mathbb{C}^{n \times n}$ and $\mathcal{S} \subseteq \mathbb{C}^n$, $A\mathcal{S}$ denotes the set $\{Ax : x \in \mathcal{S}\}$. A subspace $\mathcal{L} \subseteq \mathbb{C}^n$ is *invariant* for $A \in \mathbb{C}^{n \times n}$ (or $A-$*invariant*) if $A\mathcal{L} \subseteq \mathcal{L}$. An $A-$invariant

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† Institute of Information Technologies, Bulgarian Academy of Sciences, 1113 Sofia, Bulgaria (vmonov@iit.bas.bg).

‡ Mathematics Department, Washington State University, Pullman, WA 99164-3113, U.S.A., (tsat@math.wsu.edu).

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subspace $\mathcal{L}$ is $A-$*reducing* if there exists a direct complement $\mathcal{M}$ to $\mathcal{L}$ in $\mathbb{C}^n$ that is also $A-$invariant; the pair of subspaces $(\mathcal{L}, \mathcal{M})$ is then called a *reducing pair* for A. Clearly, $\{0\}$, $\mathbb{C}^n$ and the generalized eigenspaces of $A \in \mathbb{C}^{n\times n}$ are examples of $A-$reducing subspaces.

Let $A, B \in \mathbb{C}^{n\times n}$ and $\mathcal{L}, \mathcal{M} \subseteq \mathbb{C}^n$ be $k-$dimensional subspaces, where $0 \le k \le n$. Then $\mathcal{L}$ and $\mathcal{M}$ are called *deflating subspaces* for A and B if $A\mathcal{L} \subseteq \mathcal{M}$ and $B\mathcal{L} \subseteq \mathcal{M}$. As $k = 0$ and $k = n$ correspond to the trivial cases $\mathcal{L} = \mathcal{M} = \{0\}$ and $\mathcal{L} = \mathcal{M} = \mathbb{C}^n$, respectively, we shall consider the cases $1 \le k < n$.

The following basic notation and facts from multilinear algebra will be used; see e.g., [6]. Given positive integers $k \le n$, let $Q_{k,n}$ be the set of all $k-$tuples of $\{1,\dots,n\}$ with elements in increasing order. The members of $Q_{k,n}$ are considered ordered lexicographically.

For any matrix $X \in \mathbb{C}^{m\times n}$ and nonempty $\alpha \subseteq \{1,\dots,m\}$, $\beta \subseteq \{1,\dots,n\}$, let $X[\alpha\,|\,\beta]$ denote the submatrix of X in rows and columns indexed by α and β, respectively. Given an integer $0 < k \le \min\{m,n\}$, the $k-$*th compound* of X is defined as the $\binom{m}{k} \times \binom{n}{k}$ matrix

$$X^{(k)} = (\det X[\alpha\,|\,\beta])_{\alpha\in Q_{k,m},\beta\in Q_{k,n}}.$$

Matrix compounds satisfy $(XY)^{(k)} = X^{(k)}Y^{(k)}$. The *exterior product* of the vectors $x_i \in \mathbb{C}^n$ $(i = 1,\dots,k)$, denoted by $x_1 \wedge \dots \wedge x_k$, is the $\binom{n}{k}-$component vector equal to the $k-$th compound of $X = [x_1 \,|\, \dots \,|\, x_k]$; i.e.,

$$x_1 \wedge \dots \wedge x_k = X^{(k)}.$$

Consequently, if $A \in \mathbb{C}^{n\times n}$ and $0 < k \le n$, the first column of $A^{(k)}$ is precisely the exterior product of the first k columns of A. Exterior products satisfy the following:

$$x_1 \wedge \dots \wedge x_k = 0 \iff x_1,\dots,x_k \text{ are linearly dependent.} \tag{2.1}$$

$$\mu_1 x_1 \wedge \dots \wedge \mu_k x_k = \prod_{i=1}^{k} \mu_i (x_1 \wedge \dots \wedge x_k) \quad (\mu_i \in \mathbb{C}). \tag{2.2}$$

$$A^{(k)}(x_1 \wedge \dots \wedge x_k) = Ax_1 \wedge \dots \wedge Ax_k. \tag{2.3}$$

A vector $x \in \Lambda^k(\mathbb{C}^n)$ in the $k-$th Grassmann space over $\mathbb{C}^n$, is called *decomposable* if $x = x_1 \wedge \dots \wedge x_k$ for some $x_i \in \mathbb{C}^n$ $(i = 1,\dots,k)$. We refer to $x_1,\dots,x_k$ as the *factors* of x. By conditions (2.2) and (2.3), those decomposable vectors whose factors are linearly independent eigenvectors of $A \in \mathbb{C}^{n\times n}$ are eigenvectors of $A^{(k)}$. The spectrum of $A^{(k)}$ coincides with the set of all possible $k-$products of the eigenvalues of A. In general, not all eigenvectors of a matrix compound are decomposable.

Consider now a $k-$dimensional subspace $\mathcal{L} \subseteq \mathbb{C}^n$ spanned by $\{x_1,\dots,x_k\}$. By (2.1) and the definition of the exterior product it follows that

$$\mathcal{L} = \{x \in \mathbb{C}^n : x \wedge x_1 \wedge \dots \wedge x_k = 0\}.$$

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The vector $x_1\wedge\ldots\wedge x_k$ is known as a *Grassmann representative* of $\mathcal{L}$. As a consequence, two $k-$dimensional subspaces spanned by $\{x_1,\ldots,x_k\}$ and $\{y_1,\ldots,y_k\}$, respectively, coincide if and only if for some nonzero $\mu\in\mathbb{C}$,

$$x_1\wedge\ldots\wedge x_k=\mu(y_1\wedge\ldots\wedge y_k); \tag{2.4}$$

that is, Grassmann representatives for a subspace differ only by a nonzero scalar factor.

