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INTERPOLATION BY MATRICES*

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Abstract. Assume that two sets of k vectors in $\mathbb{R}^n$ are given, namely $\{\mathbf{x}^1,\ldots,\mathbf{x}^k\}$ and $\{\mathbf{y}^1,\ldots,\mathbf{y}^k\}$, and a class of matrices, e.g., positive definite matrices, positive matrices, strictly totally positive matrices, or P-matrices. The question considered in this paper is that of determining necessary and sufficient conditions on these sets of vectors such that there exists an $n\times n$ matrix A in the given class satisfying $A\mathbf{x}^j=\mathbf{y}^j$ $(j=1,\ldots,k)$.

Key words. Positive definite matrices, Positive matrices, Strictly totally positive matrices, P-matrices.

AMS subject classifications. 15A04, 15A57.

1. Introduction. In this paper we consider a class of matrix interpolation problems. Rather than attempt to formulate the general problem we will, for clarity of exposition, first define this problem on the class of positive definite matrices.

Assume we are given two sets of k vectors in $\mathbb{R}^n$, namely $\{\mathbf{x}^1,\ldots,\mathbf{x}^k\}$ and $\{\mathbf{y}^1,\ldots,\mathbf{y}^k\}$. What are the exact conditions on these sets of vectors such that there exists an $n\times n$ positive definite matrix A satisfying

$$A\mathbf{x}^j=\mathbf{y}^j,\qquad j=1,\ldots,k? \tag{1.1}$$

Note that we can easily reformulate this problem. For example, we might say: Assume we are given a set of k vectors $\{\mathbf{x}^1,\ldots,\mathbf{x}^k\}$ in $\mathbb{R}^n$. What are the exact conditions on another set of k vectors $\{\mathbf{y}^1,\ldots,\mathbf{y}^k\}$ in $\mathbb{R}^n$ such that there exists an $n\times n$ positive definite matrix A satisfying

$$A\mathbf{x}^j=\mathbf{y}^j,\qquad j=1,\ldots,k?$$

That is, given a subspace in $\mathbb{R}^n$ what is the possible range of the positive definite matrices on this subspace. Or alternatively, let X be a linear subspace of $\mathbb{R}^n$ and T a linear operator from X into $\mathbb{R}^n$. What are the exact conditions on X and T such that there exists an $n\times n$ positive definite matrix A for which

$$A\mathbf{x}=T\mathbf{x},\qquad \mathbf{x}\in X?$$

That is, when can T be extended to or embedded in a positive definite matrix?

We ask this same question for Hermitian positive definite matrices, positive matrices, strictly totally positive matrices and P-matrices. You may ask this same question for your favorite class of matrices. We present a complete characterization when dealing with positive definite, Hermitian positive definite, positive and strictly positive matrices. For the classes of strictly totally positive matrices and P-matrices we have

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necessary conditions that we conjecture to be sufficient, and some partial results. This paper contains many unanswered questions.

What we are studying is not a matrix completion problem. There is no matrix to start with. It is an extension or an embedding problem. We chose to term it an *interpolation* problem; see (1.1). This is also the name given in Johnson, Smith [5] where this problem is considered in the case $k = 1$ for certain classes of matrices.

We have organized the paper as follows. In section 2 we provide a characterization for the class of positive definite and Hermitian positive definite matrices. We do not have a similar characterization for the class of Hermitian positive semi-definite and positive semi-definite matrices. In section 3 we provide a characterization for the class of positive and strictly positive matrices (they need not be square matrices). In section 4 we consider strictly totally positive matrices and in section 5 P-matrices. In these latter two sections we present only partial results. Nonetheless we do present conjectures on what we believe to be the correct characterizations.

