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APPROXIMATING THE ISOPERIMETRIC NUMBER OF STRONGLY REGULAR GRAPHS*

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Abstract. A factor 2 and a factor 3 approximation algorithm are given for the isoperimetric number of Strongly Regular Graphs. One approach involves eigenvalues of the combinatorial laplacian of such graphs. In this approach, both the upper and lower bounds involve the spectrum of the combinatorial laplacian. An interesting inequality is proven between the second smallest and the largest eigenvalue of combinatorial laplacian of strongly regular graphs. This yields a factor 3 approximation of the isoperimetric number. The second approach, firstly, finds properties of the metric which is returned by the linear programming formulation of [Linial et. al, The geometry of graphs and some of its algorithmic applications, Combinatorica, vol. 15(2) (1995), pp. 215–245] and secondly, gives an explicit cut which is within factor 2 of the optimal value of the linear program. The spectral algorithm can be generalized to get a factor 3 approximation for a variant of the isoperimetric number for Strongly Regular Graphs.

Key words. Strongly Regular graphs, Laplacian, Linear programming.

AMS subject classifications. 05C50, 15A42, 52B12.

1. Introduction. Given a simple connected graph $G = (V, E)$, define for each cut (i.e $S \subset V$),

$$\text{sparsity(S)} = \frac{|\delta(S)|}{|S|\,|V - S|},$$

where $\delta(S)$ is the set of edges which go across the cut S. We want to find a cut whose sparsity is minimum. This minimum value is known as the isoperimetric number of the graph G and is denoted as $\phi(G)$. For general graphs on n vertices, using low distortion embeddings of finite metrics into $\mathbb{R}^m$ equipped with the ℓ_1 norm; Linial, London and Rabinovich [5] give an approximation algorithm which achieves a performance ratio of $O(\log n)$. For series-parallel graphs, outerplanar graphs and some more families; Gupta, Newman, Rabinovich and Sinclair [4] give a constant factor approximation algorithm. We give a factor 3 and a factor 2 approximation algorithm for Strongly Regular Graphs.

2. Preliminaries. In this section, we outline the definition and the properties of the combinatorial laplacian $L(G)$ of a graph G (which need not be Strongly Regular). We then look at the known inequalities relating the isoperimetric number $\phi(G)$, to the spectrum of the combinatorial laplacian $L(G)$ for any graph G. Finally, we give an overview of Strongly Regular Graphs and some of their properties.

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2.1. Combinatorial Laplacian.

Definition 2.1. Let A be the adjacency matrix of any graph G. Let D be the diagonal matrix whose diagonal entries are the degree's of the vertices of G. The combinatorial laplacian L is defined to be the matrix $D - A$.

The combinatorial laplacian (henceforth called the laplacian) has some fascinating properties, many of which enable one to reason about the graph G. Below, we list some facts regarding the laplacian L and the adjacency matrix A, which we use in this work. Proofs for these can be found in the text by Biggs [2].

Fact 1. *For any graph G, $L(G)$ is real and symmetric and thus has real eigenvalues with an orthonormal eigenvector basis. Being real numbers, the eigenvalues are naturally ordered and so we can use terms like 'smallest eigenvalue'.*

Fact 2. *Zero is an eigenvalue of the laplacian of any graph. Since the laplacian is positive semi definite, this means that it is the smallest eigenvalue. The second smallest eigenvalue $\lambda_2(L(G))$ is zero iff G is disconnected. Since we assume that our input graph G is connected, $\lambda_2(L(G)) > 0$.*

Fact 3. *The absolute value of any eigenvalue of the adjacency matrix of a graph G is at most the maximum degree. Thus for a k-regular graph, all the eigenvalues of the adjacency matrix have absolute value at most k.*

Fact 4. *Let $G = (V, E)$ be a graph with $|V| = n$. Let x be an n dimensional column vector. We can think of x as assigning a value to each $u \in V$. Let x^T be the transpose of x. Then $x^T Lx = \Sigma_{uv\in E}(x_u - x_v)^2$. (This establishes the positive semi definiteness of the laplacian.)*

For k-regular graphs on n vertices, by definition, the laplacian $L = k \times I - A$, where I is the $n \times n$ identity matrix. Thus, if the spectrum of the adjacency matrix is $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$, then $\lambda_n = k$ and the spectrum of the laplacian is $0 = (k - \lambda_n) \leq (k - \lambda_{n-1}) \leq \cdots \leq (k - \lambda_1)$.

