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REACHABILITY INDICES OF POSITIVE LINEAR SYSTEMS*

RAFAEL BRU†, CARMEN COLL†, SERGIO ROMERO†, AND ELENA SÁNCHEZ†

Abstract. It is well known that positive linear systems have differences in the concepts and characterizations of the structural properties of reachability and controllability. In this paper, the reachability indices of a positive system are defined and consequently they are studied. For that, a canonical form of the reachability indices is given by positive similarity. From that canonical form, it is established that the reachability indices are invariant by positive similarity. At the end, a complete sequence of invariants of a canonical reachability system is given.

Key words. Positive systems, Reachability indices, Canonical form, Sequence of invariants.

AMS subject classifications. 15A21, 93B10.

1. Introduction.

Consider a discrete–time linear system

$$x(k+1) = Ax(k) + Bu(k), \tag{1.1}$$

where $A \in \mathbb{R}^{n\times n},\ B \in \mathbb{R}^{n\times m},\ k \in \mathbb{Z}$. This system is denoted by (A, B).

The system (A, B) is *positive* if for all nonnegative initial state $x(0) \geq 0$ and for all nonnegative control or input sequences $\{u(j)\} \geq 0,\ j \geq 0$, the trajectory of the system is nonnegative, i.e. $x(k) \geq 0$, for all $k \geq 0$. As usual we denote the positive system (1.1), by $(A, B) \geq 0$. It is well–known that the system (A, B) is positive if and only if $A \in \mathbb{R}_+^{n\times n}$ and $B \in \mathbb{R}_+^{n\times m}$; see for instance [6] and [10].

In the case of positive systems, positive reachability from zero property is characterized when the reachability cone of the system at time n coincides with the positive orthant cone. In addition, it is well known that the positive controllability property of the system is equivalent to the positive reachability from zero joint with the nilpotence of the state matrix A; see [3] and [5] and the references therein.

Canonical forms have been established for positively reachable discrete–time systems; see [1]. These canonical forms characterize a positive system when it is positively reachable. The reachability indices have been studied by many authors for systems without restrictions. A summary of this topic is given in [9] and [12]. The invariance of the set of indices in a similarity class is studied in [16] and necessary and sufficient conditions to assign invariant factors of the system under state feedback are given [15]. In [4], the indices for descriptor systems are analyzed. In [11] monomial indices are used for pole–assignment of positive linear systems and in [7] a complete set of invariants for nonnegative unitary operators are introduced. It is worth noting that the reachability and controllability properties of linear time–continuous positive systems are widely studied for different authors (see for instance [10]) so for this kind of systems an extension of the results of this paper could be studied.

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†Institut de Matemàtica Multidisciplinar. Departament de Matemàtica Aplicada, Universitat Politècnica de València, València, Spain (rbru@mat.upv.es, mccoll@mat.upv.es, sromero@mat.upv.es, esanchezj@mat.upv.es). Supported by Spanish DGI grant number BFM2001–2783.

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In this paper, a set of indices related to the positive reachability property are introduced for positive systems. It is known that the construction of the reachability indices of a general system follows from Brunovsky indices but in the positive case many difficulties appear because the characterization of the positive reachability property is given in terms of cones as it is said before.

The characterization of positive similarity of two systems is given in section 2. The positive reachability indices of positive systems are introduced in section 3. From those indices a canonical form of reachability indices is constructed in section 4. Finally, a sequence of invariants (and a complete sequence) of that canonical form is given in section 5. Some examples illustrate the different concepts and results given in this work.

In order to notice the difference among properties of the system (1.1) with and without nonnegative restrictions we recall the reachability concepts in both cases.

Definition 1.1. Consider the system (1.1).

(a) (A, B) is *reachable (from 0)* if for every final state $x_f \in \mathbb{R}^n$ there exists a finite input sequence transferring the state of the system from the origin to x_f.

(b) $(A, B) \geq 0$ is *positively reachable (from 0)* if for every final state $x_f \in \mathbb{R}^n_+$ there exists a finite nonnegative input sequence transferring the state of the system from the origin to x_f.

The reachability characterizations are given as follows:

(a) the general system (A, B) is reachable if and only if, the reachability matrix

$$\mathcal{R}_n(A, B) = \begin{bmatrix} B \mid AB \mid \ldots \mid A^{n-1}B \end{bmatrix}$$

has full rank,

(b) the positive system $(A, B) \geq 0$ is positively reachable if and only if $\mathcal{R}_n(A, B)$ contains a monomial submatrix of order n, that is, there are n distinct monomial vectors; see [6]. Recall that a monomial vector is a (nonzero) multiple of some unit basis vector, and a monomial matrix M is a matrix whose columns are distinct monomial vectors, and can be decomposed as $M = DP$ where D is a diagonal matrix and P is a permutation matrix.

