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NORM ESTIMATES FOR FUNCTIONS OF TWO COMMUTING MATRICES*

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Abstract. Matrix valued analytic functions of two commuting matrices are considered. A precise norm estimate is established. As a particular case, the matrix valued functions of two matrices on tensor products of Euclidean spaces are explored.

Key words. Functions of commuting matrices, Norm estimate.

AMS subject classifications. 15A15, 15A54, 15A69, 15A60.

1. Introduction and statement of the main result. In the book [4], I.M. Gel'fand and G.E. Shilov have established an estimate for the norm of a regular matrix-valued function in connection with their investigations of partial differential equations. However that estimate is not sharp, it is not attained for any matrix. The problem of obtaining a precise estimate for the norm of a matrix-valued function has been repeatedly discussed in the literature. In the paper [5] (see also [7]) the author has derived a precise estimate for a regular matrix-valued function. It is attained in the case of normal matrices. In the present paper we generalize the main result of the paper [5] to functions of two commuting matrices. Besides, the main result of the present paper-Theorem 1.1 is improved in the case of matrices on tensor products of Euclidean spaces.

It should be noted that functions of commuting operators were investigated by many mathematicians, cf. [1, 10, 12] and references therein however the norm estimates were not considered, but as it is well-known, matrix valued functions are Green's functions and characteristic functions of various differential and difference equations. This fact allow us to investigate stability, well-posedness and perturbations of these equations by norm estimates for matrix valued functions, cf. [2, 3, 6].

Let $\mathbb{C}^n$ be a Euclidean space with a scalar product $(\cdot,\cdot)$, the unit matrix I and the Euclidean norm $\|\cdot\| = (\cdot,\cdot)^{1/2}$; $M(\mathbb{C}^n)$ denotes the set of all linear operators in $\mathbb{C}^n$. For a $A \in M(\mathbb{C}^n)$, $\|A\|$ is the operator norm; $N(A)$ is the Frobenius (Hilbert-Schmidt) norm: $N^2(A) = Trace\ (AA^*)$; A^* is the operator adjoint to A, $\lambda_j(A),\ j = 1,\dots,n$ are the eigenvalues counting with their multiplicities, $\sigma(A)$ is the spectrum.

Everywhere below A and B are commuting matrices. Let Ω_A and Ω_B be open simple connected sets containing $\sigma(A)$ and $\sigma(B)$, respectively. Let f be a scalar function analytic on $\Omega_A \times \Omega_B$. Introduce the operator valued function

$$f(A,B) := -\frac{1}{4\pi^2}\int_{L_B}\int_{L_A} f(z,w)R_z(A)R_w(B)dw\,dz, \tag{1.1}$$

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where $L_A \subset \Omega_A, L_B \subset \Omega_B$ are closed contour surrounding $\sigma(A)$ and $\sigma(B)$, respectively. Note that if the series

$$f(z,w) = \sum_{j,k=0}^{\infty} c_{jk} z^j w^k$$

converges for $|z| \leq r_s(A), |w| \leq r_s(w)$, where $r_s(A)$ denotes the spectral radius of A, then (1.1) holds.

The following quantity plays a key role in the sequel

$$g(A) = (N^2(A) - \sum_{k=1}^{n} |\lambda_k(A)|^2)^{1/2}.$$

Since

$$\sum_{k=1}^{n} |\lambda_k(A)|^2 \geq |Trace\ A^2|, \text{ we have } g^2(A) \leq N^2(A) - |Trace\ A^2|.$$

If A is a normal matrix, i.e. if $AA^* = A^*A$, then $g(A) = 0$. Also the inequality

$$g^2(A) \leq \frac{1}{2} N^2(A^* - A) \tag{1.2}$$

is valid, cf. [7, Section 2.1]. Introduce the numbers

$$\eta_k := \frac{1}{k!} \sqrt{\frac{(n-1)!}{(n-k-1)!(n-1)^k}} \text{ for } k = 1, ..., n-1;\ \eta_0 = 1.$$

It is simple to check that

$$\eta_k^2 \leq \frac{1}{(k!)^3} \ \ (k = 1, ..., n-1). \tag{1.3}$$

Denote by $co(A), co(B)$ the closed convex hulls of $\sigma(A)$ and $\sigma(B)$, respectively. Put

$$f^{(j,k)}(z,w) = \frac{\partial^{j+k} f(z,w)}{\partial z^j \partial w^k}.$$

Now we are in a position to formulate the main result of the paper.