Finally, let $A\in\mathbb{C}^{n\times n}$ and let $\mathcal{L}\subseteq\mathbb{C}^n$ be an $A-$invariant subspace with basis $\{x_1,\ldots,x_k\}$. We shall use the fact that any Grassmann representative of $\mathcal{L}$ is an eigenvector of $A^{(k)}$. This is seen by noting that if $A\mathcal{L}\subseteq\mathcal{L}$, then properties (2.1) and (2.3) imply that $A^{(k)}(x_1\wedge\ldots\wedge x_k)$ is either 0 or a Grassmann representative of $\mathcal{L}$; that is, $A^{(k)}(x_1\wedge\ldots\wedge x_k)$ is indeed a scalar multiple of $x_1\wedge\ldots\wedge x_k\neq 0$.

3. Reducing subspaces. In this section, we present reducibility conditions for two matrices based on a relationship between Grassmann representatives of reducing subspaces and eigenvectors of matrix compounds. First is an auxiliary result characterizing complementary subspaces.

LEMMA 3.1. *Let $\mathcal{L},\mathcal{M}\subset\mathbb{C}^n$ be subspaces with* $\dim\mathcal{L}=k$ *and* $\dim\mathcal{M}=n-k$, $1\leq k<n$, *and let $x,y\in\Lambda^k(\mathbb{C}^n)$ be Grassmann representatives of $\mathcal{L}$ and $\mathcal{M}^\perp$, respectively. The following are equivalent.*

(i) *$\mathcal{L}$ and $\mathcal{M}$ are direct complements in $\mathbb{C}^n$.*

(ii) *Vectors x and y satisfy $\langle x,y\rangle\neq 0$.*

Proof. Since $\dim\mathcal{L}+\dim\mathcal{M}=n$, condition (i) is equivalent to

$$\mathcal{L}\cap\mathcal{M}=\{0\}. \tag{3.1}$$

Let $\{x_1,\ldots,x_k\}$ and $\{y_1,\ldots,y_k\}$ be bases of $\mathcal{L}$ and $\mathcal{M}^\perp$, respectively, and consider the $n\times k$ matrices $X=[x_1\,|\,\ldots\,|\,x_k]$ and $Y=[y_1\,|\,\ldots\,|\,y_k]$. Then, up to nonzero scalar multiples, $x=x_1\wedge\ldots\wedge x_k$ and $y=y_1\wedge\ldots\wedge y_k$. By the Cauchy-Binet formula for the determinant it can be seen that $\langle x,y\rangle=\det X^*Y$. Hence, in order to prove the lemma, we need only show that (3.1) is equivalent to $\det X^*Y\neq 0$.

Assume first that X^*Y is singular. Then there exists a nonzero vector $u\in\mathbb{C}^k$ such that $u^*X^*Yv=\langle Xu,Yv\rangle=0$ for all $v\in\mathbb{C}^k$. Thus, $0\neq Xu\in\mathcal{L}$ and since $\mathcal{M}^\perp=\{Yv:\,v\in\mathbb{C}^k\}$, it follows that $Xu\in\mathcal{M}$, which contradicts (3.1). Conversely, if $\mathcal{L}$ and $\mathcal{M}$ have a common nonzero vector z, then $z=Xu$ for some $u\in\mathbb{C}^k$ and also z is orthogonal to all vectors in $\mathcal{M}^\perp$; i.e., $u^*X^*Yv=0$ for all $v\in\mathbb{C}^k$. This implies that X^*Y is singular. ∎

Recall that if $A\in\mathbb{C}^{n\times n}$ and λ,μ are distinct eigenvalues of A, then by the biorthogonality principle (see e.g., [5]), each left eigenvector of A corresponding to μ is orthogonal to each right eigenvector of A corresponding to λ.

Given matrices $A,B\in\mathbb{C}^{n\times n}$ and vectors $x,y\in\mathbb{C}^n$, we shall say that (x,y) is a *common pair of right and left eigenvectors* of A and B if x is a common right eigenvector of A and B and y is a common left eigenvector of A and B.

THEOREM 3.2. *Let $A,B\in\mathbb{C}^{n\times n}$ and $1\leq k<n$. The following are equivalent.*

(i) *There exist subspaces $\mathcal{L},\mathcal{M}\subset\mathbb{C}^n$ of dimensions k and $n-k$, respectively, such that $(\mathcal{L},\mathcal{M})$ is a common reducing pair for A and B.*

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(ii) *For all* $s \in \mathbb{C}$, $(A+sI)^{(k)}$ *and* $(B+sI)^{(k)}$ *have a common pair* (x,y) *of right and left eigenvectors* $x, y \in \Lambda^k(\mathbb{C}^n)$ *that are decomposable and satisfy* $\langle x, y\rangle \neq 0$.

(iii) *There exists* $\hat{s} \in \mathbb{C}$ *such that* $(A+\hat{s}I)$ *and* $(B+\hat{s}I)$ *are nonsingular, and such that* $(A+\hat{s}I)^{(k)}$, $(B+\hat{s}I)^{(k)}$ *have a common pair* (x,y) *of right and left eigenvectors* $x, y \in \Lambda^k(\mathbb{C}^n)$ *that are decomposable and satisfy* $\langle x, y\rangle \neq 0$.