The sequence of positively reachable vectors at time j is the cone $\mathsf{R}_j(A, B)$ generated by the column vectors of the matrix

$$\mathcal{R}_j(A, B) = \begin{bmatrix} B \mid AB \mid \ldots \mid A^{j-1}B \end{bmatrix}$$

and a positive system is reachable if and only if $\mathsf{R}_n(A, B)$ is the positive orthant.

The general reachability property is preserved under similarity transformations, and canonical systems of each equivalent class of reachable systems can be constructed; see [2]. However, as is pointed out in the following section, two similar positive systems are not necessarily both positively reachable.

2. Similar positively reachable systems. It is well-known that the system (A, B) is similar to the system $(\hat{A}, \hat{B})$ if there exists a nonsingular matrix T such that

$$\hat{A} = T^{-1}AT, \quad \hat{B} = T^{-1}B.$$

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As it was mentioned the reachability property, for general systems, is transferred under similarity transformations. However, things are different with positive restrictions. Let consider the following example.

EXAMPLE 2.1. Consider the positive system

$$A = \begin{bmatrix} 4 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

It is easy to check that this system is positively reachable, since the reachability matrix $\mathcal{R}_2(A,B) = [B \mid AB]$ contains a monomial submatrix of order 2.

If for instance, we use the transformation matrix

$$T = \begin{bmatrix} 2 & 0 \\ -3 & 1 \end{bmatrix},$$

whose inverse is

$$T^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{3}{2} & 1 \end{bmatrix},$$

then, the similar system $(T^{-1}AT, T^{-1}B)$, given by

$$T^{-1}AT = \begin{bmatrix} \frac{5}{2} & \frac{1}{2} \\ \frac{15}{2} & \frac{3}{2} \end{bmatrix} \text{ and } T^{-1}B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

remains positive but, it is not positively reachable (the reachability matrix does not contain a monomial submatrix of order 2). Note that the transformation T preserves the positiveness of the system, but it does not transfer the positive reachability property. However, considering both systems as general systems, without restrictions, both of them are reachable.

This fact, together with the construction of canonical systems in [1] by similarity permutation, motivates us to consider a special similarity concept for positive systems (in [8] the concept of similar matrices by a monomial matrix was introduced).

DEFINITION 2.2. Two positive systems (A,B) and $(\hat{A},\hat{B})$ are *positively similar* if there exists a square nonnegative monomial matrix M satisfying

$$\hat{A} = M^{-1}AM \quad \text{and} \quad \hat{B} = M^{-1}B.$$

The following property, of invertible nonnegative matrices, is used in the proof of Theorem 2.4.

REMARK 2.3. (see [13]) The only nonnegative matrices having nonnegative inverses are monomial.

Next, we give a characterization of two positively similar systems.

THEOREM 2.4. *Let $(A,B) \geq 0$ be a positively reachable system similar to the system $(\hat{A},\hat{B}) \geq 0$. Then, the system $(\hat{A},\hat{B})$ is positively reachable if and only if both systems are positively similar.*

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Proof. First, since the positive system $(\hat{A}, \hat{B})$ is similar to the positive system (A, B), there exits an invertible matrix M such that $\hat{A} = M^{-1}AM$ and $\hat{B} = M^{-1}B$. Therefore, the reachability matrices of both systems are related by

$$\begin{aligned} \mathcal{R}_n(\hat{A}, \hat{B}) = \ & \left[M^{-1}B \mid M^{-1}AMM^{-1}B \mid \ldots \mid M^{-1}A^{n-1}MM^{-1}B\right] \\ & = M^{-1}\left[B \mid AB \mid \ldots \mid A^{n-1}B\right] \\ & = M^{-1}\mathcal{R}_n(A, B). \end{aligned}$$

Thus,

$$\mathcal{R}_n(\hat{A}, \hat{B}) = M^{-1}\mathcal{R}_n(A, B) \geq 0. \tag{2.1}$$

Since (A, B) is positively reachable, $\mathcal{R}_n(A, B)$ contains a monomial submatrix of size n, and hence the matrix M^{-1} is nonnegative.

Suppose that the two considered systems are positively similar, in which case, M is a nonnegative monomial matrix. By Definition 2.2 and by the remark, M^{-1} is monomial. Then, from this fact and equation (2.1) the reachability matrix of the system $(\hat{A}, \hat{B})$ contains a monomial submatrix of order n, and hence that system is positively reachable.

Conversely, consider the positively reachable system $(\hat{A}, \hat{B})$. By the above remark, it suffices to prove that the matrix M is nonnegative. Since $\mathcal{R}_n(\hat{A}, \hat{B})$ contains a monomial submatrix of size n, using equation (2.1) there exist n columns of the type $M^{-1}col(A^kB) = \alpha_i e_i$, with $\alpha_i > 0$, $i = 1, \ldots, n$. Therefore, $\alpha_i col_i M = \alpha_i M e_i = col(A^kB) \geq 0$. Hence, $M \geq 0$. ☐

3. Positive reachability indices. Recall that for a system without restrictions (A, B) the r-numbers or Brunovsky numbers are defined as (see [2])

$$r_j = rank\mathcal{R}_j(A, B) - rank\mathcal{R}_{j-1}(A, B), \quad j = 1, 2, \ldots, n,$$

where $\mathcal{R}_0(A, B) = 0$. It is clear that $r_1 \geq r_2 \geq \cdots \geq r_n$. From this sequence the reachability indices are defined as

$$k_i = card\{j : \ r_j \geq i\}, \quad i = 1, 2, \ldots, m,$$

where the symbol "*card*" denotes the cardinal of a sequence. The nonnegative sequence $\{k_1, k_2, \ldots, k_m\}$ is a nonincreasing sequence, and it is the dual sequence of the Brunovsky numbers; see for instance [12]. The sum of the reachability indices is less than or equal to the dimension of the space n. When that sum is n the pair is reachable.

