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AN ANALYSIS OF GCD AND LCM MATRICES VIA THE LDL^T-FACTORIZATION*

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Abstract. Let $S = \{x_1, x_2, \ldots, x_n\}$ be a set of distinct positive integers such that $\gcd(x_i, x_j) \in S$ for $1 \le i, j \le n$. Such a set is called GCD-closed. In 1875/1876, H.J.S. Smith showed that, if the set S is "factor-closed", then the determinant of the matrix $e_{ij} = \gcd(x_i, x_j)$ is $\det(E) = \prod_{m=1}^{n} \phi(x_m)$, where ϕ denotes Euler's Phi-function. Since the early 1990's there has been a rebirth of interest in matrices defined in terms of arithmetic functions defined on S. In 1992, Bourque and Ligh conjectured that the matrix $f_{ij} = \mathrm{lcm}(x_i, x_j)$ is nonsingular. Several authors have shown that, although the conjecture holds for $n \le 7$, it need not hold in general. At present there are no known necessary conditions for F to be nonsingular, but many have offered sufficient conditions.

In this note, a simple algorithm is offered for computing the LDL^T-Factorization of any matrix $b_{ij} = f(\gcd(x_i, x_j))$, where $f : S \to \mathbb{C}$. This factorization gives us an easy way of answering the question of singularity, computing its determinant, and determining its inertia (the number of positive negative and zero eigenvalues). Using this factorization, it is argued that E is positive definite regardless of whether or not S is GCD-closed (a known result), and that F is indefinite for $n \ge 2$. Also revisited are some of the known sufficient conditions for the invertibility of F, which are justified in the present framework, and then a few new sufficient conditions are offered. Similar statements are made for the reciprocal matrices $g_{ij} = \gcd(x_i, x_j)/\mathrm{lcm}(x_i, x_j)$ and $h_{ij} = \mathrm{lcm}(x_i, x_j)/\gcd(x_i, x_j)$.

Key words. GCD and LCM matrices, Reciprocal GCD and LCM matrices, Determinants, Singularity, Inertia, LDL^T-Factorization, Partially ordered sets.

AMS subject classifications. 06A07, 06A12, 11A05, 15A18, 15A23, 15A36.

1. Introduction. Let $S = \{x_1, x_2, \ldots, x_n\}$ be a set of distinct positive integers which is GCD-closed, $\gcd(x_i, x_j) \in S$ for $1 \le i, j \le n$. In 1992, Bourque and Ligh [4] conjectured that the matrix $f_{ij} = \mathrm{lcm}(x_i, x_j)$ is nonsingular. Several authors [5,6,7,12] have shown that, although the conjecture holds for $n \le 7$, it need not hold in general. At present there are no known necessary conditions for F to be nonsingular, but many have offered sufficient conditions ([6,12] for instance). In this paper, we address the question of singularity of the matrices $f_{ij} = \mathrm{lcm}(x_i, x_j)$ and $h_{ij} = \mathrm{lcm}(x_i, x_j)/\gcd(x_i, x_j)$ by looking at their LDL^T-Factorizations, offering several new sufficient conditions for nonsingularity.

The paper is structured as follows:

- In section 2, we present an algorithm for finding the LDL^T-Factorization of any matrix of the form $b_{ij} = f(\gcd(x_i, x_j))$, where $f : S \to \mathbb{C}$. This factorization provides an easy way of answering the question of singularity of B, computing its determinant, and determining its inertia (the number of positive negative and zero eigenvalues).
- In section 3, we show that both the GCD and reciprocal GCD/LCM matrices are positive definite (regardless of whether or not S is GCD closed) by embedding them in larger matrices which are shown to be positive definite by

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looking at their LDL^T-Factorizations. The LCM and reciprocal LCM/GCD matrices are shown to be indefinite for $n \geq 2$.

- Section 4 contains several examples of sufficient conditions for the invertibility of LCM and reciprocal LCM/GCD matrices on S, culminating in an example that includes many of the sufficient conditions given in the literature as special cases, with the notable exception of H. J. Smith's condition that S be factor-closed [10]. Many of the sufficient conditions presented here are described in terms of properties of the Hasse diagram associated with S.

2. The Factorization Theorem. Let $\mathcal{P} = \{1, 2, \ldots, n\}$ and $\preceq$ be a partial ordering on $\mathcal{P}$ such that the greatest lower bound (with respect to $\preceq$) of any pair of elements in $\mathcal{P}$ is defined and in $\mathcal{P}$, $\mathrm{glb}(i,j) \in \mathcal{P}$. Such a partially ordered set (poset), $(\mathcal{P}, \preceq)$ is called a meet semi-lattice. We will assume here that $i \preceq j$ implies $i \leq j$.