Moreover, when either of these conditions hold, x *and* y *in* (ii) *and* (iii) *are Grassmann representatives of* $\mathcal{L}$ *and* $\mathcal{M}^\perp$*, respectively.*

Proof. (i) $\Rightarrow$ (ii). Let $(\mathcal{L}, \mathcal{M})$ be a reducing pair for A and B with dim $\mathcal{L} = k$ and dim $\mathcal{M} = n-k$; i.e., $\mathcal{L}$ and $\mathcal{M}$ are common invariant subspaces of A and B that are complementary in $\mathbb{C}^n$. Let $\{x_1, \ldots, x_k\}$ and $\{y_1, \ldots, y_k\}$ be bases of $\mathcal{L}$ and $\mathcal{M}^\perp$, respectively. Since $A\mathcal{L} \subseteq \mathcal{L}$ and $B\mathcal{L} \subseteq \mathcal{L}$ if and only if $(A+sI)\mathcal{L} \subseteq \mathcal{L}$ and $(B+sI)\mathcal{L} \subseteq \mathcal{L}$ for all $s \in \mathbb{C}$, it follows by the discussion in Section 2 that $x = x_1 \wedge \ldots \wedge x_k$ is a common right eigenvector of $(A+sI)^{(k)}$ and $(B+sI)^{(k)}$. Similarly, $(A+sI)\mathcal{M} \subseteq \mathcal{M}$ and $(B+sI)\mathcal{M} \subseteq \mathcal{M}$ for all $s \in \mathbb{C}$, which is also equivalent to $(A+sI)^*\mathcal{M}^\perp \subseteq \mathcal{M}^\perp$ and $(B+sI)^*\mathcal{M}^\perp \subseteq \mathcal{M}^\perp$. Thus, $y = y_1 \wedge \ldots \wedge y_k$ is a common right eigenvector of $((A+sI)^*)^{(k)}$ and $((B+sI)^*)^{(k)}$; due to the compound matrix property $((X)^*)^{(k)} = ((X)^{(k)})^*$, we have that y is a common left eigenvector of $(A+sI)^{(k)}$ and $(B+sI)^{(k)}$. Since $\mathcal{L}$ and $\mathcal{M}$ are complementary, it follows by Lemma 3.1 that $\langle x, y\rangle \neq 0$.

(ii) $\Rightarrow$ (iii). Follows trivially.

(iii)$\Rightarrow$ (i). Let $\hat{s}$ be such that $(A+\hat{s}I)$ and $(B+\hat{s}I)$ are nonsingular and let (x, y) with $x = x_1 \wedge \ldots \wedge x_k$, $y = y_1 \wedge \ldots \wedge y_k$ be a common pair of right and left eigenvectors of $(A+\hat{s}I)^{(k)}$ and $(B+\hat{s}I)^{(k)}$ such that $\langle x, y\rangle \neq 0$. Then $(A+\hat{s}I)^{(k)}$ is nonsingular and there exists nonzero $\lambda \in \mathbb{C}$ such that

$$\begin{aligned} (A+\hat{s}I)^{(k)}x &= (A+\hat{s}I)x_1 \wedge \ldots \wedge (A+\hat{s}I)x_k \\ &= \lambda(x_1 \wedge \ldots \wedge x_k). \end{aligned} \tag{3.2}$$

By the biorthogonality principle, $\langle x, y\rangle \neq 0$ implies that y corresponds to the same eigenvalue λ of $(A+\hat{s}I)^{(k)}$, i.e.,

$$\begin{aligned} ((A+\hat{s}I)^{(k)})^*y &= ((A+\hat{s}I)^*)^{(k)}y \\ &= (A+\hat{s}I)^*y_1 \wedge \ldots \wedge (A+\hat{s}I)^*y_k \\ &= \overline{\lambda}(y_1 \wedge \ldots \wedge y_k). \end{aligned} \tag{3.3}$$

By (3.2) it follows that the subspace spanned by $\{x_1, \ldots, x_k\}$ coincides with the subspace spanned by $\{(A+\hat{s}I)x_1, \ldots, (A+\hat{s}I)x_k\}$. Thus, $\mathcal{L} = \operatorname{span}\{x_1, \ldots, x_k\}$ is an invariant subspace of $(A+\hat{s}I)$ and hence of A. Similarly, it follows from (3.3) that $\operatorname{span}\{y_1, \ldots, y_k\}$ is an invariant subspace of A^* or, equivalently, that $\mathcal{M} = (\operatorname{span}\{y_1, \ldots, y_k\})^\perp$ is an invariant subspace of A. Since x and y are right and left eigenvectors of $(B+\hat{s}I)^{(k)}$, respectively, the same argument as above shows that $\mathcal{L}$ and $\mathcal{M}$ also are invariant subspaces of B. By Lemma 3.1, the inequality $\langle x, y\rangle \neq 0$ implies that $\mathcal{L}$ and $\mathcal{M}$ are direct complements, completing the proof. □

Obviously,the above result can be easily extended to the case of any number of matrices having a common pair of reducing subspaces. In the special case of $\mathcal{M} = \mathcal{L}^\perp$

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in condition (i) of Theorem 3.2, we obtain conditions for simultaneous reducibility of A and B by orthogonal complements.

