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TWO CHARACTERIZATIONS OF INVERSE-POSITIVE MATRICES: THE HAWKINS-SIMON CONDITION AND THE LE CHATELIER-BRAUN PRINCIPLE*

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Dedicated to the late Professors David Hawkins and Hukukane Nikaido

Abstract. It is shown that (a weak version of) the Hawkins-Simon condition is satisfied by any real square matrix which is inverse-positive after a suitable permutation of columns or rows. One more characterization of inverse-positive matrices is given concerning the Le Chatelier-Braun principle. The proofs are all simple and elementary.

Key words. Hawkins-Simon condition, Inverse-positivity, Le Chatelier-Braun principle.

AMS subject classifications. 15A15, 15A48.

1. Introduction. In economics as well as other sciences, the inverse-positivity of real square matrices has been an important topic. The Hawkins-Simon condition [9], so called in economics, requires that every principal minor be positive, and they showed the condition to be necessary and sufficient for a Z-matrix (a matrix with nonpositive off-diagonal elements) to be inverse-positive. One decade earlier, this was used by Ostrowski [12] to define an M-matrix (an inverse-positive Z-matrix), and was shown to be equivalent to some of other conditions; see Berman and Plemmons [1, Ch.6] for many equivalent conditions. Georgescu-Roegen [8] argued that for a Z-matrix it is sufficient to have only *leading* (upper left corner) principal minors positive, which was also proved in Fiedler and Ptak [5]. Nikaido's two books, [10] and [11], contain a proof based on mathematical induction. Dasgupta [3] gave another proof using an economic interpretation of indirect input.

In this paper, the Hawkins-Simon condition is defined to be the one which requires that all the *leading* principal minors should be positive, and we shall refer to it as the *weak* Hawkins-Simon condition (WHS for short). We prove that the WHS condition is necessary for a real square matrix to be inverse-positive after a suitable permutation of columns (or rows). The proof is easy and simple and uses the Gaussian elimination method. One more characterization of inverse-positive matrices is given: Each element of the inverse of the leading $(n-1)\times$ $(n-1)$ principal submatrix is less than or equal to the corresponding element in the inverse of the original matrix. This property is related to the Le Chatelier-Braun principle in thermodynamics.

Section 2 explains our notation, then in section 3 we present our theorems and their proofs, finally giving some numerical examples and remarks in section 4.

2. Notation. The symbol $\mathbb{R}^n$ means the real Euclidean space of dimension n $(n \geq 2)$, and $\mathbb{R}^n_+$ the non-negative orthant of $\mathbb{R}^n$. A given real $n \times n$ matrix A is a

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map from $\mathbb{R}^n$ into itself. The (i, j) entry of A is denoted by a_{ij}, $x \in \mathbb{R}^n$ stands for a column vector, and x_i denotes the i-th element of x. The symbol $(A)_{*,\,j}$ means the j-th column of A, and $(A)_{i,*}$ means the i-th row. We also use the symbol $x_{(i)}$, which represents the column vector in $\mathbb{R}^{n-1}$ formed by deleting the i-th element from x. Similarly, the symbol $A_{(i,\,j)}$ means the $(n-1) \times (n-1)$ matrix obtained by deleting the i-th row and the j-th column from A. Likewise, $A_{(,\,j)}$ shows the $n \times (n-1)$ matrix obtained by deleting the j-th column from A. The symbol $(A)_{i,*(n)}$ shall denote the row vector formed by deleting the n-th element from $(A)_{i,*}$, and $(A)_{*(n),\,j}$ is the column vector in $\mathbb{R}^{n-1}$ formed by deleting the n-th element from $(A)_{*,\,j}$. The symbol $e_i \in \mathbb{R}^n_+$ denotes a column vector whose i-th element is unity with all the remaining entries being zero. $|A|$ denotes the determinant of A.

The inequality signs for vector comparison are as follows:

$x \geq y$ iff $x - y \in \mathbb{R}^n_+$;
$x > y$ iff $x - y \in \mathbb{R}^n_+ - \{0\}$;
$x \gg y$ iff $x - y \in \mathrm{int}(\mathbb{R}^n_+)$,

where $\mathrm{int}(\mathbb{R}^n_+)$ means the interior of $\mathbb{R}^n_+$. These inequality signs are applied to matrices in a similar way.

