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ON TWO CONJECTURES REGARDING AN INVERSE EIGENVALUE PROBLEM FOR ACYCLIC SYMMETRIC MATRICES*

FRANCESCO BARIOLI† AND SHAUN M. FALLAT†‡

Abstract. For a given acyclic graph G, an important problem is to characterize all of the eigenvalues over all symmetric matrices with graph G. Of particular interest is the connection between this standard inverse eigenvalue problem and describing all the possible associated ordered multiplicity lists, along with determining the minimum number of distinct eigenvalues for a symmetric matrix with graph G. In this note two important open questions along these lines are resolved, both in the negative.

Key words. Symmetric matrices, Acyclic matrices, Eigenvalues, Graphs, Binary trees.

AMS subject classifications. 15A18, 15A48, 05C50.

1. Introduction. Spectral Graph Theory is the study of the eigenvalues of various structured matrices associated with graphs. This subject lies at the crossroads of Linear Algebra and Graph Theory, and has become a prominent area of study for both disciplines. Of particular interest here is the so-called "inverse eigenvalue problem."

Essentially our goal is to construct a certain type of matrix from some specified spectral information. In our case part of this spectral information will be contained in an underlying graph. If A is any $n \times n$ symmetric matrix, then the *graph of* A, denoted by $G(A)$, is the graph with vertex set $V = \{1, 2, \dots, n\}$ and edge $\{i, j\}$ $(i \neq j)$ whenever $a_{ij} \neq 0$. Further, if $G = (V, E)$ is a given graph, then $S(G)$ denotes the set of all $n \times n$ symmetric matrices A such that $G(A) = G$.

Our main (general) problem of interest is the following inverse eigenvalue problem:

Given a graph G, describe or characterize all possible sets of eigenvalues that can be realized by symmetric matrices A with $A \in S(G)$.

For arbitrary (connected) graphs G, this problem is not only unresolved but is also extremely difficult. As a result the focus of this problem has been narrowed (by many researchers) by restricting the class of graphs to *connected acyclic graphs* or *trees*. Even for trees the general inverse eigenvalue problem above is open. However, there has been a significant amount of research accomplished in this case. In particular, much is known about specific classes of trees (e.g., paths, stars, generalized and double generalized stars), and about certain properties of (multiple) eigenvalues associated with matrices A for which $G(A)$ is a tree (see references below). From this point on if A is a symmetric matrix with $G(A)$ a tree, we refer to A as an *acyclic symmetric matrix*.

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†Department of Mathematics and Statistics, University of Regina, Regina, Saskatchewan, S4S 0A2, Canada (barioli@math.uregina.ca, sfallat@math.uregina.ca).

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The first specific eigenvalue results for acyclic matrices (for our purposes at least) come from Parter [9] and Gantmacher and Krein [2] (in the tridiagonal case). Later Wiener [10] extended a result of Parter's in [9] regarding multiple eigenvalues of acyclic symmetric matrices (see also Theorem 2.2).

More recently, there has been numerous substantial advances on the eigenvalues of acyclic symmetric matrices (see [1, 4, 5, 6, 8] for examples), most of which has been fueled by the connection between the inverse eigenvalue problem for trees and so-called *ordered multiplicity lists.* For any symmetric matrix A with distinct eigenvalues, $\lambda_1 < \lambda_2 < \cdots < \lambda_k$ and corresponding multiplicities $m_1, m_2, \ldots, m_k$, we can associate an *ordered multiplicity list* $\langle m_1, m_2, \ldots, m_k \rangle$. In [4] it is suspected that the problem of characterizing all possible eigenvalues of matrices in $S(G)$ when G is a tree is equivalent to determining all possible ordered multiplicity lists for matrices in $S(G)$. This claim is the content of our first unsettled problem.