Let $\mathcal{I}$ denote the set of pairs of comparable elements in $\mathcal{P}$, $\mathcal{I} = \{(k,l) \in \mathcal{P} \times \mathcal{P} : k \preceq l\}$, and $\mu : \mathcal{I} \to \mathbb{Z}$ denote the corresponding Möbius function. We have the following well-known variant of the Möbius inversion formula.

Theorem 2.1 *For $m \in \mathcal{P}$ and $g : \mathcal{P} \to \mathbb{C}$,*

$$g(m) = \sum_{d \preceq m} \sum_{c \preceq d} \mu(c,d) g(c).$$

Remark 2.2 Under the partial ordering $i \preceq j$ iff $i|j$, $\mu(c,d) = \mu(d/c)$ where $\mu(\cdot)$ is the Möbius function from elementary number theory.

Recognizing that $\{d \in \mathcal{P} : d \preceq \mathrm{glb}(i,j)\} = \{d \in \mathcal{P} : d \preceq i \text{ and } d \preceq j\}$, we rephrase Theorem 2.1 as a statement about the factorization of $b_{ij} = g(\mathrm{glb}(i,j))$.

Theorem 2.3 [The Factorization Theorem] *Let $b_{ij} = g(\mathrm{glb}(i,j))$ and $v_g = [g(1)\ g(2)\ \ldots\ g(n)]^T$. Let L and M be given by*

$$l_{id} = \begin{cases} 1 & \text{if } \ d \preceq i \\ 0 & \text{otherwise,} \end{cases} \qquad m_{dj} = \begin{cases} \mu(j,d) & \text{if } \ j \preceq d \\ 0 & \text{otherwise.} \end{cases}$$

Then $B = LDL^T$, where $D = diag(Mv_g)$.

Remark 2.4 The factorization given above is similar to that given by Bhat [3].

Remark 2.5 The condition that g map into $\mathbb{C}$ is more restrictive than necessary. We can take g to map into any vector space over $\mathbb{C}$ ($\mathbb{C}^{m \times m}$ for instance), and more generally into any abelian group, although the usefulness of such generality is questionable.

Remark 2.6 That L and M are inverses is a well-known consequence of the definition of μ. In fact, in some treatments μ is defined in terms of the inverse of L.

We now use the factorization theorem to analyze matrices of the form $b_{ij} = f(\gcd(x_i, x_j))$, where $f : S \to \mathbb{C}$. This will give us a powerful tool for answering the

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question of singularity for the LCM and reciprocal LCM/GCD matrices. We define the partial ordering on $\mathcal{P}$, $i \preceq j$ iff $x_i|x_j$. Given $f : S \to \mathbb{C}$, let $g = f \circ p$ where $p(k) = x_k$. Note that $p(\text{glb}(i,j)) = \gcd(x_i, x_j)$. So we have that $f(\gcd(x_i, x_j)) = g(\text{glb}(i,j))$, and the factorization algorithm for $b_{ij} = f(\gcd(x_i, x_j))$ follows immediately:

Algorithm 2.7 The Factorization Algorithm.

1. Form the vector $v_f = [f(x_1)\ f(x_2)\ \ldots\ f(x_n)]^T$ and the matrix
$$l_{ij} = \begin{cases} 1 & \text{if} \quad x_j|x_i \\ 0 & \text{otherwise.} \end{cases}$$
2. Solve the (lower-triangular) system $Lu = v_f$ via forward substitution.
3. $B = LDL^T$ where $D = \text{diag}(u)$.[1]

Corollary 2.8 *We have* $\det(B) = \prod_{m=1}^n u_m$ *and the number of positive, negative and zero eigenvalues of B (its inertia) is equal to the number of positive, negative, and zero entries of u.*

Remark 2.9 The assumption that S is GCD-closed is what made $(\mathcal{P}, \preceq)$ a meet semi-lattice, and allowed this factorization.

3. Basic Results on Inertia. Let the matrices E, F, G and H be defined by

$$e_{ij} = \gcd(x_i, x_j), \quad f_{ij} = \text{lcm}(x_i, x_j), \quad g_{ij} = \frac{\gcd(x_i, x_j)}{\text{lcm}(x_i, x_j)}, \quad h_{ij} = \frac{\text{lcm}(x_i, x_j)}{\gcd(x_i, x_j)}.$$

The following result concerning the inertias of E and G is well-known, being shown in [2] and [8] respectively (and elsewhere). We offer a different, simple proof based on the factorization algorithm.