2. Positive Definite. In this section we consider both Hermitian positive definite and positive definite matrices. An $n \times n$ complex matrix A is said to be *Hermitian positive definite* if

$$(A\mathbf{x}, \mathbf{x}) > 0 \tag{2.1}$$

for all $\mathbf{x} \in \mathbb{C}^n \backslash \{\mathbf{0}\}$. An $n \times n$ real matrix A is said to be *positive definite* if

$$(A\mathbf{x}, \mathbf{x}) > 0 \tag{2.2}$$

for all $\mathbf{x} \in \mathbb{R}^n \backslash \{\mathbf{0}\}$. Here, and in what follows, $(\,\cdot\,,\,\cdot\,)$ denotes the usual inner product, i.e., for $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$ (or $\mathbb{R}^n$),

$$(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} x_i \overline{y}_i.$$

Note that the inequalities (2.1) imply that for $A = (a_{ij})_{i,j=1}^n$ we have $a_{ij} = \overline{a}_{ji}$. However the inequalities (2.2) do not imply the symmetry of A. We say that the $n \times n$ matrix A is *Hermitian positive semi-definite* and *positive semi-definite* if weak inequalities hold in (2.1) and (2.2), respectively.

In this section we always assume, for ease of exposition, that the $\mathbf{x}^1, \ldots, \mathbf{x}^k$ are linearly independent. From the above definition of Hermitian positive definiteness it immediately follows that given two sets of k vectors in $\mathbb{C}^n$, namely $\{\mathbf{x}^1, \ldots, \mathbf{x}^k\}$ and $\{\mathbf{y}^1, \ldots, \mathbf{y}^k\}$, then a necessary condition for the existence of an $n \times n$ Hermitian positive definite matrix A satisfying

$$A\mathbf{x}^j = \mathbf{y}^j\,, \qquad j = 1, \ldots, k\,,$$

is that

$$\left(\sum_{j=1}^{k} c_j \mathbf{y}^j, \sum_{j=1}^{k} c_j \mathbf{x}^j \right) > 0$$

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for all $(c_1, \ldots, c_k)^T \in \mathbb{C}^k \backslash \{\mathbf{0}\}$. The analogous necessary condition over the reals holds for the existence of a positive definite matrix.

We prove that this necessary condition is also sufficient. This is hardly surprising. We were only surprised not to have so far found this result in the literature.

THEOREM 2.1. *Assume we are given two sets of k vectors in $\mathbb{C}^n$, namely $\{\mathbf{x}^1, \ldots, \mathbf{x}^k\}$ and $\{\mathbf{y}^1, \ldots, \mathbf{y}^k\}$. The necessary and sufficient condition for the existence of an $n \times n$ Hermitian positive definite matrix A satisfying*

$$A\mathbf{x}^j = \mathbf{y}^j \,, \qquad j = 1, \ldots, k \,,$$

is that

$$\left(\sum_{j=1}^{k} c_j \mathbf{y}^j, \sum_{j=1}^{k} c_j \mathbf{x}^j \right) > 0 \tag{2.3}$$

for all $(c_1, \ldots, c_k)^T \in \mathbb{C}^k \backslash \{\mathbf{0}\}$. Similarly, given two sets of k vectors in $\mathbb{R}^n$, namely $\{\mathbf{x}^1, \ldots, \mathbf{x}^k\}$ and $\{\mathbf{y}^1, \ldots, \mathbf{y}^k\}$, the necessary and sufficient condition for the existence of an $n \times n$ positive definite matrix A such that

$$A\mathbf{x}^j = \mathbf{y}^j \,, \qquad j = 1, \ldots, k \,,$$

is that

$$\left(\sum_{j=1}^{k} c_j \mathbf{y}^j, \sum_{j=1}^{k} c_j \mathbf{x}^j \right) > 0$$

for all $(c_1, \ldots, c_k)^T \in \mathbb{R}^k \backslash \{\mathbf{0}\}$.

Proof. As stated above, the fact that the above conditions are necessary is an immediate consequence of the definition. We prove that these conditions are also sufficient. Our proof is essentially the same in both cases. As such we shall assume we are in the Hermitian positive definite case. For $k = n$ there is nothing to prove. Assume $k < n$.

Set

$$X = \mathrm{span}\{\mathbf{x}^1, \ldots, \mathbf{x}^k\} \,,$$

and

$$Y = \mathrm{span}\{\mathbf{y}^1, \ldots, \mathbf{y}^k\} \,.$$

Let $U = X^\perp$ and $V = Y^\perp$. As $\dim X = k$ and $\dim Y = k$ (from (2.3)), it follows that $\dim U = \dim V = n - k$.