QUESTION 1.1. *Suppose G is a tree. Then is a collection $\lambda_1 < \lambda_2 < \cdots < \lambda_k$ of scalars with corresponding multiplicities $m_1, m_2, \ldots, m_k$ the spectrum of some A in $S(G)$ whenever the list $\langle m_1, m_2, \ldots, m_k \rangle$ is the ordered multiplicity list of some A in $S(G)$?*

The crux of this issue is that the actual values of the desired eigenvalues are not vital, but rather their associated multiplicities are the key to resolving this inverse eigenvalue problem. Indeed, up to current knowledge this connection seems to hold. However, part of the two-fold purpose of this note is to demonstrate that the general validity of this equivalence does **not** hold for all trees (see section 3 within).

A natural question to ask when considering multiplicity lists is: Can a bound on the maximum possible multiplicity be computed in terms of the graph? This matter was solved in [5], where it was shown that the maximum multiplicity of an eigenvalue for any matrix in $S(G)$ (G a tree) is given by the *path cover number.* The path cover number of a tree is the smallest number of vertex-disjoint paths needed to cover all of the vertices in that tree. Furthermore, it is shown in [8] that the minimum number of distinct eigenvalues for any matrix in $S(G)$ when G is a tree is bounded below by one plus the length of the longest path in that tree. In this note the longest path in a tree is called the *diameter* of the tree. Our next unresolved problems deals with the connection between the diameter of G and the minimum number of distinct eigenvalues for any matrix in $S(G)$.

QUESTION 1.2. *Is the minimum number of distinct eigenvalues over all A in $S(G)$ when G is a tree equal to the diameter of G plus one?*

It is conjectured in [8] that, in fact equality holds in this case, which has been verified for many specific examples of trees. Our second objective here is to show that in fact this conjecture is also false in general for trees (see section 3).

The class of trees that are considered in this paper are binary trees. A tree T is called a *binary tree* if every vertex in T has degree at most 3. A special case of a binary tree is the *path*, and since any (symmetric) matrix whose graph is a path is tridiagonal, the corresponding eigenvalues must be distinct (see also [2]). This fact will be employed throughout this note.

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2. Complete Wiener sets. A consequence of the Courant-Fischer theorem are the well known interlacing conditions on the eigenvalues of a real symmetric matrix and its principal submatrices (see for instance [3, Thm 4.3.8]). For any matrix A and collection of indices $\{v_1, \dots, v_k\}$, we let $A(v_1, \dots v_k)$ (resp. $A[v_1, \dots v_k]$) denote the principal submatrix of A obtained by deleting (resp. keeping) rows and columns $v_1, \dots, v_k$.

THEOREM 2.1. *Let A be a symmetric matrix of order n, and let v be any vertex of $G(A)$. If we denote by $\lambda_1 \leqslant \cdots \leqslant \lambda_n$ the eigenvalues of A, and by $\mu_1 \leqslant \cdots \leqslant \mu_{n-1}$ the eigenvalues of $A(v)$, then*

$$\lambda_1 \leqslant \mu_1 \leqslant \lambda_2 \leqslant \mu_2 \leqslant \cdots \leqslant \lambda_{n-1} \leqslant \mu_{n-1} \leqslant \lambda_n.$$

Let $m_A(\lambda)$ denote the multiplicity of a scalar λ as an eigenvalue of A. For convenience, we introduce the convention that $m_A(\lambda) = 0$ means λ is not an eigenvalue of A. By virtue of Theorem 2.1, if $m_A(\lambda) = k > 0$, then $m_{A(v)}(\lambda)$ must be either $k-1$ or k or $k+1$, and, in general, any of these possibilities may occur. However, if A is an acyclic matrix, we have the following important result, which can be obtained collectively from [9, 10]. Note that if A is acyclic, then $A(v)$ will be a direct sum of matrices, and we refer to the direct summands as *blocks of $A(v)$*.