For the pair (A, B), where $B = [b_1 \,|\, b_2 \,|\, b_3 \,|\, \ldots \,|\, b_m]$ the reachability indices can be obtained from the *linearly independent vectors* with respect to the precedent rows in the table:

	b_1	b_2	b_3	$\ldots$	b_m
B	$\times$	$\times$	$\otimes$	$\ldots$	$\times$
AB	$\times$	$\times$		$\ldots$	$\otimes$
A^2B	$\times$	$\otimes$		$\ldots$	
$\vdots$	$\vdots$				
$A^{n-1}B$	$\otimes$				

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where the symbol $\times$ denotes a linearly independent vector with respect to the previously considered vectors (in the same and the previous rows) and the symbol $\otimes$ stands for the linearly dependent vectors. Then, we can consider the following sequence

$$S = \{b_1,\ Ab_1,\ \ldots,\ A^{k_1'-1}b_1,\ b_2,\ Ab_2,\ \ldots,\ A^{k_2'-1}b_2,\\ \ldots,\ b_m,\ Ab_m,\ \ldots,\ A^{k_m'-1}b_m\}$$

formed by m chains of length k_i' of linearly independent vectors, obtained from the columns b_i, for all $i = 1, 2, \ldots, m$, in the reachability matrix. Then, the reachability indices $\{k_1,\ k_2,\ \ldots,\ k_m\}$ are the ordered sequence obtained from the sequence $\{k_1',\ k_2',\ \ldots,\ k_m'\}$. Note that the way of constructing the reachability indices is similar to the characterization of the reachability property in terms of the rank of the reachability matrix. Then, when the pair (A, B) is reachable the sequence of vectors S is a basis of $\mathbb{R}^n$ constructed from the column vectors of $\mathcal{R}_n(A, B)$.

Now, let us focus on a positive pair (A, B). In this case, as we mentioned in the introduction the characterization of positive reachability is given in terms of the monomial vectors of the $\mathcal{R}_n(A, B)$. Then, the attention must be addressed to detect the monomial columns in this matrix.

Denoting by $mon\mathsf{R}_j(A, B)$ the number of distinct monomial columns (up to scalar multiples) of the matrix $\mathcal{R}_j(A, B)$, we give the following definition.

Definition 3.1. Consider the positive system (A, B). The differences

$$m_j = mon\mathsf{R}_j(A, B) - mon\mathsf{R}_{j-1}(A, B), \quad j = 1, 2, \ldots, n,$$

where $\mathsf{R}_0(A, B) = 0$ are called *the m–numbers* of the system (A, B).

The dual sequence of the m–numbers is denoted by

$$d_i = card\{j :\ m_j \geq i\}, \quad i = 1, 2, \ldots, m.$$

In the next example we show that the reachability indices $\{k_1, k_2, \ldots, k_m\}$ cannot coincide with the sequence $\{d_1, d_2, \ldots, d_m\}$.

Example 3.2. Let the system (A, B) where

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

For each j, $j = 1, 2, 3$, we construct the reachable matrix $\mathcal{R}_j(A, B)$. Then, it is easy to check that the r-numbers are $\{2, 1, 0\}$. Hence the sequence of reachability indices, $\{k_1, k_2\}$ is $\{2, 1\}$.

If now we consider the different monomial vectors in the reachability matrices, then the m-numbers are $\{1, 1, 0\}$ and its dual sequence $\{d_1, d_2\}$ is $\{2, 0\}$.

Note that this system is reachable in the general sense (without restrictions) but is not positively reachable.

