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PERTURBING NON-REAL EIGENVALUES OF NONNEGATIVE REAL MATRICES*

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Abstract. Let A be an (entrywise) nonnegative $n \times n$ matrix with spectrum σ and Perron eigenvalue ρ. Guo Wuwen [Linear Algebra and its Applications 266 (1997), pp. 261–267] has shown that if λ is another real eigenvalue of A, then, for all $t \geq 0$, replacing ρ, λ in σ by $\rho + t$, $\lambda - t$, respectively, while keeping all other entries of σ unchanged, again yields the spectrum of a nonnegative matrix. He poses the question of whether an analogous result holds in the case of non-real λ. In this paper, it is shown that this question has an affirmative answer.

Key words. Nonnegative matrix, Spectrum, Realizable list, Perturbation, Perron eigenvector.

AMS subject classifications. 15A48, 15A18.

1. Introduction.

Our main result is:

THEOREM 1.1. *Let A be an $n \times n$ nonnegative real matrix with spectrum σ and Perron root ρ. Let $\lambda = b + ic$, where b and c are real and $i = \sqrt{-1}$, be a non-real eigenvalue of A. Then, for all $t \geq 0$, replacing ρ, $b + ic$, $b - ic$ in σ by $\rho + 2t$, $b - t + ic$, $b - t - ic$, respectively, while keeping the other entries of σ unchanged, again yields the spectrum of an $n \times n$ nonnegative matrix.*

We use the notation of Guo's paper [1]. In particular a list σ of n complex numbers is said to be *realizable* if σ is the spectrum of a nonnegative real $n \times n$ matrix. In proving the result, we first show that the theorem holds for all sufficiently small $t > 0$, and then we use a compactness argument to extend the result to all nonnegative t.

2. Small perturbation case.

Let A be an $n \times n$ nonnegative matrix and let $\sigma = (a, b+ic, b-ic, d_4, ..., d_n)$ be the spectrum of A, where a is the Perron eigenvalue, b and c are real, and $i = \sqrt{-1}$. We assume that c is not zero. We will show that for all sufficiently small $t \geq 0$, the list

$$(a + 2t, b - t + ic, b - t - ic, d_4, ..., d_n)$$

is realizable. If A is not (permutation) irreducible, then A is permutationally similar to a block upper-triangular matrix with irreducible diagonal blocks. Let B be a diagonal block having $b+ic$ as an eigenvalue and let s be the Perron eigenvalue of B. If the result holds for B, then we can modify the list σ by replacing s by $s + 2t$ and $b + ic$, $b - ic$ by $b - t + ic$, $b - t - ic$, respectively, and obtain a realizable list for all sufficiently small $t \geq 0$. But then, by the result of Guo [1], referred to in the abstract, if s is different from a, we can now replace a by $a + 2t$ and $s + 2t$ by s and still get a realizable list. Hence we can assume that $A = B$ is irreducible. Performing a diagonal

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similarity on A, we can assume that $Ae = ae$, where e is the vector $(1,1,...,1)^T$ of all ones. Let

$$u = (u_1, ..., u_n)^T,\ v = (v_1, ..., v_n)^T$$

be real vectors such that $u+iv$ is an eigenvector of A corresponding to the eigenvalue $b+ic$. Since c is not zero,e, u, v are linearly independent. Extend e, u, v to a basis $e, u, v, w_4, ..., w_n$ of the space of real column n-tuples and let T be the $n \times n$ matrix

$$(e, u, v, w_4, ..., w_n).$$

Observe that $AT = TC$ where C is the 2×2 block matrix

$$\begin{pmatrix} C_{11} & C_{12} \\ 0 & C_{22} \end{pmatrix}$$

and C_{11} is the 3×3 matrix

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & c \\ 0 & -c & b \end{pmatrix}.$$

Since the vector e has all its components equal, we can assume, without loss of generality, that

$$u_1^2 + v_1^2 = \text{maximum}\{u_j^2 + v_j^2 : j = 1, 2, ..., n\}. \tag{2.1}$$

Let $u_0 = u_1/(u_1^2+v_1^2), v_0 = v_1/(u_1^2+v_1^2)$. Let N be the $n\times n$ matrix with first column $e - u_0u - v_0v + yu_0v - yv_0u$, where y is a real variable, and with all its other columns equal to 0. Because of (2.1) and the Cauchy-Schwarz inequality, N is nonnegative for all sufficiently small y. Also, N has trace zero and rank one. Observe that $NT = TD$ where D is a block matrix of the form

$$\begin{pmatrix} D_{11} & D_{12} \\ 0 & D_{22} \end{pmatrix}$$

compatibly blocked with C above. Here D_{11} is the 3×3 matrix

$$\begin{pmatrix} 1 & u_1 & v_1 \\ -u_0 - yv_0 & -u_1u_0 - yu_1v_0 & -v_1u_0 - yv_1v_0 \\ -v_0 + yu_0 & -u_1v_0 + yu_1u_0 & -v_1v_0 + yv_1u_0 \end{pmatrix}.$$