THEOREM 2.2. *Let A be an acyclic symmetric matrix and λ be a multiple eigenvalue (i.e., $m_A(\lambda) \geqslant 2$). Then there exists a vertex v in $G(A)$ such that*

$$\begin{array}{ll} (2.1) \qquad & (i)\ m_{A(v)}(\lambda) = m_A(\lambda) + 1; \\ & (ii)\ \lambda \textit{ is an eigenvalue of at least three blocks of } A(v). \end{array}$$

If v satisfies (2.1i) for some λ, then we call v a *Wiener vertex for λ*. If, in addition, (2.1ii) holds, v is called a *strong Wiener vertex*. So, by Theorem 2.2, each multiple eigenvalue of an acyclic symmetric matrix has at least one strong Wiener vertex. In general, after the deletion of a Wiener vertex v for some λ, there may be another Wiener vertex w for λ in one (or more) of the blocks of $A(v)$, and hence $m_{A(v,w)}(\lambda) = m_A(\lambda) + 2$. More generally, a set of vertices $\{v_1, \dots, v_q\}$ is called a *Wiener set* for λ, if $m_{A(v_1,\dots,v_q)}(\lambda) = m_A(\lambda) + q$. Obviously all the vertices in a Wiener set are themselves Wiener vertices. However a collection of Wiener vertices does not necessarily form a Wiener set, as shown in [6, Ex. 2.4].

A Wiener set $\{v_1, \dots, v_q\}$ for the eigenvalue λ is said to be *maximal* if it is not strictly contained in any other Wiener set for λ; is *complete* if λ has multiplicity at most 1 in each of the blocks of $A(v_1, \dots, v_q)$. For any multiple eigenvalue a complete Wiener set always exists, by repeated application of Theorem 2.2. Observe that a maximal Wiener set is always complete, but not conversely. Moreover a Wiener set containing a complete one, is not necessarily complete. Indeed, if we consider the matrix in Fig. 2.1 on the following page, $\{3\}$ is a complete Wiener set for $\lambda = 2$, while $\{3, 8\}$ is a Wiener set, but it is not complete. Finally $\{3, 5, 8\}$ is a maximal Wiener set.

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FIG. 2.1.

$$A = \begin{bmatrix} 2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 3 \end{bmatrix}$$

We are now in a position to state an important result on complete Wiener sets for binary trees.

THEOREM 2.3. *Let T be a binary tree, A a symmetric matrix with graph $G(A) = T$, and suppose λ is an eigenvalue of A with multiplicity $k \geqslant 1$. Then there exists a complete Wiener set W for λ such that:*

(2.2) (*i*) $|W| = k - 1$;

(*ii*) *all the vertices in W have degree 3 and are not adjacent to each other;*

(*iii*) *λ is an eigenvalue of multiplicity 1 in every block of $A(W)$.*

Proof. The proof is by induction on k. If $k = 1$, then it suffices to define $W = \varnothing$. So assume $k > 1$. By Theorem 2.2, since T is binary, we can find a vertex v (of degree 3) such that $A(v) = A_1 \oplus A_2 \oplus A_3$, and, for each $i = 1, 2, 3$,

$$k_i = m_{A_i}(\lambda) \geqslant 1; \qquad k_1 + k_2 + k_3 = k + 1. \tag{2.3}$$

Notice that (2.3) yields $k_i < k$ for each i. So, by the inductive assumption, in each block A_i we can construct a complete Wiener set W_i for λ satisfying (2.2). Finally let $W = W_1 \cup W_2 \cup W_3 \cup \{v\}$. Then conditions (2.2i) and (2.2iii) follow by the inductive hypothesis. Concerning (2.2ii), it suffices to note that all the vertices in W_i have degree 3 in $G(A_i)$, so they cannot be adjacent to v. □

Let A be a matrix whose graph T_0 is presented in Fig. 2.2 on the next page. Let λ be an eigenvalue of A. Since the path covering number of T_0 is 4, by [5, p. 141] we have $m_A(\lambda) \leqslant 4$.