We first claim that $\mathrm{span}\{X, V\} = \mathrm{span}\{Y, U\} = \mathbb{C}^n$. To see this assume, in the negative, that $\mathrm{span}\{X, V\} \neq \mathbb{C}^n$. Then there exists an $\mathbf{x} \in X \backslash \{\mathbf{0}\}$ and a $\mathbf{v} \in V \backslash \{\mathbf{0}\}$ such that $\mathbf{x} + \mathbf{v} = \mathbf{0}$. Assume $\mathbf{x} = \sum_{j=1}^{k} c_j \mathbf{x}^j$. Thus

$$0 = \left(\sum_{j=1}^{k} c_j \mathbf{y}^j, \sum_{j=1}^{k} c_j \mathbf{x}^j + \mathbf{v} \right) = \left(\sum_{j=1}^{k} c_j \mathbf{y}^j, \sum_{j=1}^{k} c_j \mathbf{x}^j \right) + \left(\sum_{j=1}^{k} c_j \mathbf{y}^j, \mathbf{v} \right).$$

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The first bracket on the right-hand side is strictly positive from (2.3) while the second bracket is zero since $\mathbf{v} \in V = Y^{\perp}$. This is a contradiction. Thus $\mathrm{span}\{X, V\} = \mathbb{C}^n$. This same reasoning gives us $\mathrm{span}\{Y, U\} = \mathbb{C}^n$.

Let $\mathbf{u}^1, \ldots, \mathbf{u}^{n-k}$ be a basis for U. No $\mathbf{v} \in V$, $\mathbf{v} \neq \{\mathbf{0}\}$, is in $U^{\perp}$. This follows from the fact that $U^{\perp} = X$ and $\mathrm{span}\{X, V\} = \mathbb{C}^n$. This, in turn, implies that there exist $\mathbf{v}^1, \ldots, \mathbf{v}^{n-k}$ that span a basis for V and satisfy

$$(\mathbf{v}^j, \mathbf{u}^i) = \delta_{ij}, \qquad i, j = 1, \ldots, k. \tag{2.4}$$

We define A as the unique $n \times n$ matrix satisfying

$$A\mathbf{x}^j = \mathbf{y}^j\,, \qquad j = 1, \ldots, k\,,$$

and

$$A\mathbf{v}^i = \mathbf{u}^i\,, \qquad i = 1, \ldots, n-k\,.$$

Now

$$\begin{aligned}
&\left(A\big(\textstyle\sum_{j=1}^k c_j\mathbf{x}^j + \sum_{i=1}^{n-k} d_i\mathbf{v}^i \big), \sum_{j=1}^k c_j\mathbf{x}^j + \sum_{i=1}^{n-k} d_i\mathbf{v}^i \right) \\
&\quad = \left(\textstyle\sum_{j=1}^k c_j\mathbf{y}^j + \sum_{i=1}^{n-k} d_i\mathbf{u}^i, \sum_{j=1}^k c_j\mathbf{x}^j + \sum_{i=1}^{n-k} d_i\mathbf{v}^i \right) \\
&\quad = \left(\textstyle\sum_{j=1}^k c_j\mathbf{y}^j, \sum_{j=1}^k c_j\mathbf{x}^j \right) + \left(\textstyle\sum_{j=1}^k c_j\mathbf{y}^j, \sum_{i=1}^{n-k} d_i\mathbf{v}^i \right) \\
&\qquad + \left(\textstyle\sum_{i=1}^{n-k} d_i\mathbf{u}^i, \sum_{j=1}^k c_j\mathbf{x}^j \right) + \left(\textstyle\sum_{i=1}^{n-k} d_i\mathbf{u}^i, \sum_{i=1}^{n-k} d_i\mathbf{v}^i \right) \\
&\quad = \left(\textstyle\sum_{i=1}^k c_j\mathbf{y}^j, \sum_{j=1}^k c_j\mathbf{x}^j \right) + \textstyle\sum_{i=1}^{n-k} |d_i|^2
\end{aligned} \tag{2.5}$$

since from the orthogonality

$$\left(\sum_{j=1}^k c_j\mathbf{y}^j, \sum_{i=1}^{n-k} d_i\mathbf{v}^i \right) = \left(\sum_{i=1}^{n-k} d_i\mathbf{u}^i, \sum_{j=1}^k c_j\mathbf{x}^j \right) = 0,$$

and from (2.4),

$$\left(\sum_{i=1}^{n-k} d_i\mathbf{u}^i, \sum_{i=1}^{n-k} d_i\mathbf{v}^i \right) = \sum_{i=1}^{n-k} |d_i|^2\,.$$