Theorem 3.1 *The matrices E and G are positive definite, regardless of whether or not S is GCD-closed.*

Proof. Let $N = x_n = \max S$. The GCD matrix E is a principal submatrix of the $N \times N$ matrix $a_{ij} = \gcd(i,j)$; so, if A is positive definite, then E is as well. Using the factorization theorem on A, we deduce that its inertia is determined by the set of numbers $\left\{\sum_{d|m} \mu(m/d)d\right\}_{m=1}^N = \{\phi(m)\}_{m=1}^N$, which are clearly positive.

After recognizing that $g_{ij} = \frac{\gcd(x_i,x_j)^2}{x_i x_j}$ and $\tilde{g}_{ij} = \gcd(x_i, x_j)^2$ have the same inertia $\{G = K^T \tilde{G} K$, for $K = \text{diag}(1/x_1, 1/x_2, \ldots, 1/x_n)\}$, we can use the same argument as above with $\tilde{G}$ being a principal submatrix of the $N \times N$ matrix $a_{ij} = \gcd(i,j)^2$. Its inertia is determined by the set of numbers $\left\{\sum_{d|m} \mu(m/d)d^2\right\}_{m=1}^N$, which are also positive. We note that S did not have to be GCD-closed for this argument. ∎

Analyzing the inertias of F and H by embedding each in larger related matrices in the manner above fails because these larger matrices are indefinite, so they cannot

[1] We have $u_k = \Psi_{S,f}(x_k)$ in the notation of Bhat.

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give us useful information about the inertias of their principal submatrices. We can see directly from the factorization algorithm, however, that

Theorem 3.2 *The matrices F and H are indefinite for $n \geq 2$.*

Proof. Since

$$f_{ij} = \frac{x_i x_j}{\gcd(x_i, x_j)} \quad \text{and} \quad h_{ij} = \frac{x_i x_j}{\gcd(x_i, x_j)^2}$$

have the same inertias as

$$\tilde{f}_{ij} = \frac{1}{\gcd(x_i, x_j)} \quad \text{and} \quad \tilde{h}_{ij} = \frac{1}{\gcd(x_i, x_j)^2},$$

respectively, we need merely consider the two systems $Lu = v_1$ and $Lw = v_2$, where $v_1 = [1/x_1 \; 1/x_2 \cdots 1/x_n]^T$ and $v_2 = [1/x_1^2 \; 1/x_2^2 \cdots 1/x_n^2]^T$.

Regardless of the partial ordering that divisibility induces on S, we can be certain that the first two equations in the systems are

$$\begin{aligned} u_1 &= \frac{1}{x_1} \quad \text{and} \quad u_1 + u_2 = \frac{1}{x_2} \\ w_1 &= \frac{1}{x_1^2} \quad \text{and} \quad w_1 + w_2 = \frac{1}{x_2^2}. \end{aligned}$$

It is clear that $u_1, w_1 > 0$ and $u_2, w_2 < 0$, hence F and H are indefinite for $n \geq 2$. □

4. Some Sufficient Conditions for the Invertibility of F and H. We have seen that inertias of F and H are the same as those of $\tilde{F}$ and $\tilde{H}$, so we need merely consider the systems $Lu = v_1$ and $Lw = v_2$, where $v_1 = [1/x_1 \; 1/x_2 \cdots 1/x_n]^T$ and $v_2 = [1/x_1^2 \; 1/x_2^2 \cdots 1/x_n^2]^T$, to determine whether or not F and H are nonsingular. In particular, they will be nonsingular if and only if the solutions u and w have no zero components. Below, we give several examples of sufficient conditions for invertibility of F and H. We only explicitly treat F, but the arguments are easily adapted for H as well - we need only replace $1/x_i$ by $1/x_i^2$ in each of the equations and inequalities. The first two examples were given by Smith [10] and Wang [12], but the others are new.