Now, for every sufficiently small value of y and for every nonnegative number $z, A + zN$ is a nonnegative matrix whose characteristic polynomial is that of A with the factor

$$(x - a)((x - b)^2 + c^2)$$

replaced by $\det(xI - (C_{11} + zD_{11}))$. By direct calculation,

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$$\begin{aligned} f(x) &:= \det(xI - (C_{11} + zD_{11}))) \\ &= x^3 - (a+2b)x^2 + (b^2 + c^2 + 2ab - az + bz - czy)x \\ &- (ab^2 + ac^2 + b^2 z + c^2 z - azb - aczy). \end{aligned}$$

Observe that

$$\begin{aligned} g(x) &:= (x - (a+2t))((x - b + t)^2 + c^2) \\ &= x^3 - (a+2b)x^2 + (-3t^2 + c^2 + b^2 + 2ab - 2at + 2bt)x \\ &- (ab^2 + ac^2 + at^2 + 2b^2 t + 2c^2 t - 2abt - 4bt^2 + 2t^3). \end{aligned}$$

Taking

$$z = \frac{(2a - 2b + 3t)t}{(a - b + cy)} \tag{2.2}$$

ensures that the coefficients of x in f(x) and g(x) are equal and then putting

$$y = \frac{-(a^2 + b^2 - 3c^2 + 2at - 2ab - 2bt)t}{2c(a^2 + b^2 + c^2 + t^2 + 2at - 2ab - 2bt)} \tag{2.3}$$

ensures that $f(x) = g(x)$. Since a is the Perron eigenvalue of A, $a - b > 0$, so $z > 0$ for all small $t > 0$. The denominator of y is bounded away from zero for small t, while the numerator is small for all small t. Since N is nonnegative for all sufficiently small y, it follows that for all sufficiently small $t \geq 0$, defining z and y by equations (2.2) and (2.3), respectively, the corresponding matrix $A + zN$ is nonnegative and has the desired spectrum.

3. Global case. Suppose that $\sigma = (a, b + ic, b - ic, d_4, ..., d_n)$ is realizable with a as Perron eigenvalue, but that for some $m > 0$,

$$\sigma\prime = (a + 2m, b - m + ic, b - m - ic, d_4, ..., d_n)$$

is not realizable.

Let $Z = \{t \geq 0 : (a + 2s, b - s + ic, b - s - ic, d_4, ..., d_n)$ is realizable for all s with $0 \leq s < t\}$.

Note that Z is nonempty and that it is bounded above by m, so it has a supremum, z, say. Also, $z > 0$, by the result of Section 2. Let $\{z_k : k = 1, 2, ...\}$ be a strictly increasing sequence of positive real numbers with limit z. Let A_k be a nonnegative matrix having the spectrum

$$(a + 2z_k, b - z_k + ic, b - z_k - ic, d_4, ..., d_n).$$

Using [1] Lemma 2.2, we can assume that the vector e of all ones is a Perron eigenvector of A_k for all k, and, therefore, that the entries of all the A_k lie in the interval $[0, a+2z]$.

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But then the sequence $\{A_k\}$ must have a convergent subsequence. Let A_0 be the limit of such a subsequence. Then, by continuity of the spectrum, A_0 has spectrum

$$(a + 2z, b - z + ic, b - z - ic, d_4, ..., d_n).$$

But now, the result of Section 2 implies that

$$(a + 2z + 2t, b - z - t + ic, b - z - t - ic, d_4, ..., d_n)$$

is realizable, for all sufficiently small $t > 0$. But this contradicts the fact that $z = \sup(Z)$. Thus no such m exists and the theorem is proved.

Remark 3.1. Since

$$(u_1^2 + v_1^2 - u_1 u_j - v_1 v_j) \geq [(u_1 - u_j)^2 + (v_1 - v_j)^2]/2,$$

one sees that the matrix N in Section 2 is nonnegative for all y with

$$| y | \leq \min\{[(u_1 - u_j)^2 + (v_1 - v_j)^2]/(2 \mid u_1 v_j - u_j v_1 \mid) : \text{all } j \text{ with } (u_1, v_1) \neq (u_j, v_j)\}.$$

Using equations (2) and (3) of Section 2, we can deduce explicit bounds on the size of t for which the given rank one perturbation $A + zN$ gives the desired realization of

$$(a + 2t, b - t + ci, b - t - ci, d_4, ..., d_n).$$

It would be interesting to know whether a rank one perturbation of A can be found to realize the list for larger values of t.

Remark 3.2. As in the case of Guo's result quoted in the abstract, the result presented here is useful in studying the boundary of the set of realizable spectra. These applications will appear elsewhere. Meehan's thesis [2] also contains some interesting observations on Guo's question and proofs of some special cases of the theorem here.

REFERENCES

[1] Guo, Wuwen. Eigenvalues of nonnegative matrices. *Linear Algebra and its Applications*, 206:261–270, 1997.

[2] Meehan, Mary E. Some results on matrix spectra. PhD Thesis. National University of Ireland, Dublin, 1998.