Let the system $(A, B) \geq 0$ where $B = [b_1 \,|\, b_2 \,|\, b_3 \,|\, \ldots \,|\, b_m]$. We try to proceed as before with the systems without restrictions. Tracking the columns of B, consider

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all *distinct monomial vectors* (up to scalar multiples), with respect to the previously considered vectors (in the same and the previous rows). Note that the considered vectors are in the generator vector sequences of the cones $\mathsf{R}_j(A,B),\ j=1,2,\ldots,n$. In the following table

	b_1	b_2	b_3	$\ldots$	b_m
B	$\times$	$\times$	$\otimes$	$\ldots$	$\times$
AB	$\times$	$\times$	$\times$	$\ldots$	$\otimes$
A^2B	$\otimes$	$\times$		$\ldots$	$\otimes$
A^3B	$\times$				$\times$
$\vdots$				$\vdots$	
$A^{n-1}B$					

the symbol $\times$ denotes distinct monomial vectors and $\otimes$ denotes the remaining vectors (monomial or nonmonomial). As is displayed in the table, there are examples where a monomial vector can appear after a nonmonomial vector; see Example 3.4. In this case, α will denote the first power of A which provides the new monomial vector. Then, we consider the following sequence of distinct monomial vectors

$$S = S_1 \cup S_2 \cup \cdots \cup \cdots S_m,$$

where, for $i=1,2,\ldots,m$,

$$\begin{aligned} S_i = \{&\underbrace{A^{\alpha_{1i}}b_i,\ A^{\alpha_{1i}+1}b_i,\ \ldots,\ A^{\alpha_{1i}+p_{1i}-1}b_i}_{S_{1i}}, \\ &\underbrace{A^{\alpha_{2i}}b_i,\ A^{\alpha_{2i}+1}b_i,\ \ldots,\ A^{\alpha_{2i}+p_{2i}-1}b_i}_{S_{2i}}, \\ &\ldots \\ &\underbrace{A^{\alpha_{l_ii}}b_i,\ A^{\alpha_{l_ii}+1}b_i,\ \ldots,\ A^{\alpha_{l_ii}+p_{l_ii}-1}b_i}_{S_{l_ii}}\} \end{aligned} \tag{3.1}$$

is the sequence of all distinct monomial vectors obtained from the column vector b_i in the reachability matrix, and it is the union of l_i subsequences S_{ki}, $i=1,2,\ldots,m$, $k=1,\ldots,l_i$. Each subsequence S_{ki} is formed by a chain of length p_{ki} of distinct monomial vectors, $k=1,\ldots,l_i$, and α_{ki} denotes the first power of A which provides the first monomial vector.

Note that

$$p_i = p_{1i} + p_{2i} + \cdots + p_{l_ii},\ i=1,\ldots,m, \tag{3.2}$$

is the number of distinct monomial vectors obtained from the ith column b_i. Then we give the following definition.

Definition 3.3. Given the system $(A,B) \geq 0$. The indices

$$\{p_{11},\ p_{21},\ \ldots,\ p_{l_11}; p_{12},\ p_{22},\ \ldots,\ p_{l_22}; \ldots; p_{1m},\ p_{2m},\ \ldots,\ p_{l_mm}\}$$

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are called *the p–numbers* of the positive system (A, B). In the following example, note that the dual sequence of m–numbers and the sequence of p–numbers can be distinct. This fact shows again the differences between systems with or without nonnegative restrictions.

EXAMPLE 3.4. Let $(A, B) \geq 0$ where,

$$A = \begin{bmatrix} 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&1&1&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&1&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&1&1&0\\ 0&0&0&0&0&0&0&0&1&1&0&0 \end{bmatrix} \qquad B = \begin{bmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&1\\ 0&0&0&0\\ 0&0&0&0 \end{bmatrix}$$

with $n = 12$ and $m = 4$. Constructing the above table, the different monomial vectors are (e_i denotes the *ith* canonical vector of $\mathbb{R}^n$)

	b_1	b_2	b_3	b_4
B	$\boxed{e_1}$	$\boxed{e_2}$	$\boxed{e_3}$	$\boxed{e_{10}}$
AB	$\boxed{e_4}$	$\boxed{e_6}$	$\boxed{e_5}$	$e_{11} + e_{12}$
A^2B	e_5	$e_7 + e_3$	$\boxed{e_9}$	$\boxed{e_{11}}$
A^3B	e_9	$e_8 + e_5$	$\boxed{e_{12}}$	
A^4B	e_{12}	$e_7 + e_9$		
A^5B		$e_8 + e_{12}$		
A^6B		$\boxed{e_7}$		
A^7B		$\boxed{e_8}$		

which are in the reachability matrix $\mathcal{R}_{12}(A, B)$.

According to the Definition 3.1, the m–numbers are $\{4, 3, 2, 1, 0, 0, 1, 1, 0, 0, 0, 0\}$. The corresponding dual sequence is $\{6, 3, 2, 1\}$.

The sequences $S_i,\ i = 1, 2 \ldots, 4$ are

$$S_1 = \{b_1, Ab_1\} = \{e_1, e_4\},$$

$$S_2 = S_{12} \bigcup S_{22} = \{b_2, Ab_2 \vdots A^6b_2, A^7b_2\} = \{e_2, e_6, e_7, e_8\},$$

$$S_3 = \{b_3, Ab_3, A^2b_3, A^3b_3\} = \{e_3, e_5, e_9, e_{12}\},$$

$$S_4 = S_{14} \bigcup S_{24} = \{b_4 \vdots A^2b_4\} = \{e_{10}, e_{11}\},$$

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and the p–numbers are $\{2; 2, 2; 4; 1, 1\}$; see Definition 3.3. Observe that the sequence of p–numbers does not coincide with the dual sequence of the m–numbers.