3. Propositions. Let us first note that the condition "A is inverse-positive" is equivalent to the following property:

Property 1. For any $b \in \mathrm{int}(\mathbb{R}^n_+)$, the equation $Ax = b$ has a solution $x \in \mathrm{int}(\mathbb{R}^n_+)$.

This property was used in Dasgupta and Sinha [4] to establish the nonsubstitution theorem, and in Bidard [2].

Now we can prove the following theorem.

THEOREM 3.1. *Let A be inverse-positive. Then the WHS condition is satisfied when a suitable permutation of columns (or rows) is made.*

Proof. The outline of our proof is as follows. We eliminate, step by step, a variable whose coefficient is positive. The existence of such a variable is guaranteed at each step by Property 1 above. By performing a suitable permutation of columns if necessary, this coefficient can be shown to be positively proportional to a leading principal minor of A.

Because of Property 1 above, there should be at least one positive entry in the first row of A. So, such a column and the first column can be exchanged. We assume the two columns have been permuted so that

$$a_{11} > 0.$$

Next at the second step, we divide the first equation of the system $Ax = b$ by a_{11} and subtract this equation side by side from the i-th($i \geq 2$) equation after multiplying this by a_{i1}, to obtain

$$\begin{bmatrix} 1 & a_{12}/a_{11} & \cdots & a_{1n}/a_{11} \\ 0 & a_{22} - a_{12}a_{21}/a_{11} & \cdots & a_{2n} - a_{1n}a_{21}/a_{11} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n2} - a_{12}a_{n1}/a_{11} & \cdots & a_{nn} - a_{1n}a_{n1}/a_{11} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1/a_{11} \\ b_2 - b_1a_{21}/a_{11} \\ \vdots \\ b_n - b_1a_{n1}/a_{11} \end{bmatrix}.$$

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Notice that the $(2,2)$-entry of the coefficient matrix above is

$$\frac{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}{a_{11}},$$

and the corresponding entry on the RHS is

$$\frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{a_{11}}.$$

We continue this type of elimination up to the k-th step, having at the (k,k)-position

$$\frac{\begin{vmatrix} a_{11} & \cdots & \cdots & a_{1,k} \\ \vdots & \ddots & & \vdots \\ a_{k,1} & \cdots & \cdots & a_{k,k} \end{vmatrix}}{\begin{vmatrix} a_{11} & \cdots & a_{1,k-1} \\ \vdots & \ddots & \vdots \\ a_{k-1,1} & \cdots & a_{k-1,k-1} \end{vmatrix}},$$

and the RHS of the k-th equation is given as

$$\frac{\begin{vmatrix} a_{11} & \cdots & a_{1,k-1} & b_1 \\ \vdots & \ddots & & \vdots \\ a_{k,1} & \cdots & a_{k,k-1} & b_k \end{vmatrix}}{\begin{vmatrix} a_{11} & \cdots & a_{1,k-1} \\ \vdots & \ddots & \vdots \\ a_{k-1,1} & \cdots & a_{k-1,k-1} \end{vmatrix}}.$$

The denominator of these equations is known to be positive at the $(k-1)$-th step, and when b_k is large enough, the RHS of the k-th equation becomes positive. Thus, by Property 1, there is at least one positive coefficient in the k-th equation. Again, we assume a suitable permutation has been made so that the (k,k)-position is positive, giving

$$\begin{vmatrix} a_{11} & \cdots & \cdots & a_{1,k} \\ \vdots & \ddots & & \vdots \\ a_{k,1} & \cdots & \cdots & a_{k,k} \end{vmatrix} > 0 \quad \text{for} \quad k = 2, 3, \ldots, n.$$

Therefore, our theorem is proved for a permutation of columns. A similar result can be obtained by a suitable permutation of rows - just transpose the given matrix and apply the same proof. $\square$

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Corollary 3.2. *When A is a Z-matrix, the WHS condition is necessary and sufficient for A to be inverse-positive.*

Proof. First we show the necessity. Let us consider the elimination method used in the proof of Theorem 3.1. When A is a Z-matrix it is easy to notice that as elimination proceeds, a positive entry is always given at the upper left corner with the other entries (or coefficients) on the top equation being all non-positive, while the RHS of each equation always remains positive. This implies that the WHS condition holds (without any permutation).