THEOREM 1.1. *Let A and B be commuting $n \times n$-matrices and $f(z,w)$ be regular on a neighborhood of $co(A) \times co(B)$. Then*

$$\|f(A,B)\| \leq \sum_{j,k=0}^{j+k \leq n-1} \eta_j \eta_k g^j(A) g^k(B) \sup_{z \in co(A), w \in co(B)} |f^{(j,k)}(z,w)|.$$

The proof of this theorem is divided into a series of lemmas which are presented in the next section. If both A and B are normal operators, and

$$\sup_{z \in co(A), w \in co(B)} |f(z,w)| = \sup_{z \in \sigma(A), w \in \sigma(B)} |f(z,w)|,$$

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then Theorem 1.1 gives us the exact relation

$$\|f(A,B)\| = \sup_{z\in\sigma(A),w\in\sigma(B)} |f(z,w)|.$$

Taking into account (1.3), we get

COROLLARY 1.2. *Under the hypothesis of* Theorem 1.1, *the estimate*

$$\|f(A,B)\| \le \sum_{j,k=0}^{j+k\le n-1} \frac{g^j(A)g^k(B)}{(j!k!)^{3/2}} \sup_{z\in co(A),w\in co(B)} |f^{(j,k)}(z,w)|$$

is true.

Let A be a normal matrix. Then $g(A)=0$. Now Corollary 1.2 implies

$$\|f(A,B)\| \le \sum_{0\le k\le n-1} \frac{g^k(B)}{(k!)^{3/2}} \sup_{z\in co(A),w\in co(B)} |\frac{\partial^k f(z,w)}{\partial w^k}|.$$

Let us evaluate the error of Theorem 1.1 in the case of non-normal matrices. Certainly, we can obtain the exact value of the norm of a function of two matrices in very simple cases only. Consider the 2×2-matrices

$$A = \begin{pmatrix} a & 1/3 \\ 0 & a \end{pmatrix}.$$

and $B=2A$ with $a>0$. Construct a function of two variables by setting $(x,y)\to f(x+y)$. Direct calculations show that

$$f(A+B) = \begin{pmatrix} f(3a) & f'(3a) \\ 0 & f(3a) \end{pmatrix}.$$

Assume that $f(3a)=0$. Then $\|f(A+B)\| = |f'(3a)|$. At the same time, Theorem 1.1 gives us the relations,

$$\|f(A+B)\| \le |f(a)| + |f'(3a)|(g(A)+g(B)) = |f'(3a)|,$$

since $g(A)=1/3,\ g(B)=2/3$. Thus in the considered case Theorem 1.1 gives us the exact result.

EXAMPLE 1.3. Consider the polynomial

$$P(z,w) = \sum_{\nu=0}^{m_1}\sum_{l=0}^{m_2} c_{\nu l} z^\nu w^l$$

with complex, in general, coefficients. Then

$$P^{(j,k)}(z,w) = \sum_{\nu=0}^{m_1-j}\sum_{l=0}^{m_2-k} c_{\nu l}\nu(\nu-1)...(\nu-j+1)z^{\nu-j} l...(l-k+1)w^{l-k}.$$

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Now Theorem 1.1 implies

$$\|P(A,B)\| \le \sum_{j,k=0}^{j+k\le n-1} \eta_j\eta_k g^j(A)g^k(B) \sum_{\nu=0}^{m_1-j}\sum_{l=0}^{m_2-k} \frac{\nu!l!|c_{\nu l}|}{(l-k)!(\nu-j)!} r_s^{\nu-j}(A) r_s^{l-k}(B).$$

Recall that $r_s(.,)$ denotes the spectral radius. If both A and B are normal operators, then

$$\|P(A,B)\| \le \sum_{\nu=0}^{m_1}\sum_{l=0}^{m_2} |c_{\nu l}| r_s^{\nu}(A) r_s^{l}(B).$$

EXAMPLE 1.4. Consider the function

$$f(z,w) = cos\ (xz+yw) \ \ (y,x\ge 0).$$

Note that the function $U(x,y) = cos(xA+yB)$ is a solution of the equation

$$\partial^2 U(x,y)/\partial x^2 + \partial^2 U(x,y)/\partial y^2 + (A^2+B^2)U(x,y) = 0.$$

In the considered case

$$|f^{(j,k)}(z,w)| = x^j y^k |cos\ (xz+yw)| \ \ (j+k \text{ is even }),$$

$$|f^{(j,k)}(z,w)| = x^j y^k |sin\ (xz+yw)| \ \ (j+k \text{ is odd }).$$

For simplicity assume that the spectra of both A and B are real. Then thanks to Theorem 1.1,

$$\|U(x,y)\| \le \sum_{j,k=0}^{j+k\le n-1} \eta_j\eta_k g^j(A) g^k(B) x^j y^k \ \ (x,y\ge 0).$$

2. Proof of Theorem 1.1.

The following lemma is needed.