Both of the last summands in (2.5) are nonnegative and they equal zero if and only if the coefficients c_j and d_i are all zero. This proves the result. $\square$

As stated, in the real case this same analysis proves the existence of an $n \times n$ real matrix A satisfying

$$(A\mathbf{x}, \mathbf{x}) > 0$$

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for all $\mathbf{x} \in \mathbb{R}^n \backslash \{\mathbf{0}\}$. It is worth recalling, however, that this does not imply the symmetry of A. What additional conditions on the $\mathbf{x}^j$ and the $\mathbf{y}^j$, $j = 1, \ldots, k$, are necessary so that the resulting A is also symmetric? To obtain a symmetric $n \times n$ real matrix A satisfying

$$(A\mathbf{x}, \mathbf{x}) > 0$$

for all $\mathbf{x} \in \mathbb{R}^n \backslash \{\mathbf{0}\}$ it is necessary and sufficient that

$$\left(\sum_{j=1}^{k} c_j \mathbf{y}^j, \sum_{j=1}^{k} c_j \mathbf{x}^j \right) > 0$$

for all $(c_1, \ldots, c_k)^T \in \mathbb{C}^k \backslash \{\mathbf{0}\}$. If A is real and symmetric and satisfies

$$\left(\sum_{j=1}^{k} c_j \mathbf{y}^j, \sum_{j=1}^{k} c_j \mathbf{x}^j \right) > 0$$

for all $(c_1, \ldots, c_k)^T \in \mathbb{R}^k \backslash \{\mathbf{0}\}$, then it also satisfies

$$\left(\sum_{j=1}^{k} c_j \mathbf{y}^j, \sum_{j=1}^{k} c_j \mathbf{x}^j \right) > 0$$

for all $(c_1, \ldots, c_k)^T \in \mathbb{C}^k \backslash \{\mathbf{0}\}$. Conversely, if the above set of inequalities is satisfied for all $(c_1, \ldots, c_k)^T \in \mathbb{C}^k \backslash \{\mathbf{0}\}$ and the $\mathbf{x}^j$ and $\mathbf{y}^j$ are all real, then from the construction in the proof of Theorem 2.1 we may obtain a real Hermitian positive definite A, i.e., a symmetric real positive definite matrix.

The situation for Hermitian positive semi-definite and positive semi-definite matrices is more complicated. We do not have a characterization in this case. The conditions

$$\left(\sum_{j=1}^{k} c_j \mathbf{y}^j, \sum_{j=1}^{k} c_j \mathbf{x}^j \right) \geq 0$$

for all $(c_1, \ldots, c_k)^T \in \mathbb{C}^k$ and $\mathbb{R}^k$, respectively, are necessary. However they are not sufficient. Let $\mathbf{x} = (1, 1)$ and $\mathbf{y} = (1, -1)$. Then

$$(c\mathbf{y}, c\mathbf{x}) = 0$$

for all $c \in \mathbb{C}$. However there is no 2×2 Hermitian positive semi-definite matrix A for which $A\mathbf{x} = \mathbf{y}$.

3. Positive. In this section we consider positive and strictly positive (not necessarily square) matrices. An $m \times n$ matrix

$$A = (a_{ij})_{i=1}^{m}{}_{j=1}^{n}$$

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is said to be *positive* if

$$a_{ij} \geq 0$$

for all $i = 1, \ldots, m$, $j = 1, \ldots, n$. We will say it is *strictly positive* if

$$a_{ij} > 0$$

for all $i = 1, \ldots, m$, $j = 1, \ldots, n$.