Next we show the sufficiency. We assume that $b \gg 0$. When A is a Z-matrix, as elimination proceeds, a positive coefficient can appear only at the upper left corner with the remaining coefficients being all non-positive, while the RHS of each equation is always positive. So, finally we reach the equation of a single variable x_n with the two coefficients on both sides being positive. Thus, $x_n > 0$. Now moving backward, we find $x \gg 0$. Since $b \gg 0$ is arbitrary, this proves that A is inverse-positive. □

This corollary is well known and the reader is referred to Nikaido [10, p.90, Theorem 6.1], Nikaido [11, p.14, Theorem 3.1], or Berman and Plemmons [1, p.134]. (In the diagram of Berman and Plemmons [1, p.134], the N conditions (inverse-positivity) are not connected with the E conditions (WHS) for general matrices.)

Next, we present a theorem which is related to the Le Chatelier-Braun principle; see Fujimoto [6]. This theorem is valid for a class of matrices which is more general than that of inverse-positive matrices.

Theorem 3.3. *Suppose that the inverse of A has its last column and the bottom row non-negative, and that $\left|A_{(n,n)}\right| > 0$. Then each element of the inverse of $A_{(n,n)}$ is less than or equal to the corresponding element of the inverse of A.*

Proof. It is clear that $|A| > 0$. The first column of the inverse of A can be obtained as a solution vector $x \in \mathbb{R}^n$ to the system of equations $Ax = e_1$, while the first column of the inverse of $A_{(n,n)}$ is a solution vector $y \in \mathbb{R}^{n-1}$ to the system $A_{(n,n)}\, y = e_{1(n)}$. Adding these two systems with some manipulations, we get the following system:

$$A \begin{bmatrix} x_1 + y_1 \\ \vdots \\ x_{n-1} + y_{n-1} \\ x_n \end{bmatrix} = d \equiv \begin{bmatrix} 2 \\ 0 \\ 0 \\ (A)_{n,*(n)} \cdot y \end{bmatrix}. \tag{3.1}$$

By Cramer's rule, it follows that

$$x_n = \frac{\left|A_{(,n)}\, d\right|}{|A|} = 2x_n + \frac{\left|A_{(n,n)}\right|}{|A|} \cdot (A)_{n,*(n)} \cdot y.$$

Thus, if $x_n = (A^{-1})_{n1} > 0$, then $(A)_{n,*(n)} \cdot y < 0$, and if $x_n = 0$, then $(A)_{n,*(n)} \cdot y = 0$, because $\frac{\left|A_{(n,n)}\right|}{|A|} > 0$.

For the i-th ($i < n$) equation of (3.1), Cramer's rule gives us

$$x_i + y_i = 2x_i + \frac{\left|A_{(n,i)}\right|}{|A|} \cdot (A)_{n,*(n)} \cdot y.$$

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From this, we have

$$y_i = x_i + (A^{-1})_{in} \cdot (A)_{n,*(n)} \cdot y.$$

Therefore we can assert

$$\begin{cases} y_i < x_i & \text{when } (A^{-1})_{n1} > 0 \text{ and } (A^{-1})_{in} > 0, \\ y_i = x_i & \text{when } (A^{-1})_{n1} = 0 \text{ or } (A^{-1})_{in} = 0. \end{cases}$$

For the other columns, we can proceed in a similar way by replacing e_1 with the appropriate e_i. ☐

As a special case, we have

COROLLARY 3.4. *Suppose that A is inverse-positive, and the WHS condition is satisfied. Then each element of the inverse of $A_{(n,n)}$ is less than or equal to the corresponding element of the inverse of A.*

4. Numerical Examples and Remarks. The first example is given by

$$A = \begin{bmatrix} -2 & 1 \\ 7 & -3 \end{bmatrix} \text{ and } A^{-1} = \begin{bmatrix} 3 & 1 \\ 7 & 2 \end{bmatrix}.$$

By exchanging two columns, we have the M-matrix

$$\begin{bmatrix} 1 & -2 \\ -3 & 7 \end{bmatrix}, \text{ whose inverse is } \begin{bmatrix} 7 & 2 \\ 3 & 1 \end{bmatrix}.$$

This satisfies the normal Hawkins-Simon condition. The inverse of (1) is (1), and the entry 1 is smaller than 7, thus verifying Corollary 3.4.