2.2. Spectral bounds on $\phi(G)$ and a lemma. The laplacian spectrum gives us bounds on the isoperimetric number of any graph. It is well known (see the survey by Mohar and Poljak [6]) that the isoperimetric number is sandwiched by the inequalities

$$\frac{\lambda_2}{n} \leq \phi \leq \frac{\lambda_{\max}}{n}, \tag{2.1}$$

where λ_2 and $\lambda_{\max}$ are the second smallest and largest eigenvalues respectively, of the laplacian L of G. Thus, if we can output a cut whose sparsity is $O(\frac{\lambda_{\max}}{n})$ and further prove that $\lambda_{\max} \leq c \times \lambda_2$, then we have factor $O(c)$ approximation algorithm for the isoperimetric number. This is what will be done.

We first give an algorithm which outputs a cut whose sparsity is at most $\lambda_{\max}/n$. For this, we need a definition of an *equi-cut* and a claim about the sparsity of *equi-cuts*.

Definition 2.2. Given a graph $G = (V, E)$ on n vertices, an *equi-cut* is a cut with partitions of size $\lfloor \frac{n}{2} \rfloor$ and $\lceil \frac{n}{2} \rceil$.

Claim 1. *For any graph G, any equi-cut S, has sparsity at most $\frac{\lambda_{\max}}{n}$, where $\lambda_{\max}$ is the largest eigenvalue of the laplacian L of G.*

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Proof. If we let $x \in \mathbb{R}^n$ be a vector with entries ± 1, setting $x_v = 1$ iff $v \in S$, then, it is easy to verify that $\Sigma_{ij \in E}(x_i - x_j)^2 = 4 * |\delta(S)|$, and $\Sigma_{i=1}^n \Sigma_{j=1}^n (x_i - x_j)^2 = 8 * |S| * |V - S|$. Therefore,

$$\text{sparsity}(S) = \frac{2\Sigma_{ij \in E}(x_i - x_j)^2}{\Sigma_{i=1}^n \Sigma_{j=1}^n (x_i - x_j)^2}$$

Using the identity $\frac{\Sigma_{i=1}^n \Sigma_{j=1}^n (x_i - x_j)^2}{2} = n(\Sigma_{i=1}^n x_i^2) - (\Sigma_{i=1}^n x_i)^2$, we rewrite

$$\text{sparsity}(S) = \frac{\Sigma_{ij \in E}(x_i - x_j)^2}{n(\Sigma_{i=1}^n x_i^2) - (\Sigma_{i=1}^n x_i)^2}$$

For a ± 1 characteristic vector x of an equi-cut S, $\Sigma_{i=1}^n x_i \equiv n \pmod 2$ and $x^T x = n$. By using Fact 4, the above becomes

$$\text{sparsity}(S) = \frac{x^T L x}{n^2 - \epsilon}$$

where $\epsilon \equiv n \pmod 2$.

By the Courant-Fisher theorem, $\lambda_{\max}(L) = \max_{x \in \mathbb{R}^n} \frac{x^T L x}{x^T x}$. As $x^T x = n$, the above statement is equivalent to $x^T L x / n \leq \lambda_{\max}$. So, we get $x^T L x \leq n \lambda_{\max}$. Hence, the sparsity of an equi-cut is at most $\frac{n\lambda_{\max}}{n^2 - \epsilon}$. If n is even, then $\epsilon = 0$ and we are done, if n is odd, then we get a cut whose sparsity is at most $\frac{\lambda_{\max}}{n - 1/n}$, and for large n, the term $\frac{1}{n}$ in the denominator will be insignificant. $\square$

Lemma 2.3. *There exists an algorithm which outputs a cut whose sparsity is within a factor* $\frac{\lambda_{\max}}{\lambda_2}$ *of* ϕ.