From the above definition it immediately follows that given two sets of k vectors, namely $\{\mathbf{x}^1, \ldots, \mathbf{x}^k\}$ in $\mathbb{R}^n$ and $\{\mathbf{y}^1, \ldots, \mathbf{y}^k\}$ in $\mathbb{R}^m$, then a necessary condition for the existence of an $m \times n$ positive matrix A satisfying

$$A\mathbf{x}^j = \mathbf{y}^j\,, \qquad j = 1, \ldots, k\,,$$

is that if

$$\sum_{j=1}^k c_j \mathbf{x}^j \geq \mathbf{0},$$

then

$$\sum_{j=1}^k c_j \mathbf{y}^j \geq \mathbf{0}.$$

(This implies that the zero vector is mapped to the zero vector.) A necessary condition for the existence of an $m \times n$ strictly positive matrix A satisfying

$$A\mathbf{x}^j = \mathbf{y}^j\,, \qquad j = 1, \ldots, k\,,$$

is that if

$$\sum_{j=1}^k c_j \mathbf{x}^j = \mathbf{0},$$

then

$$\sum_{j=1}^k c_j \mathbf{y}^j = \mathbf{0},$$

while if $\sum_{j=1}^k c_j \mathbf{x}^j \neq \mathbf{0}$ and

$$\sum_{j=1}^k c_j \mathbf{x}^j \geq \mathbf{0},$$

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then necessarily

$$\sum_{j=1}^{k} c_j \mathbf{y}^j > \mathbf{0}.$$

We prove that these necessary conditions are also sufficient.

THEOREM 3.1. *Assume we are given two sets of k vectors, namely $\{\mathbf{x}^1, \ldots, \mathbf{x}^k\}$ in $\mathbb{R}^n$ and $\{\mathbf{y}^1, \ldots, \mathbf{y}^k\}$ in $\mathbb{R}^m$.*

a) *The necessary and sufficient condition for the existence of an $m \times n$ positive matrix A satisfying*

$$A\mathbf{x}^j = \mathbf{y}^j\,, \qquad j = 1, \ldots, k\,,$$

is that if

$$\sum_{j=1}^{k} c_j \mathbf{x}^j \geq \mathbf{0},$$

then

$$\sum_{j=1}^{k} c_j \mathbf{y}^j \geq \mathbf{0}.$$

b) *The necessary and sufficient condition for the existence of an $m \times n$ strictly positive matrix A satisfying*

$$A\mathbf{x}^j = \mathbf{y}^j\,, \qquad j = 1, \ldots, k\,,$$

is that if $\sum_{j=1}^{k} c_j \mathbf{x}^j = \mathbf{0}$, then $\sum_{j=1}^{k} c_j \mathbf{y}^j = \mathbf{0}$, and if $\sum_{j=1}^{k} c_j \mathbf{x}^j \neq \mathbf{0}$ and

$$\sum_{j=1}^{k} c_j \mathbf{x}^j \geq \mathbf{0},$$

then

$$\sum_{j=1}^{k} c_j \mathbf{y}^j > \mathbf{0}.$$

REMARK 3.2. *If the set span$\{\mathbf{x}^1, \ldots, \mathbf{x}^k\}$ does not contain a non-trivial non-negative vector, then there exists a strictly positive vector $\mathbf{a} \in \mathbb{R}^n$ orthogonal to each of the $\mathbf{x}^j$. In this case we can take any $m \times n$ matrix B satisfying $B\mathbf{x}^j = \mathbf{y}^j$, $j = 1, \ldots, k$, and add to each row of B a suitable positive multiple of $\mathbf{a}$ to obtain the desired result.*

Proof. The fact that the above conditions are necessary is, as previously stated, an immediate consequence of the definition. Let us prove that these conditions are also sufficient.

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We first prove (a). We may assume that the $\{\mathbf{x}^1, \ldots, \mathbf{x}^k\}$ are linearly independent. For if these vectors are linearly dependent and $\sum_{j=1}^k c_j \mathbf{x}^j = \mathbf{0}$, then from the above conditions we must have $\sum_{j=1}^k c_j \mathbf{y}^j = \mathbf{0}$.