COROLLARY 3.3. *Let $A, B \in \mathbb{C}^{n\times n}$ and $1 \leq k < n$. The following are equivalent.*

(i) *There exists a subspace $\mathcal{L} \subset \mathbb{C}^n$ of dimension k such that $(\mathcal{L}, \mathcal{L}^\perp)$ is a common reducing pair for A and B.*

(ii) *There exists decomposable $x \in \Lambda^k(\mathbb{C}^n)$ such that for all $s \in \mathbb{C}$, x is a common eigenvector of $(A + sI)^{(k)}$, $(A^* + sI)^{(k)}$, $(B + sI)^{(k)}$ and $(B^* + sI)^{(k)}$.*

(iii) *There exists decomposable $x \in \Lambda^k(\mathbb{C}^n)$ and $\hat{s} \in \mathbb{C}$ such that $(A + \hat{s}I)$ and $(B + \hat{s}I)$ are nonsingular and x is a common eigenvector of $(A + \hat{s}I)^{(k)}$, $(A^* + \hat{s}I)^{(k)}$, $(B + \hat{s}I)^{(k)}$ and*
$(B^* + \hat{s}I)^{(k)}$.

Moreover, when either of these conditions hold, x in (ii) *and* (iii) *is a Grassmann representative of $\mathcal{L}$.*

It should be noted that the above corollary also follows from [9, Theorem 2.2] by recalling that $(\mathcal{L}, \mathcal{L}^\perp)$ is a common reducing pair for A and B if and only if $\mathcal{L}$ is a common invariant subspace of A, A^*, B and B^*.

As another special case, we note that Theorem 3.2 and Corollary 3.3 provide reducibility conditions for a single matrix. It is clear that not every $A \in \mathbb{C}^{n\times n}$ has an $A-$reducing subspace of arbitrary dimension k. For instance, if A is an $n-$dimensional Jordan block, it can be shown that the only $A-$reducing subspaces are $\{0\}$ and $\mathbb{C}^n$. In the next example, we illustrate how Theorem 3.2 can be employed to rule out the existence of reducing subspaces of a given matrix.

EXAMPLE 3.4. Let $n = 4$, $k = 1$ and

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

The distinct eigenvalues of A are $\lambda_1 = 1$ and $\lambda_2 = -1$. $A^{(1)} = A$ has a pair of right and left eigenvectors $x_1 = [1\ 0\ 0\ 0]^T$ and $y_1 = [0\ 1\ 0\ 0]^T$ corresponding to $\lambda_1 = 1$, and a pair of right and left eigenvectors $x_2 = [0\ 0\ 1\ 0]^T$ and $y_2 = [0\ 0\ 0\ 1]^T$ corresponding to $\lambda_2 = -1$. Since we consider the first compound of A, the requirement for decomposability of x_1, x_2, y_1 and y_2 is trivially satisfied. Notice, however, that $\langle x_1, y_1\rangle = \langle x_2, y_2\rangle = 0$, showing that conditions (ii) and (iii) of Theorem 3.2 are not satisfied and thus A does not have any reducing subspaces of dimension $k = 1$; neither does any matrix similar to A. A similar argument applies to the case $k = 3$.

4. Deflating subspaces. Given two matrices $A, B \in \mathbb{C}^{n\times n}$, the generalized Schur theorem [4] shows that for each $1 \leq k < n$ there exist $k-$dimensional deflating subspaces for A and B. The problem of computing these subspaces is well-studied from a numerical point of view due to its application in solving generalized eigenvalue problems [7] and a large class of matrix algebraic equations [1]. In this section, we give a characterization of the non-trivial deflating subspaces for a regular matrix pencil using basically the multilinear approach. This characterization also enables us to

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formulate conditions for reducibility of a matrix pencil by equivalence transformations. We begin with the following necessary condition.

PROPOSITION 4.1. *Let $A, B \in \mathbb{C}^{n\times n}$ and $\mathcal{L}, \mathcal{M} \subset \mathbb{C}^n$ be subspaces of dimension k $(1 \le k < n)$. If $\mathcal{L}, \mathcal{M}$ are deflating subspaces for A and B, then there exist nonzero decomposable vectors $x, y \in \Lambda^k(\mathbb{C}^n)$ and $r, q \in \mathbb{C}$ such that*

$$A^{(k)}x = ry \quad \text{and} \quad B^{(k)}x = qy. \tag{4.1}$$

Moreover, x and y are Grassmann representatives of $\mathcal{L}$ and $\mathcal{M}$, respectively.

Proof. Let $\{x_1, \dots, x_k\}$ and $\{y_1, \dots, y_k\}$ be bases of $\mathcal{L}$ and $\mathcal{M}$, respectively. Augment these into bases of $\mathbb{C}^n$, thus forming invertible matrices

$$L = [x_1 \,|\, \dots \,|\, x_k \,|\, x_{k+1} \,|\, \dots \,|\, x_n] \quad \text{and} \quad M = [y_1 \,|\, \dots \,|\, y_k \,|\, y_{k+1} \,|\, \dots \,|\, y_n].$$

Then, as by assumption $A\mathcal{L} \subseteq \mathcal{M}$ and $B\mathcal{L} \subseteq \mathcal{M}$,

$$R := M^{-1}AL = \begin{bmatrix} A'_{11} & A'_{12} \\ 0 & A'_{22} \end{bmatrix} \quad \text{and} \quad Q := M^{-1}BL = \begin{bmatrix} B'_{11} & B'_{12} \\ 0 & B'_{22} \end{bmatrix},$$

where A'_{11} and B'_{11} are $k \times k$ matrices. Taking k-th compounds of the equations above, we obtain

$$A^{(k)}L^{(k)} = M^{(k)}R^{(k)} \quad \text{and} \quad B^{(k)}L^{(k)} = M^{(k)}Q^{(k)}.$$