Proof. Output an equi-cut. Its sparsity by Claim 1 is at most $\lambda_{\max}/n$. By equation 2.1, $\phi \geq \lambda_2/n$. Thus, the algorithm outputs a cut whose sparsity is within a factor $\lambda_{\max}/\lambda_2$ of ϕ. $\square$

2.3. Strongly Regular Graphs.

Definition 2.4. A *strongly regular graph* (henceforth SRG) G, with parameters (n, k, λ, μ) is a graph on n vertices such that

- each vertex has degree k
- each pair of adjacent vertices in G have exactly λ common neighbours and
- each pair of non-adjacent vertices have exactly μ common neighbours.

(This is the standard notation and we will ensure that the λ in the $G(n, k, \lambda, \mu)$ does not cause confusion with the eigenvalues of the combinatorial laplacian.) We will use the following notation : for any graph G, its adjacency matrix will be denoted by $A(G)$ and its combinatorial laplacian by $L(G)$. If the graph is clear from the context, we simply write these as A and L respectively.

There is a nice theory of strongly regular graphs to read about which, we refer the reader to the books by van Lint and Wilson [9] or Cameron and van Lint [3]. We list some basic facts about SRG's and point the reader to literature where proofs for these can be found.

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Let G be a connected SRG.

Fact 5. (Cameron and vanLint [3, p. 37]) *The adjacency matrix A of $G(n, k, \lambda, \mu)$ has three distinct eigenvalues $s < r < k$. Since the trace of a simple graph is zero and since the trace equals the sum of the eigenvalues, $s < 0$. Let the multiplicities of s and r be 'g' and 'f' respectively. Since G is connected, k occurs as an eigenvalue with multiplicity exactly* 1. *Thus $f + g + 1 = n$ and $fr + gs + k = 0$.*

Fact 6. (Cameron and vanLint [3, p. 37])

$$r + s = \lambda - \mu \quad \textit{and} \quad rs = \mu - k.$$

Fact 7. (Cameron and vanLint [3, p. 38]) *There are two types of* SRG*'s depending on whether $f = g$ or not. The former is called* Type-I *and the latter* Type-II. *For Type-II* SRG*'s, the numbers r and s are integers with opposite sign.*

Fact 8. (Cameron and vanLint [3, p. 33])

$$k(k - \lambda - 1) = \mu(n - k - 1).$$

Fact 9. (Neumaier's Claw Bound [7]) *Let G be an* $\mathrm{SRG}(n, k, \lambda, \mu)$ *and let the eigenvalues of $A(G)$ be $k > r > s$. Then, at least one of the following holds.*

1. $r \le \max\left\{2(-s-1)(\mu+1+s), \frac{s(s+1)(\mu+1)}{2} - s - 1\right\}$.
2. $\mu = s^2$, *in which case, G is a* Steiner Graph.
3. $\mu = s(s+1)$ *in which case, G is a* Latin Square Graph.

3. First Approach. We provide some intuition for choosing SRG's. Since the adjacency matrix (and consequently the laplacian matrix) of SRG's have three distinct eigenvalues and since the smallest eigenvalue of the laplacian occurs with multiplicity one for connected graphs, at least one of the non zero eigenvalue occurs with high multiplicity. Let us order the eigenvalues of $L(G)$ for an n vertex graph G in non decreasing order, i.e. $0 = \lambda_1 < \lambda_2 \le \cdots \lambda_n$. Thus, though we find the ratio $\frac{\lambda_{\max}}{\lambda_2}$; because of multiplicities, we actually find the ratio of $\frac{\lambda_{r+1}}{\lambda_r}$ for some $r < n$. We prove a bound on this ratio for SRG's.

Theorem 3.1. *Let G be an* SRG *on n vertices. Let λ_2 and $\lambda_{\max}$ be the second smallest and largest eigenvalue of the laplacian L. Then, for large n, $\lambda_{\max} \le 3 \times \lambda_2$*

Proof. The general strategy for the proof is as follows : let $k > r > s$ be the eigenvalues of the adjacency matrix A. Therefore, the eigenvalues of the laplacian, L are $0 < k - r < k - s$. Therefore $\lambda_{\max}(L) = k - s$ and $\lambda_2(L) = k - r$. Thus

$$\frac{\lambda_{\max}}{\lambda_2} = \frac{k-s}{k-r} = 1 + \frac{r-s}{k-r}.$$

We will prove that $\frac{r-s}{k-r} \le 2$.