Denote the ith coordinate of $\mathbf{y}^j$ by y_i^j, $i = 1, \ldots, m$. The existence of the desired matrix A is equivalent to the existence, for each $i \in \{1, \ldots, m\}$, of a vector $\mathbf{a}^i \in \mathbb{R}^n$ satisfying $\mathbf{a}^i \geq \mathbf{0}$ and

$$(\mathbf{a}^i, \mathbf{x}^j) = y_i^j, \qquad j = 1, \ldots, k.$$

This $\mathbf{a}^i$ will then be the ith row of A. The existence of $\mathbf{a}^i$ is an immediate consequence of Farkas' Lemma; see e.g. Bazarra, Shetty [2, p. 46].

Recall from Farkas' Lemma that if X is an $n \times k$ matrix and $\mathbf{y}$ a vector in $\mathbb{R}^k$, then either there exists an $\mathbf{a} \geq \mathbf{0}$ satisfying

$$\mathbf{a}^T X = \mathbf{y}$$

or there exists a vector $\mathbf{b} \in \mathbb{R}^k$ such that

$$X\mathbf{b} \geq \mathbf{0} \quad \text{and} \quad (\mathbf{b}, \mathbf{y}) < 0. \tag{3.1}$$

In our case we let X denote the $n \times k$ matrix whose columns are $\mathbf{x}^1, \ldots, \mathbf{x}^k$, respectively, and $\mathbf{y} = (y_i^1, \ldots, y_i^k)^T$. We wish to prove the existence of an $\mathbf{a} \in \mathbb{R}^n$ satisfying $\mathbf{a} \geq \mathbf{0}$ and

$$\mathbf{a}^T X = \mathbf{y}.$$

If not, then there exists a $\mathbf{b} \in \mathbb{R}^k$ satisfying (3.1), i.e.,

$$\sum_{j=1}^k b_j \mathbf{x}^j \geq \mathbf{0},$$

where $\mathbf{b} = (b_1, \ldots, b_k)^T$, and

$$\sum_{j=1}^k b_j y_i^j = \left(\sum_{j=1}^k b_j \mathbf{y}^j \right)_i < 0.$$

But this contradicts the condition given in (a).

The proof of (b) is similar. From the assumption in (b) we may assume that the $\{\mathbf{x}^1, \ldots, \mathbf{x}^k\}$ are linearly independent. Based on the above Remark 3.2 we also assume that there exists a non-trivial non-negative vector in span $\{\mathbf{x}^1, \ldots, \mathbf{x}^k\}$.

From a simple variation on Farkas' Lemma it follows that if X is an $n \times k$ matrix of rank k and $\mathbf{y}$ a vector in $\mathbb{R}^k \backslash \{\mathbf{0}\}$, then either there exists an $\mathbf{a} \in \mathbb{R}^n$ satisfying $\mathbf{a} > \mathbf{0}$ and

$$\mathbf{a}^T X = \mathbf{y}$$

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or there exists a vector $\mathbf{b} \in \mathbb{R}^k \backslash \{\mathbf{0}\}$ such that

$$X\mathbf{b} \geq \mathbf{0} \ \text{ and } \ (\mathbf{b}, \mathbf{y}) \leq 0. \tag{3.2}$$

As previously, we let X denote the $n \times k$ matrix whose columns are $\mathbf{x}^1, \ldots, \mathbf{x}^k$, respectively. By assumption X is of rank k. Let $\mathbf{y} = (y_i^1, \ldots, y_i^k)^T$, as above. Since $\sum_{j=1}^k c_j \mathbf{y}^j > \mathbf{0}$ for some choice of $c_1, \ldots, c_k$, we have that $\mathbf{y} \neq \mathbf{0}$. We wish to prove the existence of an $\mathbf{a} \in \mathbb{R}^n$ satisfying $\mathbf{a} > \mathbf{0}$ and

$$\mathbf{a}^T X = \mathbf{y}.$$

If not, there then exists a $\mathbf{b} = (b_1, \ldots, b_k)^T \in \mathbb{R}^k \backslash \{\mathbf{0}\}$ satisfying (3.2), i.e.,