REMARK 3.5. As the above example shows, in general, the sequence of p–numbers does not coincide with the dual sequence of the m–numbers. However, both sequences coincide when, in each column of the table, all distinct monomial vectors are obtained consecutively.

Following the construction of the reachability indices of a general system from the Brunovsky numbers and bearing in mind the above remark, the positive reachability indices are introduced as follows.

DEFINITION 3.6. Let (A, B) be a positive system. Consider the numbers p_i, $i = 1, 2, \dots, m$ given in the equation (3.2) ordered as $p_{i_1} \geq p_{i_2} \geq \cdots \geq p_{i_m}$. The sequence of p–numbers of this system ordered for each $r = 1, 2, \dots, m$ in a nonincreasing order

$$p_{j_1 i_r} \geq p_{j_2 i_r} \geq \cdots \geq p_{j_l i_r},$$

is said to be *the sequence of positive reachability indices* of the positive system (A, B).

REMARK 3.7. If $card(S_i) = card(S_j)$, $i \neq j$, the indices of S_i will be reordered before those of S_j when $i \geq j$ or S_i has less subsequences than S_j.

We have the following positive reachability characterization.

THEOREM 3.8. *The system $(A, B) \geq 0$ is positively reachable if and only if, $p_1 + p_2 + \cdots + p_m = n$.*

Proof. If the pair (A, B) is positively reachable the reachable matrix $\mathcal{R}_n(A, B)$ contains n distinct monomial vectors (up to scalar) which are considered in the sequences S_i, $i = 1, 2, \dots, m$, then $p_1 + p_2 + \cdots + p_m = n$.

Conversely, if $p_1 + p_2 + \cdots + p_m = n$, it is clear that the sequence of distinct monomial vectors

$$S = S_1 \cup S_2 \cup \cdots \cup S_m$$

is a generator sequence of the cone $\mathbb{R}^n_+$, and hence the system (A, B) is positively reachable. $\square$

Let us illustrate the above definition and theorem with an example.

EXAMPLE 3.9. Consider the system from the Example 3.4. Then, the positive reachability indices are $\{4; 2, 2; 2; 1, 1\}$. The sum of all reachability indices is 12 and thus, the system (A, B) is positively reachable. The reachability matrix $\mathcal{R}_{12}(A, B)$ contains a monomial submatrix of order 12.

4. Canonical form. The choice of the positive reachability indices given in Definition 3.6 is basic for the study of canonical forms of positively reachable systems. Due to the canonical forms given in the literature (see [1]) were constructed for characterizing when a positive system is positively reachable, they are not related to these positive reachability indices. In this section a canonical form is constructed such that the sequence of the sizes of its diagonal blocks coincides with the sequence of positive reachability indices. Moreover, in the last section, using this canonical form we will show that the positive reachabilility indices are a sequence of invariants under monomial transformations.

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THEOREM 4.1. *Let $(A,B) \geq 0$ be positively reachable and the sequence*

$$\{p_{11},\ p_{21},\ \ldots,\ p_{l_1 1}; p_{12},\ p_{22},\ \ldots,\ p_{l_2 2}; \ldots; p_{1m},\ p_{2m},\ \ldots,\ p_{l_m m}\}$$

its positive reachability indices. Then, there exists a nonnegative monomial matrix M_S such that the positive similar system $A_c = M_S^{-1} A M_S$ and $B_c = M_S^{-1} B$ has $A_c = \left[A_{c_{ij}}\right]_{i,j=1}^{m}$ structured in blocks as follows:

a) for each $i = 1,2,\ldots,m$, the diagonal block $A_{c_{ii}}$ has order p_i from (3.2). Moreover, $A_{c_{ii}} = \left[A_{c_{ii}}^{hk}\right]_{h,k}^{l_i}$, where for each $h,k = 1,2,\ldots,l_i$, the diagonal block $A_{c_{ii}}^{hh}$ has order p_{hi} and it is

$$\begin{bmatrix} 0 & 0 & \cdots & 0 & * \\ 1 & 0 & & 0 & * \\ 0 & 1 & \ddots & 0 & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & * \end{bmatrix}$$

and the nondiagonal block $A_{c_{ii}}^{hk}$ has size $p_{hi} \times p_{ki}$ and it is

$$\begin{bmatrix} 0 & \cdots & 0 & * \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & * \end{bmatrix}, \tag{4.1}$$

where p_{hi} (p_{ki}) is the hth (kth) index of the ith subsequence of the sequence of reachability indices, and the symbol $*$ denotes a nonnegative entry.

b) for each $i,j = 1,2,\ldots,m$, $i \neq j$, the off diagonal block, $A_{c_{ij}}$ is decomposed in blocks of appropriated sizes and with the structure given in (4.1).