Since R and Q are block upper triangular, and since the first columns of $L^{(k)}$ and $M^{(k)}$ are

$$x = x_1 \wedge \dots \wedge x_k \quad \text{and} \quad y = y_1 \wedge \dots \wedge y_k,$$

respectively, we have that (4.1) holds. Notice that $r = \det A'_{11}$, $q = \det B'_{11}$ and that x, y are Grassmann representatives of $\mathcal{L}$ and $\mathcal{M}$, respectively. ∎

Given deflating subspaces $\mathcal{L}, \mathcal{M}$ for A and B, and referring to the notation in Proposition 4.1, it can be seen that

$$A\mathcal{L} = \mathcal{M} \quad \text{and} \quad B\mathcal{L} = \mathcal{M} \qquad \text{if and only if} \qquad r \neq 0 \quad \text{and} \quad q \neq 0.$$

Indeed, if $r \neq 0$, $A^{(k)}x$ is a Grassmann representative of $\mathcal{M}$ and thus $\{Ax_1, \dots, Ax_k\}$ is a basis for $\mathcal{M}$; i.e., $A\mathcal{L} = \mathcal{M}$. Moreover, if $r \neq 0$ and $q \neq 0$, it follows from (4.1) that

$$\left(\frac{1}{r}A^{(k)} - \frac{1}{q}B^{(k)}\right)x = 0 \quad \text{and} \quad A^{(k)}x \neq 0. \tag{4.2}$$

Conversely, if (4.2) holds for some $r \neq 0$, $q \neq 0$ and a nonzero decomposable vector $x = x_1 \wedge \dots \wedge x_k$, then

$$A^{(k)}x = \frac{q}{r}B^{(k)}x \neq 0$$

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or

$$Ax_1 \wedge \ldots \wedge Ax_k = \frac{q}{r}\,(Bx_1 \wedge \ldots \wedge Bx_k) \neq 0,$$

which is equivalent to

$$\mathrm{span}\{Ax_1, \ldots, Ax_k\} = \mathrm{span}\{Bx_1, \ldots, Bx_k\}.$$

Since $A^{(k)}x \neq 0$, letting

$$\mathcal{L} := \mathrm{span}\{x_1, \ldots, x_k\} \quad \text{and} \quad \mathcal{M} := \mathrm{span}\{Ax_1, \ldots, Ax_k\},$$

we have that $A\mathcal{L} \subseteq \mathcal{M}$ and $B\mathcal{L} \subseteq \mathcal{M}$; i.e., $\mathcal{L}$ and $\mathcal{M}$ are k-dimensional deflating subspaces for A and B. In fact, as $r \neq 0$ and $q \neq 0$, it follows that $A\mathcal{L} = \mathcal{M}$ and $B\mathcal{L} = \mathcal{M}$. We have thus shown the following result.

THEOREM 4.2. *Let $A, B \in \mathbb{C}^{n\times n}$ and $\mathcal{L}, \mathcal{M} \subset \mathbb{C}^n$ be subspaces of dimension k $(1 \leq k < n)$. The following are equivalent.*

(i) *$A\mathcal{L} = \mathcal{M}$ and $B\mathcal{L} = \mathcal{M}$.*

(ii) *There exist nonzero $s_1, s_2 \in \mathbb{C}$ and Grassmann representative $x \in \Lambda^k(\mathbb{C}^n)$ of $\mathcal{L}$ such that*

$$\left(s_1 A^{(k)} + s_2 B^{(k)}\right) x = 0$$

and such that $A^{(k)}x \neq 0$ is a Grassmann representative of $\mathcal{M}$.

In the rest of this section, we consider matrix pencils of the form $A + sB$, denoted by (A, B). Recall that (A, B) is called *regular* if $\det(A + sB)$ is not identically zero as a function of s.

LEMMA 4.3. *Let $A, B \in \mathbb{C}^{n\times n}$ and $\mathcal{L}, \mathcal{M} \subset \mathbb{C}^n$ be subspaces of dimension k $(1 \leq k < n)$. Suppose that (A, B) is a regular pencil. The following are equivalent.*

(i) *$A\mathcal{L} \subseteq \mathcal{M}$ and $B\mathcal{L} \subseteq \mathcal{M}$.*

(ii) *$(A + \hat{s}B)\mathcal{L} = \mathcal{M}$ for some $\hat{s} \in \mathbb{C}$ and $B\mathcal{L} \subseteq \mathcal{M}$.*

(iii) *$(A + sB)\mathcal{L} \subseteq \mathcal{M}$ for all $s \in \mathbb{C}$ and $B\mathcal{L} \subseteq \mathcal{M}$.*

Proof. (i) $\Rightarrow$ (ii). Since $A + sB$ is a regular pencil, there exists $\hat{s} \in \mathbb{C}$ such that $A + \hat{s}B$ is invertible. Thus $(A + \hat{s}B)\mathcal{L}$ is also a k-dimensional subspace and so $(A + \hat{s}B)\mathcal{L} = \mathcal{M}$.

(ii) $\Rightarrow$ (iii). For any $s \in \mathbb{C}$, $(A + sB)\mathcal{L} = (A + \hat{s}B + (s - \hat{s})B)\mathcal{L} \subseteq \mathcal{M} + \mathcal{M} = \mathcal{M}$.