PROPOSITION 2.4. *Let A be a symmetric matrix with eigenvalue λ, whose graph T_0 is presented in Fig. 2.2 on the facing page, and define $A' = A(v_0, v_1, v_2, v_3)$. Then*

1. if $m_A(\lambda) = 4$, then

(a) $W = \{v_1, v_2, v_3\}$ is a complete Wiener set for λ;

(b) λ is eigenvalue of each of the 7 blocks of $A(W)$;

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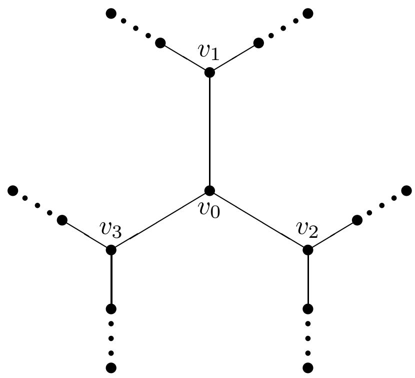
FIG. 2.2. *The graph T_0*



   (c) $m_{A'}(\lambda) = 6$.
2. if $m_A(\lambda) = 3$, *then there exist* $i, j \in \{1, 2, 3\}$ *such that*
   (a) $W = \{v_i, v_j\}$ *is a complete Wiener set for* λ*;*
   (b) λ *is eigenvalue of each of the* 5 *blocks of* $A(W)$*;*
   (c) $m_{A'}(\lambda) \geqslant 4$.

Proof. If $m_A(\lambda) = 4$, then by Theorem 2.3 there exists a complete Wiener set of size three all of whose vertices have degree three and are independent. Evidently, $W = \{v_1, v_2, v_3\}$ is the only such set satisfying all of these requirements. Further, since W is a Wiener set for λ, $m_{A(W)}(\lambda) = m_A(\lambda) + 3 = 7$, and since each block of $A(W)$ is a path it follows that (1b) must hold above. Finally, since A' is a direct sum of 6 of these 7 blocks, $m_{A'}(\lambda) = 6$ is certainly true.

On the other hand, if $m_A(\lambda) = 3$, then by Theorem 2.3 there exists a complete Wiener set of size two consisting of vertices of degree three that are independent. Hence $W = \{v_i, v_j\}$, where $i, j \in \{1, 2, 3\}$. Moreover, $m_{A(W)}(\lambda) = m_A(\lambda) + 2 = 5$, and by *(iii)* of Theorem 2.3, λ is an eigenvalue of multiplicity 1 in each of the five blocks of $A(W)$. Finally, since the four blocks in $A(W)$ that do not contain v_0 are also present as blocks in A', it follows that $m_{A'}(\lambda) \geqslant 4$. □

Both Theorem 2.3 and Proposition 2.4 are essential for resolving Questions 1.1 and 1.2 in the next section.

3. Resolution of Questions 1.1 and 1.2. The trees that are used to derive counterexamples are certain instances of binary trees (see Figures 3.1 and 3.3).

EXAMPLE 3.1. Let A be a symmetric matrix whose graph T_1 is provided in Fig. 3.1 on the next page. Observe that the diameter of T_1, which we denote by $d(T_1)$, is equal to 6. From the results in [8], A has at least $d(T_1)+1 = 7$ distinct eigenvalues. We prove that the minimum number of distinct eigenvalues of A is 8, contradicting a conjecture by Leal Duarte and Johnson in [8] and thus resolving Question 1.2. Suppose that A has only 7 distinct eigenvalues, and let $(m_1, \ldots, m_7)$ be the sequence of multiplicities in decreasing order. Since an irreducible acyclic symmetric matrix