The second example is not an M-matrix:

$$A = \begin{bmatrix} 1 & -9 & 8 \\ 0 & 12 & -12 \\ -1 & 6 & -4 \end{bmatrix} \text{ and } A^{-1} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & \frac{1}{3} & 1 \\ 1 & \frac{1}{4} & 1 \end{bmatrix}.$$

It should be noted that there does not exist a permutation matrix P such that PA or AP satisfies the normal Hawkins-Simon condition. However, the WHS condition is satisfied by A. The inverse of $A_{(3,3)}$ is calculated as

$$\begin{bmatrix} 1 & -9 \\ 0 & 12 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & \frac{3}{4} \\ 0 & \frac{1}{12} \end{bmatrix}.$$

This verifies Corollary 3.4.

The next example is again not an M-matrix:

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix} \text{ and } A^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}.$$

The inverse of $A_{(3,3)}$ is calculated as

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$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

The elements $(A^{-1})_{11}$, $(A^{-1})_{12}$, and $(A^{-1})_{22}$ are all equal to $(A^{-1}_{(3,3)})_{11}$, $(A^{-1}_{(3,3)})_{12}$, and $(A^{-1}_{(3,3)})_{22}$ because $(A^{-1})_{32} = 0$ and $(A^{-1})_{13} = 0$. The entry $(A^{-1}_{(3,3)})_{21}$ is, however, $-\frac{1}{2}$ and is smaller than the corresponding entry $(A^{-1})_{21} = 0$, confirming the statements in the proof of Theorem 3.3.

The final example illustrates Theorem 3.3:

$$A = \begin{bmatrix} -\frac{17}{24} & \frac{2}{3} & -\frac{5}{24} \\ \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \\ \frac{23}{24} & -\frac{2}{3} & \frac{11}{24} \end{bmatrix} \text{ and } A^{-1} = \begin{bmatrix} -1 & -4 & 1 \\ 2 & -3 & 2 \\ 5 & 4 & 3 \end{bmatrix}.$$

Since

$$\begin{bmatrix} -\frac{17}{24} & \frac{2}{3} \\ \frac{1}{6} & -\frac{1}{3} \end{bmatrix}^{-1} = \begin{bmatrix} -\frac{8}{3} & -\frac{16}{3} \\ -\frac{4}{3} & -\frac{17}{3} \end{bmatrix},$$

these results conform to Theorem 3.3.

REMARK 4.1. The Le Chatelier-Braun principle in thermodynamics states that when an equilibrium in a closed system is perturbed, directly or indirectly, the equilibrium shifts in the direction which can attenuate the perturbation. As is explained in Fujimoto [6], the system of equations $Ax = b$ can be solved as an optimization problem when A is an M-matrix. Thus, a solution x to the system can be viewed as a sort of equilibrium. A similar argument can be made when A is inverse-positive. That is, the solution vector x of the equations $Ax = b$ can be obtained by solving the minimization problem: min $e \cdot x$ subject to $Ax \geq b$, $x \geq 0$, where e is the row n-vector whose elements are all positive, or more simply unity. Thus, the solution vector x can be regarded as a sort of physical equilibrium. In terms of economics, the above minimization problem is to minimize the use of labor input while producing the final output vector b. (Each column of A represents a production process with a positive entry being output and a negative one input, while the vector e is the labor input coefficient vector.) Then, in our case, a perturbation is a new constraint that the n-th variable x_n should be kept constant even after the vector b shifts, destroying the n-th equation. The changes in other variables may become smaller when the increase of those variables requires x_n to be greater. This is obvious in the case of an M-matrix. What we have shown is that it is also the case with an inverse-positive matrix or even with a matrix with *positively bordered inverse* as can be seen from Theorem 3.3.

REMARK 4.2. Much more can be said about the sensitivity analysis in the case of M-matrices. We can also deal with the effects of changes in the elements of A on the solution vector x; see Fujimoto, Herrero, and Villar [7].

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