$$\sum_{j=1}^k b_j \mathbf{x}^j \geq \mathbf{0},$$

where $\sum_{j=1}^k b_j \mathbf{x}^j \neq \mathbf{0}$, and

$$\sum_{j=1}^k b_j y_i^j = \left(\sum_{j=1}^k b_j \mathbf{y}^j \right)_i \leq 0,$$

contradicting the condition given in (b). $\square$

4. Strictly Totally Positive. An $m \times n$ matrix $A = (a_{ij})_{i=1}^m{}_{j=1}^n$ is said to be *strictly totally positive* (STP) if all its minors are strictly positive. STP matrices were independently introduced by Schoenberg in 1930 (see [7]; also to be found in [8]) and by Krein and Gantmacher in the 1930's. One of the important equivalent properties defining STP matrices is that of *variation diminishing*. This was Schoenberg's initial contribution to the theory. To explain this more precisely we define, for each $\mathbf{x} \in \mathbb{R}^n$, two sign change counts. These are $S^-(\mathbf{x})$, which is simply the number of ordered sign changes in the vector $\mathbf{x}$ where zero entries are discarded, and $S^+(\mathbf{x})$, which is the maximum number of ordered sign changes in the vector $\mathbf{x}$ where zero entries are given arbitrary values. Thus, for example,

$$S^-(1, 0, 2, -3, 0, 1) = 2, \quad \text{and} \quad S^+(1, 0, 2, -3, 0, 1) = 4\,.$$

Note also that $S^-(\mathbf{0}) = 0$, while we will set $S^+(\mathbf{0}) = n$ for $\mathbf{0} \in \mathbb{R}^n$.

The following result is essentially to be found, with variants, in Ando [1], Karlin [6], and Schoenberg [7].

Theorem VD. *Let A be an $m \times n$ STP matrix. Then*

a) *for each vector $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{x} \neq \mathbf{0}$,*

$$S^+(A\mathbf{x}) \leq S^-(\mathbf{x}),$$

b) *if $S^+(A\mathbf{x}) = S^-(\mathbf{x})$, then the sign of the first (and last) component of $A\mathbf{x}$ (if zero, the sign given in determining $S^+(A\mathbf{x})$) agrees with the sign of the first (and last) nonzero component of $\mathbf{x}$. (If $A\mathbf{x} = \mathbf{0}$, then we do not concern ourselves with the signs.)*

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Conversely, if (a) and (b) hold for some $m \times n$ *matrix* A *and every* $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{x} \neq \mathbf{0}$, *then* A *is STP.*

One consequence of the above theorem is the following. *Assume we are given* k *linearly independent vectors* $\mathbf{x}^1, \ldots, \mathbf{x}^k$ *in* $\mathbb{R}^n$ *and vectors* $\mathbf{y}^1, \ldots, \mathbf{y}^k$ *in* $\mathbb{R}^m$. *A necessary condition for the existence of an* $m \times n$ *STP matrix* A *satisfying*

$$A\mathbf{x}^j = \mathbf{y}^j\,, \qquad j = 1, \ldots, k\,,$$

is that

$$S^+\left(\sum_{j=1}^k c_j \mathbf{y}^j\right) \leq S^-\left(\sum_{j=1}^k c_j \mathbf{x}^j\right) \tag{4.1}$$

for all $(c_1, \ldots, c_k)^T \in \mathbb{R}^k \backslash \{\mathbf{0}\}$, *and if equality occurs in* (4.1) *then the sign patterns necessarily agree, as in the statement of Theorem VD.*

From Theorem VD we have that this condition is also sufficient if $k = n$. We conjecture that this condition is always sufficient. We have proved this conjecture when $k = 1$. However the proof is technical and cumbersome and does not seem to generalize. As such, we will not reproduce it here. In our proof we first reduce the problem to that of a totally positive (see below) nonsingular square matrix. We then use the basic fact that all totally positive nonsingular square matrices are products of matrices with strictly positive diagonal entries, and all other entries zero aside from one positive entry in one of the first off-diagonals. These one positive off-diagonal entries permit us to add a positive multiple of any one coefficient of $\mathbf{x}$ to its neighbor. These are the essentials ingredients used in the proof. Anyone particularly interested in the proof can e-mail me and I will send him/her a TeX file with a proof.