(iii)$\Rightarrow$ (i). Choose $s = 0$ in (iii). ∎

Of course, in the above lemma, the equivalence of (i) and (iii) and the implication (ii) $\Rightarrow$ (i) hold for all matrix pencils.

THEOREM 4.4. *Let $A, B \in \mathbb{C}^{n\times n}$ and $\mathcal{L}, \mathcal{M} \subset \mathbb{C}^n$ be subspaces of dimension k $(1 \leq k < n)$. Suppose that (A, B) is a regular pencil. The following are equivalent.*

(i) *$\mathcal{L}$ and $\mathcal{M}$ are deflating subspaces for A and B, i.e., $A\mathcal{L} \subseteq \mathcal{M}$ and $B\mathcal{L} \subseteq \mathcal{M}$.*

(ii) *There exist $\hat{s}, \hat{t}, s_1, s_2 \in \mathbb{C}$, $s_1 \neq 0$, $s_2 \neq 0$, and Grassmann representative $x \in \Lambda^k(\mathbb{C}^n)$ of $\mathcal{L}$ such that*

$$\left(s_1 (A + \hat{s}B)^{(k)} + s_2 \left(B + \hat{t}(A + \hat{s}B)\right)^{(k)}\right) x = 0$$

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and such that $(A + \hat{s}B)^{(k)}x \neq 0$ *is a Grassmann representative of* $\mathcal{M}$.

Proof. As (A, B) is a regular pencil, by Lemma 4.3 applied to A and B, (i) is equivalent to

$$(A + \hat{s}B)\mathcal{L} = \mathcal{M} \text{ for some } \hat{s} \in \mathbb{C} \text{ and } B\mathcal{L} \subseteq \mathcal{M}.$$

In turn, by Lemma 4.3 applied to B and $A + \hat{s}B$, it follows that (i) is equivalent to

$$(A + \hat{s}B)\mathcal{L} = \mathcal{M} \text{ and } \left(B + \hat{t}(A + \hat{s}B)\right)\mathcal{L} = \mathcal{M} \text{ for some } \hat{t} \in \mathbb{C}. \tag{4.3}$$

Now the claimed equivalence of (i) and (ii), as well as the nature of x and $(A+\hat{s}B)^{(k)}x$ as Grassmann representatives follow from Theorem 4.2. □

Remark 4.5. Under the assumption that (A, B) is a regular pencil, $\hat{s}$ can be chosen such that $A+\hat{s}B$ is invertible and in this case, the inequality $(A+\hat{s}B)^{(k)}x \neq 0$ in (ii) is satisfied for all nonzero $x \in \Lambda^k(\mathbb{C}^n)$. However, it can be seen that in Theorem 4.4, condition (ii) implies (i) without the regularity assumption for (A, B). That is, if (ii) is satisfied then the factors of x and $(A+\hat{s}B)^{(k)}x$ are basis vectors for two $k-$dimensional deflating subspaces for A and B.

Theorem 4.4 can be used to obtain necessary and sufficient reducibility conditions for a regular matrix pencil via equivalence transformations as follows.

Proposition 4.6. *Let* $A, B \in \mathbb{C}^{n\times n}$, $\mathcal{L}, \mathcal{M} \subset \mathbb{C}^n$ *be subspaces of dimension* k $(1 \leq k < n)$ *and* $\tilde{\mathcal{L}}, \tilde{\mathcal{M}} \subset \mathbb{C}^n$ *be direct complements of* $\mathcal{L}$ *and* $\mathcal{M}$, *respectively. Suppose that* (A, B) *is a regular pencil. The following are equivalent.*

(i) $(\mathcal{L}, \mathcal{M})$ *and* $(\tilde{\mathcal{L}}, \tilde{\mathcal{M}})$ *are deflating pairs for* A *and* B, *i.e.,*

$$A\mathcal{L} \subseteq \mathcal{M},\ B\mathcal{L} \subseteq \mathcal{M} \quad \text{and} \quad A\tilde{\mathcal{L}} \subseteq \tilde{\mathcal{M}},\ B\tilde{\mathcal{L}} \subseteq \tilde{\mathcal{M}}.$$

(ii) *There exist* $\hat{s}_i, \hat{t}_i \in \mathbb{C}$ $(i = 1, 2)$, *nonzero* $s_i \in \mathbb{C}$ $(i = 1, \dots, 4)$ *and Grassmann representatives* $x, y \in \Lambda^k(\mathbb{C}^n)$ *of* $\mathcal{L}$ *and* $\tilde{M}^{\perp}$, *respectively, such that*

$$\left(s_1(A + \hat{s}_1B)^{(k)} + s_2\left(B + \hat{t}_1(A + \hat{s}_1B)\right)^{(k)}\right)x = 0, \tag{4.4}$$

$$\left(s_3(A + \hat{s}_2B)^{(k)} + s_4\left(B + \hat{t}_2(A + \hat{s}_2B)\right)^{(k)}\right)^* y = 0, \tag{4.5}$$

$$\langle y, (A + \hat{s}_1B)^{(k)}x\rangle \neq 0, \quad \langle x, \left((A + \hat{s}_2B)^{(k)}\right)^* y\rangle \neq 0, \tag{4.6}$$

and such that $(A + \hat{s}_1B)^{(k)}x \neq 0$, $\left((A + \hat{s}_2B)^{(k)}\right)^* y \neq 0$ *are Grassmann representatives of* $\mathcal{M}$ *and* $\tilde{\mathcal{L}}^{\perp}$, *respectively.* The proof of this proposition is an immediate application of Theorem 4.4 and Lemma 3.1.