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We give a proof for SRG's of each type separately. Let G be an SRG of Type-I. Any such G has $n = 4\mu + 1$, $k = 2\mu$, $\lambda = \mu - 1$ [3]. From Fact 6, we find that

$$r = \frac{1}{2}[\sqrt{n} - 1], s = \frac{1}{2}[-\sqrt{n} - 1] \text{ and } k = \frac{n-1}{2}.$$

Thus,

$$\frac{r-s}{k-r} = 1 + \frac{2\sqrt{n}}{n - 2\sqrt{n}} \leq 2$$

for $n \geq 16$.

Thus, we need to consider only SRG's of Type-II. By Fact 7, we can infer that the smallest eigenvalue of A, i.e. s will be an integer. Moreover, by Fact 5, s is negative. We will show that for each of the three possibilities in Neumaier's Theorem,

$$\frac{r-s}{k-r} \leq 2.$$

Case 1: We know $r \leq 2(-s-1)(\mu + 1 + s)$. Since $r \geq 0$ (from Fact 7), we infer that either

$[-s-1 \geq 0$ **AND** $\mu+s+1 \geq 0]$ (**OR**) $[-s-1 < 0$ **AND** $\mu+s+1 < 0]$

In the latter case, we get $s > -1$. We know that $s < 0$ and that s is an integer (Fact 7). Thus, this latter case never happens. Hence, $s \leq -1$ AND $\mu \geq -1-s$. Fact 6 states $rs = \mu - k$ which means

$$rs \geq -s - 1 - k. \tag{3.1}$$

To show that $\frac{r-s}{k-r} \leq 2$ we will show that $r \leq \frac{2k+s}{3}$ or equivalently,

$$rs \geq \frac{s(2k+s)}{3}. \tag{3.2}$$

We know inequality (3.1) and want to prove inequality (3.2). If we show $3(-1-s-k) \geq s(2k+s)$, then we are done. Rearranging, we need to show that $(s+1)^2 + (s+2) + k(2s+3) \leq 0$ for negative integers s satisfying $-k \leq s \leq -1$ (we are in the case when $s \leq -1$, and by Fact 3, $|s| \leq k$ which means $-k \leq -s$).

We distinguish two cases here. The first being when $s = -1$ and the second when $-k \leq s \leq -2$. We will rule out the possibility that $s = -1$ and it can be verified that $(s+1)^2 + (s+2) + k(2s+3) \leq 0$ for $k \leq s \leq -2$.

We now rule out the possibility that $s = -1$. We claim this because we will infer below that the only connected graphs with $s = -1$ are the complete graphs and we can handle them separately (all complete graphs K_z have $\phi(K_z) = 1$).

If $s = -1$, then Fact 6 tells us that $r = \lambda - \mu + 1 = k - \mu$, or that $k = \lambda + 1$. This when applied with Fact 8 tells us that either $\mu = 0$ or $k = n - 1$. If $k = n - 1$, then the input graph is a complete graph. So, let $\mu = 0$. Fact 6 means $r = k$. Fact

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2 implies this can happen only when the graph is disconnected. Disconnected graphs contradict the assumption that the input graph is connected.

REMARK 3.2. *In the rest of the proof, we will assume that $-k \leq s \leq -2$, as the argument ruling out $s = -1$ did not use any property specific to* **Case 1**.

Case 2: The SRG satisfies $\mu = s^2$ and is a STEINER GRAPH. This means $s = -\sqrt{\mu}$. Now, $\mu \geq 4$ as by Remark 3.2, s is a negative integer satisfying $-k \leq s \leq -2$. By Fact 6, we know

$$r = \frac{k-\mu}{\sqrt{\mu}} \Rightarrow r - s = \frac{k}{\sqrt{\mu}},$$

$$k - r = \frac{\sqrt{\mu}k - k + \mu}{\sqrt{\mu}} \Rightarrow \frac{r-s}{k-r} = \frac{k}{\sqrt{\mu}k - k + \mu} \leq \frac{k}{\sqrt{\mu}k - k} = \frac{1}{\sqrt{\mu}-1}.$$