LEMMA 2.1. *Let Ω and $\tilde\Omega$ be the closed convex hulls of complex, in general, points*

$$x_0, x_1, ..., x_n \tag{2.1}$$

and

$$y_0, y_1, \ldots, y_m, \tag{2.2}$$

respectively, and let a scalar-valued function $f(z,w)$ be regular on $D\times\tilde D$, where D and $\tilde D$ are neighborhoods of Ω and $\tilde\Omega$, respectively. In addition, let $L\subset D, \tilde L\subset\tilde D$ be Jordan closed contours surrounding the points in (2.1) *and* (2.2), *respectively. Then with the notation*

$$Y(x_0,...,x_n;\ y_0,...y_m) = -\frac{1}{4\pi^2}\int_L\int_{\tilde L} \frac{f(z,w)dz\,dw}{(z-x_0)\cdots(z-x_n)(w-y_0)\cdots(w-y_m)},$$

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we have

$$|Y(x_0,...,x_n;\ y_0,...y_m)| \leq \frac{1}{n!m!} \sup_{z\in\Omega, w\in\tilde{\Omega}} |f^{(n,m)}(z,w)|.$$

Proof. First, let all the points be distinct: $x_j \neq x_k,\ y_j \neq y_k$ for $j \neq k$. Let a function h of one variable be regular on D and $[x_0,x_1,...,x_n]h$ be a divided difference of function h at points $x_0,x_1,...,x_n$. Then

$$[x_0,x_1,...,x_n]h = \frac{1}{2\pi i}\int_L \frac{h(\lambda)d\lambda}{(\lambda-x_0)\cdots(\lambda-x_n)} \tag{2.3}$$

(see [4, formula (54)]). Thus

$$Y(x_0,...x_n;\ y_0,...,y_m) = \frac{1}{2\pi i}\int_{\tilde{L}} \frac{[x_0,...,x_n]f(.,w)dw}{(w-y_0)\cdots(w-y_n)}.$$

Now apply (2.3) to $[x_0,...x_n]f(.,w)$. Then

$$Y(x_0,...x_n;\ y_0,...,y_m) = [x_0,...x_n]\ [y_0,...y_m]f(.,.) \equiv [x_0,...x_n]\ ([y_0,...y_m]f(.,.)).$$

The following estimate is well-known [11, p. 6]:

$$|[x_0,...x_n]\ [y_0,...y_m]f(.,.)| \leq \sup_{z\in\Omega, w\in\tilde{\Omega}} |f^{(n,m)}(z,w)|.$$

It proves the required result if all the points are distinct. If some points coincide, then the claimed inequality can be obtained by small perturbations and the previous arguments. ∎

Furthermore, since A and B commute they have the same orthogonal normal basis of the triangular representation (Schur's basis) $\{e_k\}$. We can write

$$A = D_A + V_A, B = D_B + V_B, \tag{2.4}$$

where D_A, D_B are the diagonal parts, V_A and V_B are the nilpotent parts of A and B, respectively. Furthermore, let $|V_A|$ be the operator whose matrix elements in $\{e_k\}$ are the absolute values of the matrix elements of the nilpotent part V_A of A with respect to this basis. That is,

$$|V_A| = \sum_{k=1}^{n}\sum_{j=1}^{k-1} |a_{jk}|(.,e_k)e_j,$$

where $a_{jk} = (Ae_k, e_j)$. Similarly $|V_B|$ is defined.

LEMMA 2.2. *Under the hypothesis of* Theorem 1.1 *the estimate*

$$\|f(A,B)\| \leq \sum_{j,k=0}^{j+k\leq n-1} \sup_{z\in co(A), w\in co(B)} |f^{(j,k)}(z,w)| \frac{\|\ |V_A|^j\ |V_B|^k\|}{j!k!}$$

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is true, where V_A and V_B are the nilpotent part of A and B, respectively.