Example 4.1 S is factor-closed, so $\{x_i \in S : x_i | x_j\} = \{d \in \mathbb{N} : d | x_j\}$. The equations to solve here are $\sum_{x_i | x_j} u_i = 1/x_j$, and they have solutions

$$u_j = \sum_{x_i | x_j} \mu\left(\frac{x_j}{x_i}\right) \frac{1}{x_i} \neq 0.$$

Example 4.2 S has the property that $\gcd(x_i, x_j) = x_1$. The equations to solve here are $u_1 = 1/x_1, \; u_1 + u_j = 1/x_j$, and they have solutions

$$u_1 = \frac{1}{x_1} > 0, \; u_j = \frac{1}{x_j} - \frac{1}{x_1} < 0.$$

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Example 4.3 S has the property that $x_1|x_2|\cdots|x_n$. The equations to solve here are $\sum_{k=1}^{m} u_k = 1/x_k$, $1 \leq k \leq n$ and they have solutions

$$u_1 = \frac{1}{x_1} > 0,\ u_k = \frac{1}{x_k} - \frac{1}{x_{k-1}} < 0.$$

We note here that the only elements in S which have any bearing on the component u_k of the system $Lu = v_f$ are those which divide x_k. So we can, for purposes of analysis, decouple the larger system into several smaller ones - one for each of the maximal elements of S. We use this fact to demonstrate another set of sufficient conditions for the invertibility of F and H which generalizes those offered in Examples 4.2 and 4.3.

Example 4.4 The Hasse diagram for S is a tree. We can see this by decoupling the corresponding system of equations into several systems of the form in Example 4.3 - one for each of the maximal chains in S (the fact that a given x_j might belong to more than one chain is irrelevant). We already saw that the solution of such systems is completely nonzero.

Example 4.5 S has the property that $\gcd(x_i, x_j) = x_1$ for $2 \leq j \leq n-1$ and $x_1|x_j|x_n$ for $1 \leq j \leq n$. The case $n = 3$ is trivial, so we assume $n > 3$. The equations to solve here are

$$u_1 = 1/x_1,\ u_1 + u_j = 1/x_j \text{ for } 2 \leq j \leq n-1 \text{ and } \textstyle\sum_{i=1}^{n} u_i = 1/x_n.$$

It is clear that the only equation we have to be concerned about is the last one, where we solve for u_n. We have

$$u_n = \frac{n-3}{x_1} - \left(\frac{1}{x_2} + \cdots + \frac{1}{x_{n-1}}\right) + \frac{1}{x_n} > \frac{1}{x_1} - \frac{1}{x_{n-2}} - \frac{1}{x_{n-1}} + \frac{1}{x_n} > 0.$$

Example 4.6 $S = \bigcup_{j=1}^{k} C_j$ where $C_i \cap C_j = \{x_1, x_n\}$ for $i \neq j$, and C_j is a maximal chain. This is an extension of Example 4.5 and can be analyzed in the same way. As in example 4.5, we need only be concerned about u_n. Let y_j denote the second largest element (after x_n) in C_j. We have

$$u_n = \frac{k-1}{x_1} - \left(\frac{1}{y_1} + \cdots + \frac{1}{y_k}\right) + \frac{1}{x_n} > 0.$$

We have found it useful to view certain families of sets S in terms of the Hasse diagrams of their associated meet semilattices. These diagrams can provide a convenient way of classifying and visualizing the family and determining its associated matrix L. For example, the Hasse diagrams for the families in Examples 4.2 and 4.3 are given respectively in Figure 4.1. The value of the diagrams becomes more apparent in the next two examples, where a formulaic description of S would be cumbersome at best.

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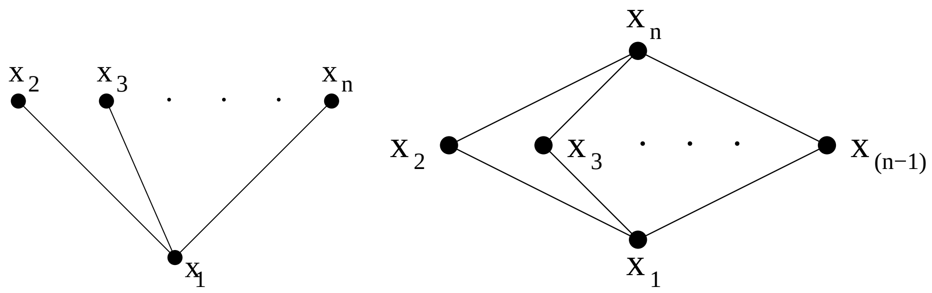