Remark 4.7. Assuming (A, B) to be a regular pencil, $\hat{s}_1$ and $\hat{t}_1$ in (4.4) can be chosen such that $A + \hat{s}_1B$ and $B + \hat{t}_1(A + \hat{s}_1B)$ are invertible; in this case one can take in (4.5) $\hat{s}_2 = \hat{s}_1$, $\hat{t}_2 = \hat{t}_1$ and omit the second inequality in (4.6). As mentioned in Remark 4.5, condition (ii) above implies (i) for every matrix pencil, without the requirement of regularity.

Remark 4.8. Proposition 4.6 with $\tilde{\mathcal{L}} = \mathcal{L}^{\perp}$ and $\tilde{\mathcal{M}} = \mathcal{M}^{\perp}$ provides reducibility conditions for (A, B) via unitary equivalence transformations.

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5. Practical considerations. In this section we illustrate strategies for finding (common) reducing pairs of subspaces and deflating subspaces. For the case of reducing pairs, we will use the following criterion for the existence of a common eigenvector among two matrices.

THEOREM 5.1. ([8]) *Let* $X, Y \in \mathbb{C}^{p\times p}$ *and*

$$K(X,Y) = \sum_{m,\ell=1}^{p-1} [X^m, Y^\ell]^*[X^m, Y^\ell],$$

where $[X^m, Y^\ell]$ *denotes the commutator* $X^mY^\ell - Y^\ell X^m$. *Then* X *and* Y *have a common eigenvector if and only if* K *is not invertible.*

EXAMPLE 5.2. Let us consider whether

$$A = \begin{bmatrix} 0.5 & -6 & -6 & -1.5 \\ 0.5 & 3 & 1 & -0.5 \\ -0.5 & 0 & 2 & 0.5 \\ -1.5 & -6 & -6 & 0.5 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2.5 & -3 & -4 & 0.5 \\ 0.5 & -2 & -4 & -0.5 \\ -0.5 & 0 & 2 & 0.5 \\ 0.5 & -3 & -2 & 2.5 \end{bmatrix}$$

have a common reducing pair of subspaces of dimension $k = 2$. For that purpose, recall Theorem 3.2 and in particular its third clause. The spectrum of A is $\{-1, 2, 3\}$ and the spectrum of B is $\{-2, 1, 3\}$. Thus A and B are nonsingular and we can take $\hat{s} = 0$. Next compute the second compounds of A and B:

$$X = A^{(2)} = \begin{bmatrix} 4.5 & 3.5 & 0.5 & 12 & 7.5 & 4.5 \\ -3 & -2 & -0.5 & -12 & -3 & 0 \\ -12 & -12 & -2 & 0 & -12 & -12 \\ 1.5 & 1.5 & 0 & 6 & 1.5 & 1.5 \\ 1.5 & -1.5 & -0.5 & -12. & -1.5 & -2.5 \\ 3 & 6 & 0.5 & 12 & 3 & 4 \end{bmatrix},$$

$$Y = B^{(2)} = \begin{bmatrix} -3.5 & -8 & -1.5 & 4 & 2.5 & 4 \\ -1.5 & 3 & 1.5 & -6 & -1.5 & -3 \\ -6 & -3 & 6 & -6 & -6 & -9 \\ -1 & -1 & 0 & -4 & -1 & -1 \\ -0.5 & 1 & 1.5 & -8 & -6.5 & -11 \\ 1.5 & 0 & -1.5 & 6 & 1.5 & 6 \end{bmatrix}.$$

Referring to Theorem 5.1, the matrices $K = K(X,Y)$ and $K' = K(X^T, Y^T)$ are singular and so X, Y have common right and left eigenvectors. Note that if either K or K' were nonsingular, Theorem 3.2 would imply that A and B do not have a common reducing pair of subspaces.

Using Matlab's `null` routine, we find that

$$\text{Nul}\,(X + 3I) = \text{span}\{x\}, \quad \text{where} \quad x = [\,-1 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0\,]^T;$$

notice that $Yx = -6x$ and thus x is a common right eigenvector of X, Y. Similarly, we see that

$$\text{Nul}\,(Y^T + 6I) = \text{span}\{y\}, \quad \text{where} \quad y = [\,-1 \quad -1 \quad 0 \quad 0 \quad 1 \quad 1\,]^T;$$

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notice that $X^T y = -3y$ and thus y is a common left eigenvector of X, Y.

Next we examine the decomposability of x, y. The quadratic Plücker relations for decomposability can be used in this instance (see [6, Vol. II, §4.1, Definition 1.1]). For example,

$$[x_1, \ldots, x_6]^T \in \Lambda^2(\mathbb{C}^4) \text{ is decomposable if and only if } x_1x_6 - x_2x_5 + x_3x_4 = 0.$$

It follows that x, y are decomposable. In fact, $x = \alpha_1 \wedge \alpha_2 \quad \text{and} \quad y = \beta_1 \wedge \beta_2$, where

$$\alpha_1 = \begin{bmatrix} 1 & 1 & 0 & 1 \end{bmatrix}^T,\ \alpha_2 = \begin{bmatrix} 1 & -1 & 0 & 1 \end{bmatrix}^T,$$

$$\beta_1 = \begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix}^T,\ \beta_2 = \begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix}^T.$$