An $m \times n$ matrix $A = (a_{ij})_{i=1}^m{}_{j=1}^n$ is said to be *totally positive* (TP) if all its minors are non-negative. An analogue of Theorem VD holds for TP matrices, except that each $S^+(A\mathbf{x})$ is replaced by $S^-(A\mathbf{x})$. We expect that something similar to the above conjecture will also hold in the case of TP matrices.

This problem, for STP matrices, is related to the problem of embedding functions in a Markov system.

5. P-matrices. An $n \times n$ (real) matrix A is said to be a P-matrix if all its principal minors are strictly positive. P-matrices were introduced by Fiedler, Pták [3], [4], and one of the equivalent definitions of a P-matrix is that A is a P-matrix if and only if for every $\mathbf{x} \in \mathbb{R}^n \backslash \{\mathbf{0}\}$ we have

$$\max_{i=1,\ldots,n} (A\mathbf{x})_i \, x_i > 0\,.$$

As such it is natural to conjecture the following. *Assume we are given* k *linearly independent vectors* $\mathbf{x}^1, \ldots, \mathbf{x}^k$ *in* $\mathbb{R}^n$ *and vectors* $\mathbf{y}^1, \ldots, \mathbf{y}^k$ *in* $\mathbb{R}^n$. *Then the necessary and sufficient conditions implying the existence of an* $n \times n$ *P-matrix* A *satisfying*

$$A\mathbf{x}^j = \mathbf{y}^j, \qquad j = 1, \ldots, k\,.$$

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is that

$$\max_{i=1,\ldots,n} (\sum_{j=1}^{k} c_j \mathbf{y}^j)_i \, (\sum_{j=1}^{k} c_j \mathbf{x}^j)_i > 0\,,$$

for every choice of $(c_1, \ldots, c_k)^T \in \mathbb{R}^k \backslash \{\mathbf{0}\}$. It is proven in Johnson, Smith [5] that this conjecture is valid in the case $k = 1$. An explanation thereof is sufficiently simple to warrant repeating. Assume we are given $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n \backslash \{\mathbf{0}\}$ with $x_m y_m > 0$. Let $A = (a_{ij})_{i,j=1}^n$ where

$$a_{ij} = \begin{cases} \frac{y_m}{x_m}, & i = j = m \\ 1, & i = j \neq m \\ \frac{y_i - x_i}{x_m}, & i \neq m, j = m \\ 0, & \text{otherwise.} \end{cases}$$

Then, as is easily checked, $A\mathbf{x} = \mathbf{y}$ and A is a P-matrix.

In Johnson, Smith [5] can also be found the solution to this same problem for M-matrices in the case $k = 1$.

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REFERENCES

[1] T. Ando. Totally positive matrices. *Lin.Alg. and Appl.*, 90:165–219, 1987.
[2] M. S. Bazarra and C. M. Shetty. *Nonlinear Programming. Theory and Algorithms.* John Wiley & Sons, New York-Chichester-Brisbane, 1979.
[3] M. Fiedler and V. Pták. On matrices with non-positive off-diagonal elements and positive principal minors. *Czech. Math. J.*, 12:382–400, 1962.
[4] M. Fiedler and V. Pták. Some generalizations of positive definiteness and monotonicity. *Numer. Math.*, 9:163–172, 1966.
[5] C. R. Johnson and R. L. Smith. Linear interpolation problems for matrix classes and a transformational characterization of M-matrices. *Lin.Alg. and Appl.*, 330:43–48, 2001.
[6] S. Karlin. *Total Positivity. Volume 1.* Stanford University Press, Stanford, CA, 1968.
[7] I. J. Schoenberg. Über variationsvermindernde lineare Transformationen. *Math. Z.*, 32:321–328, 1930.
[8] I. J. Schoenberg. *I. J. Schoenberg: Selected Papers.* Ed. C. de Boor, 2 Volumes, Birkhäuser, Basel, 1988.