Therefore the approximation ratio is $\leq 1 + \frac{1}{\sqrt{\mu}-1} \leq 2$

Case 3: The SRG is a LATIN SQUARE GRAPH. From the text by van Lint and Wilson [9, p. 414], these graphs have for some m and $2 \leq z$, have parameters $n = m^2$, $k = z(m-1)$, $\lambda = m - 2 + (z-1)(z-2)$ and $\mu = z(z-1)$. Note that the requirement $2 \leq z$ arises as otherwise $\mu = 0$ and the only graphs with $\mu = 0$ are complete graphs on $k+1$ vertices. By Fact 6, $s = -z$, $r = m - z$ and $k = z(m-1)$. (We know $r + s$ and rs and need their individual values. Its easily seen that either $r = m - z, s = -z$ or $r = -z, s = m - z$. Since $r > s$ and z is positive, we infer that $r = m - z, s = -z$.) Thus,

$$\frac{r-s}{k-r} = \frac{m}{mz - m} = \frac{1}{z-1}.$$

The approximation ratio is thus at most $1 + \frac{1}{z-1} \leq 2$. ◻

We get the following theorem from Theorem 3.1 and Claim 1.

THEOREM 3.3. *For Strongly Regular Graphs, the sparsity of any equi-cut is at most a factor* 3 *bigger than the isoperimetric number.*

4. The second approach. In this approach, we look at the linear programming formulation of Linial et al [5]. If the input is an n vertex graph, the linear program (henceforth referred to as LP) returns a metric on n points. The optimal value of the LP is not more than the isoperimetric number. This metric is then embedded into $\mathbb{R}^n_1$ (i.e. $\mathbb{R}^n$ equipped with the ℓ_1 norm) with a factor c distortion. Then by theorems in the book by Vazirani [8, Theorems 21.7, 21.12] we will get a cut whose sparsity is within a factor c from the isoperimetric number.

If the input is an SRG, we prove that the metric which the LP returns has some property and that we can easily exhibit a cut whose sparsity is within a factor 2 of the optimal objective value function without getting into low distortion embeddings.

We say that a finite metric on n points $M = (X, d)$ is *supported* on an n vertex (connected) graph $G = (V, E)$ if there is an assignment of weights $w : E \to \mathbb{R}^+ \cup \{0\}$ such that for all pairs of vertices $u, v \in V(G)$, the weight of the minimum weight path (with each edge e having weight $w(e)$) between u and v is $d(u, v)$. It is easy to see

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that given any finite metric there is a connected weighted graph (the complete graph with weights) with this property and vice versa.

The linear program of Linial et al [5] which models the sparsest cut is given below. Assume that the input is a connected SRG $G = (V, E)$. Also assume that $V = [n]$. The linear program has a variable x_{ij} for each $1 \le i < j \le n$ (i.e. each edge and non edge). The LP is:

$$
\begin{array}{lllr}
\textbf{Minimise} & \sum_{ij\in E} x_{ij} & & \\
\text{subject to} & \sum_{i<j} x_{ij} & = 1 & \\
 & x_{ij} + x_{jk} - x_{ik} & \ge 0 & \\
 & x_{ij} - x_{jk} + x_{ik} & \ge 0 & \forall\, i<j<k \in V \\
 & -x_{ij} + x_{jk} + x_{ik} & \ge 0 & \\
 & x_{ij} & \ge 0 & \forall\, i<j
\end{array}
$$

We note that the normalisation $\Sigma_{i<j} x_{ij} = 1$ is to make the objective function linear. Sparsity of cuts can also be modeled as a linear program with a fractional objective function, (minimise $\frac{\Sigma_{ij\in E} x_{ij}}{\Sigma_{i<j} x_{ij}}$) where the constraints are identical, except that we drop the normalisation.

There is a correspondence between cuts in the input graph and metrics which are feasible for the LP. Given a cut (i.e given an $S \subset V$), the cut semi-metric assigns 0/1 weights as follows : it assigns $x_{uv} = 1$ on pairs uv iff $|\{u, v\} \cap S| = 1$. This semi-metric is feasible for the above LP and the objective value function for this semi-metric is precisely sparsity(S). To see this, use the non-normalised version of the LP mentioned above. Thus any cut semi-metric and in particular the semi-metric corresponding to the sparsest cut is feasible for the LP and hence the optimal value of the LP is not more than the isoperimetric number.

On the other hand, given a feasible metric, the algorithm of Linial et al [5] embeds it into $\mathbb{R}_1^m$ with distortion $O(\log n)$ and retrieves a cut which is a factor $O(\log n)$ bigger than the optimal value of the LP.