Proof. From each column vector b_i, consider the p_i distinct monomial vectors of the sequence S_i, $i = 1,2,\ldots,m$; see (3.1). Without loss of generality, consider these sequences of vectors ordered according to Definition 3.6, that is, the sequences of vectors S_i are arranged in a nonincreasing order of its cardinals p_i. And, in each sequence S_i, the subsequences of vectors S_{ki}, $k = 1,2,\ldots,l_i$, are ordered in a nonincreasing order of its cardinals p_{ki}, $k = 1,2,\ldots,l_i$. Now, denote by M_S the $n \times n$ matrix whose columns are the vectors of all ordered sequences $S_1, S_2, \ldots, S_m$. This matrix is monomial and nonsingular; see Theorem 3.8.

Since the column vectors of M_S are of type $A^\alpha b_i$, the columns of AM_S are $A^{\alpha+1} b_i$, and thus, they are in the same sequence S_{ki}, except the last vector of each chain. Note that these last vectors are nonnegative linear combination of all columns of M_S. Therefore, the matrix $M_S^{-1} A M_S$ has a block structure with the diagonal blocks given in part a.1) and the off diagonal blocks given in part a.2).

Thus, the pair (A_c, B_c)=$(M_S^{-1} A M_S, M_S^{-1} B)$ has the desired structure. ☐

The system (A_c, B_c) obtained in above theorem will be called the *canonical form of the positive reachability indices* of the positive system (A,B). It is worth noting that the sizes of the diagonal blocks of the matrix A_c are the positive reachability indices of the system.

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Next, we illustrate Theorem 4.1 in the following example.

EXAMPLE 4.2. Consider the system given in Example 3.4. Since the positive reachability indices are $\{4; 2, 2; 2; 1, 1\}$ (see Example 3.9), the matrix M_S associated with the sequences S_i reordered according to the proof is

$$M_S = \left[e_3\, e_5\, e_9\, e_{12} \,|\, e_2\, e_6 \,\vdots\, e_7\, e_8 \,|\, e_1\, e_4 \,|\, e_{10} \,\vdots\, e_{11} \right],$$

where the sequences S_i are ordered as follows

$$M_S = \left\{ S_3 \,|\, S_{12} \,\vdots\, S_{22} | S_1 | S_{14} \,\vdots\, S_{24} \right\}.$$

Thus, the canonical form $[A_c \,\|\, B_c]$, for this example is given by

$$\left[\begin{array}{cccc|cccc|cc|cc||cccc}
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & \multicolumn{1}{|c}{0} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & \multicolumn{1}{|c}{0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\cline{5-8}
0 & 0 & 0 & 0 & 0 & 1 & \multicolumn{1}{|c}{0} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \multicolumn{1}{|c}{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \multicolumn{1}{|c||}{0} & 0 & 0 & 0 & 1 \\
\cline{11-12}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \multicolumn{1}{|c||}{1} & 0 & 0 & 0 & 0
\end{array}\right].$$

Before studying the sequences of invariants, all results are computed with a new example, in which, all possibilities appear when ordering the sequences S_i. In addition, a sequence of distinct monomial vectors is constructed from a nonmonomial column of B.

EXAMPLE 4.3. Let $(A, B) \geq 0$ where,

$$A = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0
\end{bmatrix}, \quad B = \begin{bmatrix}
0 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1
\end{bmatrix}$$

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with $n = 12$ and $m = 5$.

The different monomial vectors are

	b_1	b_2	b_3	b_4	b_5
B	$\boxed{e_{12}}$	$\boxed{2e_1}$	$e_3 + 2e_7$	$\boxed{e_3}$	$e_5 + e_{12}$
AB		$\boxed{6e_4}$	$2e_5 + 2e_8$	$\boxed{2e_5}$	$\boxed{e_9}$
A^2B		$6e_6 + 6e_{12}$	$2e_9 + 2e_7$	$2e_9$	$\boxed{e_{11}}$
A^3B		$\boxed{12e_2}$	$2e_{11} + 2e_8$	$2e_{11}$	$\boxed{2e_{10}}$
A^4B		$\boxed{12e_6}$	$4e_{10} + 2e_7$	$4e_{10}$	$2e_{12}$
A^5B		$24e_2$	$4e_{12} + 2e_8$	$4e_{12}$	
A^6B		$24e_6$	$\boxed{2e_7}$		
A^7B		$48e_2$	$\boxed{2e_8}$		

which are in the reachability matrix $\mathcal{R}_{12}(A, B)$.