FIG. 4.1. *Example 4.2 (left) and 4.5 (right).*

Example 4.7 Any set S for which the associated Hasse diagram is given in Figure 4.2 (left). The solution u of the associate system is

$$\begin{aligned}
u_1 &= \frac{1}{x_1} > 0\\
u_j &= \frac{1}{x_j} - \frac{1}{x_{j-1}} < 0 \quad \text{for} \quad 2 \le j \le k \;\text{ and }\; l+1 \le j \le n\\
u_j &= \frac{1}{x_j} - \frac{1}{x_l} < 0 \quad \text{for} \quad k+1 \le j \le l-1\\
u_l &= \frac{l-(k+2)}{x_k} - \left(\frac{1}{x_{k+1}} + \cdots + \frac{1}{x_{l-1}}\right) + \frac{1}{x_l} > 0.
\end{aligned}$$

Example 4.8 Any set S for which the associated Hasse diagram is given in Figure 4.2 (right). The solution u of the associate system is

$$\begin{aligned}
u_1 &= \frac{1}{x_1} > 0\\
u_j &= \frac{1}{x_j} - \frac{1}{x_1} < 0 \quad \text{for} \quad 2 \le j \le k-1\\
u_k &= \frac{k-3}{x_1} - \left(\frac{1}{x_2} + \cdots + \frac{1}{x_{k-1}}\right) + \frac{1}{x_k} > 0\\
u_j &= \frac{1}{x_j} - \frac{1}{x_{j-1}} < 0 \quad \text{for} \quad k+1 \le j \le l\\
u_j &= \frac{1}{x_j} - \frac{1}{x_l} < 0 \quad \text{for} \quad l+1 \le j \le n-1\\
u_n &= \frac{n-(l+2)}{x_l} - \left(\frac{1}{x_{l+1}} + \cdots + \frac{1}{x_{n-1}}\right) + \frac{1}{x_n} > 0.
\end{aligned}$$

For convenience in explanation, we will refer to the Hasse diagrams associated with Examples 4.5 and 4.6, respectively, as simple and general "bulbs". We can think of Examples 4.7 and 4.8 as having been constructed by beginning with a single maximal

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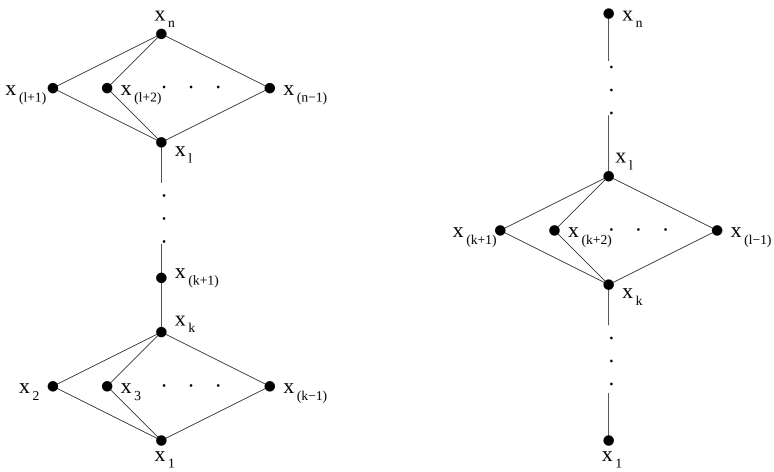

FIG. 4.2. *Example 4.7 (left) and 4.8 (right).*

chain, and replacing one or two vertices with a simple bulb in the ways shown in Figure 4.2. It is a trivial extension of these examples to recognize that any set S whose Hasse diagram can be constructed from a single maximal chain by replacing some of the vertices with simple bulbs gives rise to nonsingular matrices F and H. Actually, one could replace some of the vertices with general bulbs instead without affecting the singularity of F and H. We shall call such a Hasse diagram a "burled chain". These examples and discussion serve as lemmas for our final example, which generalizes all of the previous ones except for Smith's example (4.1).

Example 4.9 Any set S for which the associated Hasse diagram which can be constructed by beginning with a tree, and then replacing some of the vertices with general bulbs, thereby creating a "burled tree". We can decouple the associated system of equations into several smaller systems associated with the maximal "burled chains". As the previous two examples suggest, these systems have solutions which are completely nonzero.

Certainly there are many other variations on this general theme, and we merely wish to put forth a unique way of visualizing classes of sufficient conditions on S based on the associated Hasse diagrams. We do not claim that this is always a useful approach, as Smith's example is one case for which a general description of the associated Hasse diagram is probably not worth the effort. The techniques used in this paper could certainly be used to analyze many other sorts of matrices whose underlying structure is determined by a meet-semilattice. The types of matrices analyzed here were chosen because the author had seen discussion of them in the literature and thought that there might be interest in the approach presented here.

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