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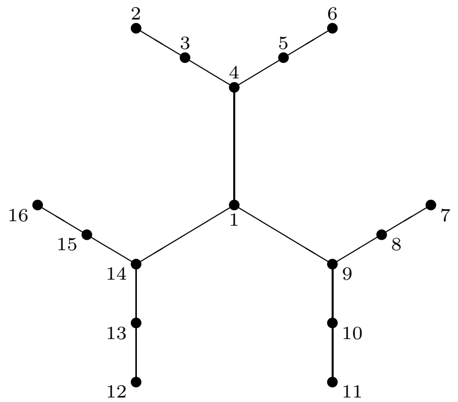
FIG. 3.1. *The graph T_1*



is diagonally similar both to a translate of an irreducible nonnegative matrix and a translate of an irreducible nonpositive matrix, both the largest and the smallest eigenvalues of A have multiplicity 1, that is, $m_6 = m_7 = 1$. This fact can easily be obtained even by Theorems 2.1 and 2.2 collectively. We have already noticed that the path cover number is 4, which implies $m_1 \leqslant 4$. However, there can be at most one eigenvalue with multiplicity 4, since, by Proposition 2.4 (1b), a_{11} must equal this eigenvalue. Therefore, the only sequences of multiplicities matching all of the above mentioned conditions are $\mathcal{S}_1 = (4,3,3,3,1,1,1)$, $\mathcal{S}_2 = (4,3,3,2,2,1,1)$ and $\mathcal{S}_3 = (3,3,3,3,2,1,1)$. Note that $\mathcal{S}_1$ is not realizable. Indeed, if it were the case that $m_A(\lambda_1) = 4$, $m_A(\lambda_2) = m_A(\lambda_3) = m_A(\lambda_4) = 3$, then by Proposition 2.4 (1c) and (2c) we would have $\sum_1^4 m_{A'}(\lambda_i) \geqslant 18$, which is impossible, since A' is equal to $A(1,4,9,14)$ and is 12×12. Similar reasoning applied to $\mathcal{S}_2$ and $\mathcal{S}_3$ yields $\sum_1^3 m_{A'}(\lambda_i) \geqslant 14$ and $\sum_1^4 m_{A'}(\lambda_i) \geqslant 16$ respectively, which are impossible as well. This proves that the minimum number of distinct eigenvalues must be greater than 7. Actually such a minimum is 8, since the matrix A in Fig 3.2 on the facing page has eigenvalues $(1,2,2,3,3,3,3,4,4,\frac{9}{2},5,5,5,6,6,\frac{13}{2})$.

EXAMPLE 3.2. Let T_2 be the binary tree in Fig. 3.3 on the next page and let

$$\underline{\lambda} = (\lambda_1,\lambda_2,\lambda_2,\lambda_3,\lambda_3,\lambda_3,\lambda_3,\lambda_4,\lambda_4,\lambda_5,\lambda_5,\lambda_5,\lambda_6,\lambda_6,\lambda_7),$$

with $\lambda_1 < \lambda_2 < \lambda_3 < \lambda_4 < \lambda_5 < \lambda_6 < \lambda_7$. The corresponding ordered sequence of multiplicities is $\langle 1,2,4,2,3,2,1\rangle$. We prove that there exists a matrix with graph T_2 realizing $\underline{\lambda}$, as, for instance, the matrix B in Fig. 3.4 on page 49, only if

$$\lambda_2 + \lambda_4 + \lambda_6 = \lambda_1 + \lambda_5 + \lambda_7.$$

This example and additional relation then represents a counterexample to the equivalence between the inverse eigenvalue problem for trees and ordered multiplicity sequences suspected by Johnson et al. in [6] and resolves Question 1.1.