The sequences $S_i,\ i = 1, 2 \ldots, 5$ are

$$S_1 = \{b_1\} = \{e_{12}\},$$

$$S_2 = S_{12} \bigcup S_{22} = \{b_2, Ab_2 \,\vdots\, A^3b_2, A^4b_2\} = \{2e_1, 6e_4, 12e_2, 12e_6\},$$

$$S_3 = \{A^6b_3, A^7b_3\} = \{2e_7, 2e_8\},$$

$$S_4 = \{b_4, Ab_4\} = \{e_3, 2e_5\},$$

$$S_5 = \{Ab_5, A^2b_5, A^3b_5\} = \{e_9, e_{11}, 2e_{10}\},$$

and the p–numbers are $\{1; 2, 2; 2; 2; 3\}$; see Definition 3.3. Ordering this sequence according to Definition 3.6, the positive reachability indices are

$$\{2, 2; 3; 2; 2; 1\}.$$

This reordering yields to the following ordered sequences $\{S_{12} \,\vdots\, S_{22}, \mid S_5 \mid S_3 \mid S_4 \mid S_1\}$. Then, the matrix M_S is

$$M_S = [2e_1\, 6e_4 \,\vdots\, 12e_2\, 12e_6 \mid e_9\, e_{11}\, 2e_{10} \mid 2e_7\, 2e_8 \mid e_3\, 2e_5 \mid e_{12}].$$

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Thus, the canonical form $[A_c \| B_c]$ of the system is

$$\left[\begin{array}{cc|cc|ccc|cc|cc|c||ccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{2} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\
\hline
0 & 6 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1
\end{array}\right].$$

5. Sequence of invariants. First, note that the positive reachability indices are invariant under positive monomial transformations.

THEOREM 5.1. *The positive reachability indices defined in Definition* 3.6 *are invariants under positive monomial transformations.*

Proof. Consider two similar positively reachable systems (A, B) and $(\hat{A}, \hat{B})$, with positive reachability indices $\{p_{ij},\ j = 1, \ldots, m,\ i = 1, \ldots, l_j\}$ and $\{\hat{p}_{ij},\ j = 1, \ldots, m,\ i = 1, \ldots, l_j\}$. These two systems are related by

$$\hat{A} = M^{-1}AM \quad \hat{B} = M^{-1}B, \tag{5.1}$$

where M is a nonnegative monomial matrix. Then, the directed digraph of $\hat{A}$ is isomorphic to the directed digraph of A, since the transformation M is a permutation matrix (up to scalars) with the sequence of vertices reordered.

Therefore, if $A^\alpha b_i$ is a monomial column of $\mathcal{R}_n(A, B)$, then $M^{-1}A^\alpha b_i$ is a monomial column of $\mathcal{R}_n(\hat{A}, \hat{B})$, and thus, monomial vectors in the two reachability matrices $\mathcal{R}_n(A, B)$ and $\mathcal{R}_n(\hat{A}, \hat{B})$ appear in the same positions. Then, each column of both matrices B and $\hat{B}$ may provide the same chain of monomial vectors, up to the reordering the vertices. Hence,

$$\{p_{ij} = \hat{p}_{ij},\ j = 1, \ldots, m,\ i = 1, \ldots, l_j\}$$

and then, the positive reachability indices are invariant under positive monomial transformations. ◻

However, the sequence of positive reachability indices

$$\{p_{ij},\ j = 1, \ldots, m,\ i = 1, \ldots, l_j\}$$

is not a complete system of invariants for nonnegative monomial transformations. There are positive systems with the same positive reachability indices, but they are not in the same equivalence class of positive similarity. We illustrate this assertion with the following example.

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EXAMPLE 5.2. Consider the two positive systems

$$A = \left[\begin{array}{cc|c} 0 & 0 & 1 \\ 1 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right], \; B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$F = \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right], \; G = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

It is easy to check that they are not positively similar. However, from the tables

(A, B)

	b_1	b_2
B	$\boxed{e_1}$	$\boxed{e_3}$
AB	$\boxed{e_2}$	e_1

(F, G)

	g_1	g_2
G	$\boxed{e_1}$	$\boxed{e_3}$
FG	$\boxed{e_2}$	0

both systems have the same positive reachability indices

$$\{p_{11} = \hat{p}_{11}, \; p_{21} = \hat{p}_{21}\} = \{2; 1\}.$$

In order to find a complete system of invariants we give the following result.

THEOREM 5.3. *Two positively similar positive reachable systems have the same canonical form constructed in Theorem* 4.1.

Proof. Consider two positively reachable systems (A, B) and $(\hat{A}, \hat{B})$ as in (5.1), and denote by M_S and $M_{\hat{S}}$ the matrices which transform the systems (A, B) and $(\hat{A}, \hat{B})$ in its canonical forms, respectively; see Theorem 4.1. Reasoning in the same way as in Theorem 5.1, the column vectors of $M_{\hat{S}}$ are the transformed column vectors of M_S under the nonnegative monomial matrix M. Thus, $M_{\hat{S}} = M^{-1}M_S$ and the canonical forms

$$\begin{aligned} \hat{A}_c = &\; M_{\hat{S}}^{-1}\hat{A}M_{\hat{S}} = M_S^{-1}M\hat{A}M^{-1}M_S = M_S^{-1}AM_S = A_c, \\ \hat{B}_c = &\; M_{\hat{S}}^{-1}\hat{B} = M_S^{-1}M\hat{B} = M_S^{-1}B = B_c \end{aligned}$$

are equal. □

We illustrate the above result with the following example.