It is easy to see that the metric which the LP returns for a graph G is supported on G itself. The *uniform metric* is defined as the metric supported on G where each edge has the same weight. (Note that for the LP above, the value of the objective function does not depend upon the value of this "uniform" weight because of the normalisation.)

THEOREM 4.1. *For* SRG*'s the uniform metric achieves the optimum value for the LP given above.*

Proof. We call the uniform metric x_u and we will construct a solution dd to the dual of this LP which with x_u will satisfy both primal and dual complimentary slackness conditions. We set up some notation to describe the dual LP.

Let us state the dual variable corresponding to a primal constraint. The dual variable ϕ corresponds to the primal constraint $\sum_{i<j} x_{ij} = 1$. In addition, there are three primal constraints for each triple of distinct vertices. All the dual variables corresponding to a triple ijk are subscripted by ijk. To distinguish among the three constraints, since there is only one negative pair in each constraint, we add that pair on the superscript. Thus, the dual variables for the constraints stated in LP (from top to bottom) are $d_{ijk}^{ik}, d_{ijk}^{jk}$ and d_{ijk}^{ij} respectively.

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Let S_{ij} be the set of all 3 subsets of V which contain both i and j. ($|S_{ij}| = n-2$.) Let p_{ij} be the indicator as to whether the edge ij is present in E or not. i.e $p_{ij} = 1$ if $ij \in E(G)$ and $p_{ij} = 0$ if $ij \notin E(G)$. The dual of LP is:

Maximise ϕ

subject to $\sum_{k \in S_{ij}} d_{ijk}^{ik} + d_{ijk}^{jk} - d_{ijk}^{ij} + \phi \leq p_{ij} \quad \forall\, i < j$

$d_{ijk}^{ij} \geq 0 \quad \forall\, i, j, k \in V.$

Let G_{ijk} be the induced subgraph of G on the three vertices i, j, k. A primal triangle inequality corresponding to the dual variable d_{ijk}^{ij} will be satisfied with equality (we call such an inequality tight) iff G_{ijk} is a path of length 2 with ij being the non edge (i.e if ij is a non edge, then x_{ij} will be 2 when $x_{ik} = x_{jk} = 1$). We set the dual variable d_{ijk}^{ij} corresponding to such a tight inequality to a non zero value $= d$ shown below while the remaining dual variables d_{ijk}^{jk} and d_{ijk}^{ik} for that triple are set to zero. (This is because for any triple ijk, at most one inequality can become tight.) We set the value for ϕ also to be non zero as below. Thus, there are three values set to the dual variables.

We set

$$d_{ijk}^{ij} = \begin{cases} d & \text{if the corresponding primal inequality is tight} \\ 0 & \text{otherwise,} \end{cases}$$

where $d = \dfrac{1}{\mu + 2(k - \lambda - 1)}$. We set $\phi = \mu \times d = \dfrac{\mu}{\mu + 2(k - \lambda - 1)}$.

We claim that these assignments are firstly a feasible solution to the dual LP and that these along with x_u satisfy all complimentary slackness conditions. To check for feasibility, it is easy to note that $d \geq 0$. Further, in the dual LP after setting the necessary d_{ijk}^{ij} to zero, each constraint has the form $\phi + d \times 2(k - \lambda - 1) \leq 1$ if $ij \in E$ or $\phi - d \times \mu \leq 0$ if $ij \notin E$

This is because each edge ij of G will have exactly $2(k - \lambda - 1)$ vertices which are adjacent to exactly one of i or j (because there are exactly λ vertices common to both i and j and the degree of each vertex is k) and each non-edge ij of G will have exactly μ vertices which are adjacent to both i and j. With these assignments to the variables, it is easy to verify that the primal and dual complimentary slackness conditions are satisfied.