Notice that $\langle x, y \rangle \neq 0$. Hence, letting

$$\mathcal{L} = \operatorname{span}\{\alpha_1, \alpha_2\} \quad \text{and} \quad \mathcal{M} = (\operatorname{span}\{\beta_1, \beta_2\})^\perp,$$

by Theorem 3.2 we have that $(\mathcal{L}, \mathcal{M})$ is a common reducing pair for A and B. Indeed, if

$$L = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 1 & 1 \end{bmatrix},$$

where the first two columns of L have been computed to be a basis for $\mathcal{M}$ as defined above, we obtain the following simultaneous reductions of A and B:

$$L^{-1}AL = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & -5 & 4 \end{bmatrix} \quad \text{and} \quad L^{-1}BL = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & -1 & 4 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

Example 5.3. In this example let us consider whether

$$A = \begin{bmatrix} 2 & -4 & 3 & 1 \\ -6 & 7 & -5 & 0 \\ 9 & -11 & 8 & 0 \\ 4 & -7 & 6 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 7 & -9 & 8 & 4 \\ 3 & -6 & 6 & 4 \\ 4 & -2 & 1 & -1 \\ 6 & -6 & 5 & 2 \end{bmatrix}$$

have a pair of deflating subspaces of dimension $k = 2$. For that purpose, compute the second compounds of A and B:

$$A^{(2)} = \begin{bmatrix} -10 & 8 & 6 & -1 & -7 & 5 \\ 14 & -11 & -9 & 1 & 11 & -8 \\ 2 & 0 & 0 & -3 & -1 & 0 \\ 3 & -3 & 0 & 1 & 0 & 0 \\ 14 & -16 & -12 & 7 & 14 & -10 \\ -19 & 22 & 18 & -10 & -22 & 16 \end{bmatrix},$$

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$$B^{(2)} = \begin{bmatrix} -15 & 18 & 16 & -6 & -12 & 8 \\ 22 & -25 & -23 & 7 & 17 & -12 \\ 12 & -13 & -10 & 3 & 6 & -4 \\ 18 & -21 & -19 & 6 & 14 & -10 \\ 18 & -21 & -18 & 6 & 12 & -8 \\ -12 & 14 & 14 & -4 & -10 & 7 \end{bmatrix}.$$

Referring to Theorem 4.2 (ii) and taking $s_1 = s_2 = 1$, we have that $A^{(2)} + B^{(2)}$ is singular and its nullspace is spanned by

$$x = [\,-1 \quad -1 \quad 0 \quad -1 \quad 1 \quad 1\,]^T .$$

Consider also $y = A^{(2)}x$. The vectors x, y are decomposable since

$$x = \alpha_1 \wedge \alpha_2 \quad \text{and} \quad y = \beta_1 \wedge \beta_2,$$

where

$$\alpha_1 = [\,-1 \quad 0 \quad 1 \quad -1\,]^T,\ \alpha_2 = [\,1 \quad 1 \quad 0 \quad 1\,]^T,$$

$$\beta_1 = [\,1 \quad 0 \quad 1 \quad 1\,]^T,\ \beta_2 = [\,1 \quad 1 \quad 0 \quad 1\,]^T.$$

It follows from Theorem 4.2 that if we let

$$\mathcal{L} = \operatorname{span}\{\alpha_1, \alpha_2\} \quad \text{and} \quad \mathcal{M} = \operatorname{span}\{\beta_1, \beta_2\}$$

then $A\mathcal{L} = \mathcal{M}$ and $B\mathcal{L} = \mathcal{M}$. By the proof of Theorem 4.2, if we choose invertible matrices L and M whose first two columns are α_1, α_2 and β_1, β_2, respectively, then a simultaneous deflation of A and B is achieved via the equivalences $M^{-1}AL$ and $M^{-1}BL$.

To conclude, our results and their illustration above raise the question of whether a subspace of the k-th Grassmann space over $\mathbb{C}^n$ contains a nonzero decomposable vector or not. Also, the question arises of how to take full advantage of our results by computing bases for the reducing or deflating subspaces; that is, how to find the factors of a decomposable vector. The answers to these questions are related to the dimension of the Grassmann variety and are examined in [9]. However, these issues present substantial challenges in theoretical and computational terms.

REFERENCES

[1] P. Benner, R. Byers, V. Mehrmann, and H. Xu. A unified deflating subspace approach for classes of polynomial and rational matrix equations. Technical report SFB393/00-05, Fakultät für Mathematik, TU Chemnitz, January 2000.

[2] A. George and K. D. Ikramov. Common invariant subspaces of two matrices. *Linear Algebra and Its Applications,* 287:171-179, 1999.

[3] I. Gohberg, P. Lancaster, and L. Rodman. *Invariant Subspaces of Matrices with Applications.* Wiley-Interscience, 1986.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 11, pp. 246-257, November 2004

ELA
http://math.technion.ac.il/iic/ela

[4] G. H. Golub and C. F. Van Loan. *Matrix Computations.* The John Hopkins University Press, Baltimore 1989.
[5] R. A. Horn and C. R. Johnson. *Matrix Analysis,* Cambridge Univ. Press, Cambridge 1985.
[6] M. Marcus. *Finite dimensional multilinear algebra.* Two volumes. Marcel Dekker, 1973-1975.
[7] C. B. Moler and G. W. Stewart. An algorithm for generalized matrix eigenvalue problems. *SIAM Journal on Numerical Analysis,* 10:241-256, 1973.
[8] D. Shemesh. Common eigenvectors of two matrices. *Linear Algebra and Its Applications,* 62:11-18, 1984.
[9] M. Tsatsomeros. Matrices with a common non-trivial invariant subspace. *Linear Algebra and Its Applications,* 322:51-59, 2001.