EXAMPLE 5.4. Consider the positive systems

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 4 & 2 \\ 3 & 0 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 3 \end{bmatrix}$$

and

$$\hat{A} = \begin{bmatrix} 0 & 0 & 3 \\ \frac{3}{5} & 0 & 0 \\ 0 & \frac{10}{3} & 0 \end{bmatrix}, \; \hat{B} = \begin{bmatrix} 2 & 0 \\ 0 & \frac{3}{5} \\ 0 & 0 \end{bmatrix}.$$

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It is easy to check that they are positively similar under the monomial matrix

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 5 & 0 \end{bmatrix}$$

From the tables

	b_1	b_2
B	$\boxed{2e_1}$	$\boxed{3e_3}$
AB	$6e_3$	$\boxed{6e_2}$

	$\hat{b}_1$	$\hat{b}_2$
$\hat{B}$	$\boxed{2e_1}$	$\boxed{\frac{3}{5}e_2}$
$\hat{A}\hat{B}$	$\frac{6}{5}e_2$	$\boxed{2e_3}$

the matrices $M_S = [3e_3\ 6e_2|2e_1]$ and $M_{\hat{S}} = \left[\frac{3}{5}e_2\ 2e_3|2e_1\right]$ transform the systems (A, B) and $(\hat{A}, \hat{B})$ into the canonical forms (A_c, B_c) and $(\hat{A}_c, \hat{B}_c)$, respectively. It can be seen that these canonical forms are equal and are given by

$$A_c = \hat{A}_c = \left[\begin{array}{cc|c} 0 & 0 & 2 \\ 1 & 4 & 0 \\ \hline 0 & 3 & 0 \end{array}\right], \; B_c = \hat{B}_c = \left[\begin{array}{cc} 1 & 0 \\ 0 & 0 \\ \hline 0 & 1 \end{array}\right].$$

It is clear that when two systems have the identical canonical form, these systems are positively similar and they have the same reachability indices and the same nonzero entries, denoted by the symbol $*$, in that canonical form. Therefore, the following corollary can be established with the help of Theorems 5.1 and 5.3.

Corollary 5.5. *A complete sequence of invariants of positively similar positive reachable systems is formed by the positive reachable indices* $\{p_{ij},\ i = 1, \ldots, l_j,\ j = 1, \ldots, m\}$ *and the possible nonzero pattern of the blocks of the canonical form given in Theorem* 4.1.

Popov in [14] provides a complete sequence of invariants for systems without restrictions. This sequence is formed by the reachability indices and the nonzero entries of the state matrix. In Corollary 5.5, a complete sequence of invariants for positive systems is obtained. The structure of this complete sequence is in the Popov sense, that is, is formed by the positive reachable indices and the nonzero entries of the canonical form.

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REFERENCES

[1] R. Bru, S. Romero, and E. Sánchez. Canonical forms for positive discrete-time linear systems. *Linear Algebra and its Applications*, 310:49–71, 2000.

[2] P. Brunovsky. A classification of linear controllable systems. *Kibernetika*, 36(6):173–188, 1970.

[3] L. Caccetta and V. G. Rumchev. A survey of reachability and controllability for positive linear systems. *Annals of Operations Research*, 98:101–122, 2000.

[4] C. Coll, M. Fullana, and E. Sánchez. Some invariants of discrete-time descriptor systems. *Applied Mathematics and Computation*, 127:277–287, 2002.

[5] P.G. Coxson and H. Shapiro. Positive reachability and controllability of positive systems. *Linear Algebra and its Applications*, 94:35–53, 1987.

[6] L. Farina and S. Rinaldi. *Positive Linear Systems*. John Wiley & Sons, New York, 2000.

[7] K.H. Förster and B. Nagy. Nonnegative unitary operators. *Proceedings of the American Mathematical Society*, 132(4):1181–1193, 2003.

[8] D.J.G. James, S.P. Kostova, and V.G. Rumchev. Pole–assignment for a class of positive linear systems. *International Journal of Systems Science*, 32(12):1377–1388, 2001.

[9] T. Kailath. *Linear Systems*. Prentice-Hall, Int., New Jersey, 1980.

[10] T. Kaczorek. *Positive 1D and 2D Systems*. Springer–Verlag, London, 2002.

[11] S. Kostova. Pole–assignment for controllable positive linear discrete time systems. *Systems Science*, 28:41–50, 2002.

[12] V. Kucera. *Analysis and Design of Discrete Linear Control Systems*. Prentice-Hall, Int., Hetfordshire, UK, 1991.

[13] H. Minc. *Non–negative Matrices*. Wiley and Sons, New York, 1988.

[14] V. M. Popov. Invariant description of linear time–invariant controllable systems. *SIAM Journal of Control*, 15(2):252–264, 1972.

[15] A. Roca and I. Zaballa. Invariant Factor assignment under state feedback and output injection. *Linear Algebra and its Applications*, 332:401–436, 2001.

[16] I. Zaballa. Controllability and Hermite indices of matrix pairs. *International Journal of Control*, 68 (1):61–86, 1997.