The dual solution dd thus constructed has objective function value (the quantity maximised) $\phi = \frac{\mu}{\mu + 2(k - \lambda - 1)}$. We need to check that the primal objective function (the function minimised) on the uniform metric also achieves the same value. Since any SRG has diameter two (because it has three distinct eigenvalues), if we place a unit weight on the edges of G, the non edges will get a weight of 2. The ratio

$$\frac{\Sigma_{ij \in E} x_{ij}}{\Sigma i < j x_{ij}} = \frac{nk/2}{nk/2 + 2(\binom{n}{2} - nk/2)} = \frac{k}{k + 2(n - k - 1)} = \phi.$$

The last equation follows by using Fact 8. □

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For an SRG(n, k, λ, μ), we exhibit a cut whose sparsity is within $2\times$ (isoperimetric number). Take any single vertex on one side of the cut and the remaining $(n-1)$ vertices on the other side. Since the graph is regular of degree k, the sparsity of this cut is $\frac{k}{n-1}$. Since the optimal value of the LP is $\frac{k}{k+2(n-k-1)}$, we see that the singleton cut has performance ratio $\leq \frac{2(n-1-k/2)}{n-1} < 2$ if $k \geq 2$.

5. A variant of the Isoperimetric Number. In literature, there is another related quantity which is called the *flux* of a graph. Let $G = (V, E)$ be a graph. Define for each cut $S \subset V$,

$$\text{sparseness}(S) = \frac{|\delta(S)|}{\min(|S|, |V-S|)}.$$

Our task is to find a cut whose sparseness is minimum. The minimum sparseness of a graph G is called the *flux* of G, and is denoted by $i(G)$. This variant is also NP-hard to compute exactly (see the survey by Mohar and Poljak [6]).

Some spectral bounds are known for the flux of a graph as well. We quote inequality 21 of Mohar and Poljak [6, p. 14].

Lemma 5.1. *Let G be a graph and let λ_2 be the second smallest eigenvalue of $L(G)$. The flux $i(G)$ satisfies the inequality $i(G) \geq \lambda_2/2$.*

Proof. Let $S \subseteq V$ with $|S| \leq n/2$ have the minimum sparseness. Thus, $i(G) = |\delta(S)|/|S|$. Now,

$$\frac{i(G)}{|V-S|} = \frac{|\delta(S)|}{|S| * |V-S|} \geq \phi(G)$$

because ϕ is the minimum sparsity among cuts. Recall the lower bound in equation 2.1, $\phi \geq \frac{\lambda_2}{n}$. Thus, $i(G) \geq \frac{\lambda_2 * |V-S|}{n}$. The proof is completed by noting that $|V-S| \geq n/2$. ∎

For SRG's, this number is also approximable to within a factor of 3 using the spectral approach outlined earlier. To see this, note that since the sparsity of an equi-cut is at most $\lambda_{\max}/n$, its sparseness is at most $\lambda_{\max}/2$. Formally, for an equi-cut S, we know

$$\frac{|\delta(S)|}{|S| * |V-S|} \leq \frac{\lambda_{\max}}{n}.$$

Since each S and $V-S$ have size $n/2$, the sparseness of an equi-cut is at most $\frac{\lambda_{\max}}{2}$. Thus, as before, by outputting an equi-cut, we can approximate its flux to within a factor of 3.

Corollary 5.2. (of Theorem 3.1) *Let G be a Strongly Regular Graph. The sparseness of an equi-cut is within a factor 3 of the flux of G.*

6. Discussion and Open Problems. We have two approaches to approximate the isoperimetric number and one for the flux of a graph. These approaches give a constant factor performance for SRG's. The running time of both the algorithms is linear (outputting an equi-cut or a singleton-cut is all that is required). The spectral

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algorithm presented here clearly does not perform in the worst case as well as the LP based algorithm of Linial et al [5]. (Indeed, this can be expected as the algorithm outputs some *equi-cut* without looking at the graph, and such a naive method can only work on special graphs.) Recall that their algorithm achieves a ratio of $O(\log n)$ for all graphs. The ratio achieved by the spectral algorithm presented in this work is $\lambda_{\max}/\lambda_2$. It is easy to construct graphs for which $\lambda_{\max} > \Omega(n^2) \times \lambda_2$.

(Let n be even and let G be two $K_{n/2}$'s joined by a single edge. To see that for such graphs $\lambda_{\max} = \Omega(n^2)\lambda_2$, we prove that $\lambda_{\max} = \Omega(n)$ and $\lambda_2 \leq \frac{4}{n}$.