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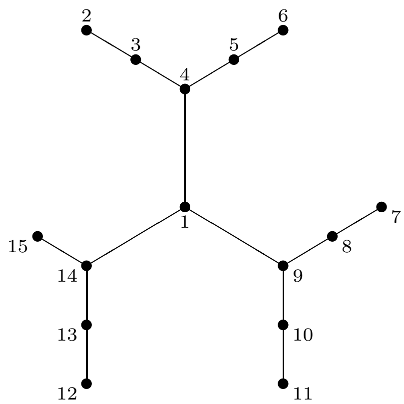
FIG. 3.2. *Matrix with graph T_1 and 8 distinct eigenvalues*

$$A = \left[\begin{array}{c|ccccc|ccccc|ccccc}
3 & 0 & 0 & \frac{\sqrt{5}}{2} & 0 & 0 & 0 & 0 & \frac{\sqrt{5}}{2} & 0 & 0 & 0 & 0 & \frac{\sqrt{5}}{2} & 0 & 0 \\ \hline
0 & 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 4 & \frac{\sqrt{6}}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{\sqrt{5}}{2} & 0 & \frac{\sqrt{6}}{2} & 4 & \frac{\sqrt{6}}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{\sqrt{6}}{2} & 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 4 & \frac{\sqrt{6}}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{\sqrt{5}}{2} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{6}}{2} & 4 & \frac{\sqrt{6}}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{6}}{2} & 4 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 4 & 0 & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 4 & \frac{\sqrt{6}}{2} & 0 & 0 \\
\frac{\sqrt{5}}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{6}}{2} & 4 & \frac{\sqrt{6}}{2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{6}}{2} & 4 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 4
\end{array}\right]$$

FIG. 3.3. *The graph T_2*



Suppose A is a matrix with graph T_2 realizing $\underline{\lambda}$. In order to simplify the notation, the spectrum of the submatrix $A[2,3,4,5,6]$ will be denoted by $\sigma_{2:6}$. Similar meaning is given to the symbols $\sigma_{7:8}$, $\sigma_{7:11}$ and so on. Since $m_A(\lambda_3) = 4$, by Proposition 2.4 (1b) we have

$$a_{1,1} = a_{15,15} = \lambda_3, \tag{3.1}$$

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and

$$\sigma_{2:3} \cap \sigma_{5:6} \cap \sigma_{7:8} \cap \sigma_{10:11} \cap \sigma_{12:13} \supseteq \{\lambda_3\}. \tag{3.2}$$

Let us consider λ_5, which has multiplicity 3. Notice that a complete Wiener set as defined in item 2 of Proposition 2.4 cannot be $\{4, 14\}$ nor $\{9, 14\}$, since in each case we would have $a_{15,15} = \lambda_5$, which is in conflict with (3.1). Therefore the complete Wiener set for λ_5 must be $\{4, 9\}$. Hence Proposition 2.4 (2b) together with (3.2) yields

$$\sigma_{2:3} = \sigma_{5:6} = \sigma_{7:8} = \sigma_{10:11} = \{\lambda_3, \lambda_5\}. \tag{3.3}$$

Applying Theorem 2.1 to $A[2, 3, 4, 5, 6]$ and $A[2, 3, 5, 6]$ it follows that λ_3 and λ_5 are both in $\sigma_{2:6}$. Similar arguments can be applied to $A[7, 8, 9, 10, 11]$, $A[7, 8, 10, 11]$, and to $A[12, 13, 14, 15]$, $A[12, 13, 15]$ to obtain

$$\sigma_{2:6} \cap \sigma_{7:11} \supseteq \{\lambda_3, \lambda_5\}, \tag{3.4}$$

and

$$\sigma_{12:15} \supseteq \{\lambda_3\}. \tag{3.5}$$

Now consider the eigenvalue λ_2. Vertex 4 (or similarly 9) cannot be a Wiener vertex for λ_2, since we would have $\lambda_2 \in \sigma_{2:3}$ (or $\lambda_2 \in \sigma_{7:8}$), which would be in conflict with (3.3). Finally, it cannot be 14 either, since in this case we would have $a_{15,15} = \lambda_2$, which again is impossible. Therefore $\{1\}$ must be the complete Wiener set for the eigenvalue λ_2, and similarly, $\{1\}$ must be the complete Wiener set for the eigenvalues λ_4 and λ_6. In other words we have

$$\sigma_{2:6} \cap \sigma_{7:11} \cap \sigma_{12:15} \supseteq \{\lambda_2, \lambda_4, \lambda_6\}. \tag{3.6}$$