To show $\lambda_{\max}$ is large, we use Rayleigh Quotients. By the Courant-Fisher inequality, $\lambda_{\max} = \max_{x \in \mathbb{R}^n} \frac{x^T Lx}{x^T x}$. We will exhibit a vector $x \in \mathbb{R}^n$ with a large Rayleigh's quotient. Let us number the vertices of a $K_{n/2}$ as $\{1, 2, \ldots, n/2\}$ and the second $K_{n/2}$ as $\{n/2+1, n/2+2, \ldots, n\}$. Consider the vector x with only 0/1 components. x has only two ones at x_1 and $x_{n/2+1}$ and the remaining $(n-2)$ components are zeroes. Clearly, $x^T x = 2$. By Fact 4, $x^T Lx = \sum_{ij \in E}(x_i - x_j)^2 = n - 2$. Thus $\lambda_{\max} \geq \frac{x^T Lx}{x^T x} \geq \frac{n-2}{2} = \Omega(n)$.

To show that λ_2 is small, note that $\phi(G) = \frac{4}{n^2}$ ($\frac{4}{n^2}$ is the smallest value of ϕ for an n vertex graph). By inequality 2.1, $\phi \geq \frac{\lambda_2}{n}$ which implies $\lambda_2 \leq \frac{4}{n}$.)

Several questions are open, and we list a few of them.

1. Can the spectral bounds be tightened? Does the isoperimetric number $\phi(G)$ satisfy some equation like
$$\phi(G) \geq \frac{\lambda_{f(G)}}{n},$$
where $\lambda_{f(G)}$ is the $f(G)^{th}$ smallest eigenvalue of the laplacian for some function $f(G)$ of the graph G? (The known lower bound says $f(G) = 2$ for all graphs G.) Is there a similar inequality for the upper bound?

2. (Weighted Version) If in the problem, we are given a weighted SRG, and suppose for $S \subset V$ the definition of sparsity was changed to
$$\text{sparsity}(S) = \frac{\text{weight}(\delta(S))}{|S| * |V - S|},$$
where weight$(\delta(S))$ is the sum of the weights of the edges which go across the cut S, and we now want to find the sparsest cut. For SRG's is the weighted version approximable to within a constant factor?

3. What about approximating the weighted flux of a graph? Can we say something for SRG's?

4. The algorithm of Linial et al [5] outputs a cut whose sparsity is within an $O(\log n)$ factor of the isoperimetric number. They also show that their algorithm, on a constant degree expander on n vertices, does achieve a ratio of $\Omega(\log n)$. For such graphs, the spectral algorithm presented here will get us

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within a constant factor of the optimal answer. Can a best-of-two algorithm improve on the $O(\log n)$ bound?
Recently Arora, Rao and Vazirani [1] have shown an $O(\sqrt{\log n})$ ratio algorithm for the flux of a graph. Their algorithm uses semi-definite programming and they exhibit hypercubes as graphs where a ratio of $\Omega(\sqrt{\log n})$ ratio is achieved.

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REFERENCES

[1] S. Arora, S. Rao, and U. V. Vazirani. Expander flows, geometric embeddings and graph partitioning. In *Proceedings of Annual Symposium on Theory of Computing*, pp. 222–231, 2004.
[2] N. Biggs. *Algebraic Graph Theory.* Cambridge University Press, 1974.
[3] P. J. Cameron and J. H. van Lint. *Designs, Graphs, Codes and their Links.* Cambridge University Press, 1991.
[4] A. Gupta, I. Newman, Y. Rabinovich, and A. Sinclair. Cuts, trees and ℓ_1 embeddings of graphs. In *Proceedings of the 40th Annual IEEE Symposium on Foundations of Computer Science*, pp. 399–408, 1999.
[5] N. Linial, E. London, and Y. Rabinovich. The geometry of graphs and some of its algorithmic applications. *Combinatorica*, 15(2):215–245, 1995.
[6] B. Mohar and S. Poljak. Eigenvalues in combinatorial optimization. In *Combinatorial and graph-theoretical problems in linear algebra*, IMA Volumes in Mathematics and Its Applications, Vol. 50, pp. 107–151. Springer-Verlag, 1993.
[7] A. Neumaier. Strongly regular graphs with least eigenvalue $-m$. *Arch. Math*, 4:392–400, 1979.
[8] V. V. Vazirani. *Approximation Algorithms.* Springer Verlag, 2001.
[9] J. H. van Lint and R. M. Wilson. *A Course in Combinatorics.* Cambridge University Press, 1992.