By (3.4), (3.5) and (3.6) we then obtain $\sigma_{2:6} = \sigma_{7:11} = \{\lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6\}$, and $\sigma_{12:15} = \{\lambda_2, \lambda_3, \lambda_4, \lambda_6\}$, so that

$$\begin{aligned} \operatorname{tr}(A) &= a_{11} + \operatorname{tr}(A[2, 3, 4, 5, 6]) + \operatorname{tr}(A[7, 8, 9, 10, 11]) + \operatorname{tr}(A[12, 13, 14, 15]) \\ &= 3\lambda_2 + 4\lambda_3 + 3\lambda_4 + 2\lambda_5 + 3\lambda_6. \end{aligned} \tag{3.7}$$

Finally, since the spectrum of A is $\underline{\lambda}$, we also have that

$$\operatorname{tr}(A) = \lambda_1 + 2\lambda_2 + 4\lambda_3 + 2\lambda_4 + 3\lambda_5 + 2\lambda_6 + \lambda_7 \tag{3.8}$$

Comparing (3.7) with (3.8) we then find the additional necessary condition

$$\lambda_2 + \lambda_4 + \lambda_6 = \lambda_1 + \lambda_5 + \lambda_7. \tag{3.9}$$

So, for instance, $\underline{\lambda} = (1, 2, 2, 3, 3, 3, 3, 4, 4, 5, 5, 5, 6, 6, 7)$ does not satisfy (3.9), and hence is not realizable, while $\underline{\lambda}' = (0, 2, 2, 3, 3, 3, 3, 4, 4, 5, 5, 5, 6, 6, 7)$ satisfies (3.9) and is realized by the matrix B in Fig. 3.4 on the next page. Hence for the tree T_2

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FIG. 3.4. *Matrix with graph T_2 realizing $\underline{\lambda}'$*

$$B = \left[\begin{array}{c|ccccc|ccccc|cccc}
3 & 0 & 0 & \frac{\sqrt{6}}{2} & 0 & 0 & 0 & 0 & \frac{\sqrt{6}}{2} & 0 & 0 & 0 & 0 & \sqrt{6} & 0 \\
\hline
0 & 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 4 & \frac{\sqrt{6}}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{\sqrt{6}}{2} & 0 & \frac{\sqrt{6}}{2} & 4 & \frac{\sqrt{6}}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{\sqrt{6}}{2} & 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 4 & \frac{\sqrt{6}}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{\sqrt{6}}{2} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{6}}{2} & 4 & \frac{\sqrt{6}}{2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{6}}{2} & 4 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 4 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & \frac{\sqrt{6}}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{6}}{2} & \frac{9}{2} & \frac{\sqrt{7}}{2} & 0 \\
\sqrt{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{7}}{2} & \frac{7}{2} & \frac{\sqrt{2}}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{2}}{2} & 3
\end{array}\right]$$

not every list of scalars with corresponding ordered multiplicity list $\langle 1,2,4,2,3,2,1\rangle$ is realizable, but at least one such list, namely $\underline{\lambda}'$, is realizable by a symmetric matrix in $S(T_2)$.

Part of the ambition of this note was to resolve Questions 1.1 and 1.2, and to indicate the importance of the class of trees presented in Figure 2.2. In our estimation this class of trees needs to be studied further in connection with these kinds of inverse eigenvalue problems. Moreover, it seems that many of the techniques developed up to this point for other types of trees do not apply in a natural way to this class of trees. Thus a new approach is required when considering an inverse eigenvalue problem for these trees.

Another important question that is still critical to an inverse eigenvalue problem for trees is: What parameter should be in place of diameter in connection with the minimum number of distinct eigenvalues of symmetric acyclic matrices?

A natural place to begin this study is the class of trees in Figure 2.2, and, in particular, the examples in Figures 3.1 and 3.3.

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