Proof. It is not hard to see that the representation (2.4) implies the equality

$$(A - I\lambda)^{-1} = (D_A + V_A - \lambda I)^{-1} = (I + R_\lambda(D_A)V_A)^{-1} R_\lambda(D_A)$$

for all regular λ. According to Lemma 1.7.1 from [7] $R_\lambda(D_A)V_A$ is a nilpotent operator because V and $R_\lambda(D_A)$ the same invariant subspaces. Hence, $(R_\lambda(D_A)V_A)^n = 0$. Therefore,

$$R_z(A) = \sum_{k=0}^{n-1} (R_z(D_A)V_A)^k (-1)^k R_z(D_A).$$

Similarly,

$$R_\mu(B) = \sum_{k=0}^{n-1} (R_\mu(D_B)V_B)^k (-1)^k R_\mu(D_B).$$

So from (1.1) it follows

$$f(A, B) = \sum_{j,k=0}^{n-1} C_{jk}, \tag{2.5}$$

where

$$C_{jk} = \frac{(-1)^{k+j}}{4\pi^2} \int_{L_B} \int_{L_A} f(z, w)(R_z(D_A)V_A)^j R_z(D_A)(R_w(D_B)V_B)^k R_w(D_B) dz\, dw.$$

Since D_A is a diagonal matrix with respect to the Schur basis $\{e_k\}$ and its diagonal entries are the eigenvalues of A, then

$$R_z(D_A) = \sum_{j=1}^{n} \frac{Q_j}{\lambda_j(A) - z},$$

where $Q_k = (., e_k)e_k$. Similarly,

$$R_z(D_B) = \sum_{j=1}^{n} \frac{Q_j}{\lambda_j(B) - z}.$$

Taking into account that $Q_s V_A Q_m = 0,\ Q_s V_B Q_m = 0\ (s \geq m)$, we get

$$C_{jk} = \sum_{1 \leq s_1 < s_2 < \ldots < s_{j+1} < m_1 < m_2 < \ldots < m_{k+1} < n} Q_{s_1} V_A Q_{s_2} V_A \cdots$$

$$V_A Q_{s_{j+1}} Q_{m_1} V_B Q_{m_2} V_B \ldots V_B Q_{m_{k+1}} J(s_1, \ldots, s_{j+1}, m_1, \ldots, m_{k+1}),$$

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where

$$J(s_1, \ldots, s_{j+1}, m_1, \ldots m_{k+1}) =$$

$$\frac{(-1)^{k+j}}{4\pi^2} \int_{L_A} \int_{L_B} \frac{f(z,w)dz\, dw}{(\lambda_{s_1}(A) - z) \cdots (\lambda_{s_{k+1}}(A) - z)(\lambda_{m_1}(B) - w) \cdots (\lambda_{m_{k+1}}(B) - w)}.$$

Lemma 2.5.1 from [7] gives us the estimate

$$\|C_{jk}\| \le \max_{1 \le s_1 < \ldots < s_{j+1} < m_1 < \ldots < m_{k+1} < n} |J(s_1, \ldots, s_{j+1}, m_1, \ldots m_{k+1})| \| \, |V_A|^j \; |V_B|^k \, \|.$$

Due to Lemma 2.1,

$$|J(s_1, \ldots, s_{j+1}, m_1, \ldots, m_{k+1})| \le \sup_{z \in co(A), w \in co(B)} \frac{|f^{(j,k)}(z,w)|}{j!k!}.$$

This inequality and (2.5) imply the required result. □

Proof of Theorem 1.1: Theorem 2.5.1 from [7] implies

$$\|V^k\| \le \eta_k k! N^k(V) \tag{2.6}$$

for any nilpotent matrix $V \in M(\mathbb{C}^n)$. Take into account that $N(|V_A|) = N(V_A)$. Moreover, thanks to Lemma 2.3.2 from [7], $N(V_A) = g(A)$. Thus

$$\| \, |V_A|^k \, \| \le k! \eta_k g^k(A) \; (k = 1, ..., n-1). \tag{2.7}$$

The similar inequality holds for V_B. Now the previous lemma yields the required result. □

3. Functions of matrices on tensor products. Let $E_1 = \mathbb{C}^{n_1}, E_2 = \mathbb{C}^{n_2}$, be the Euclidean spaces of the dimensions n_1 and n_2, with the scalar products $< .,. >_1$ and $< .,. >_2$, respectively, and the norms $\|.\|_l = \sqrt{< .,. >_l}$ $(l = 1, 2)$. Let $H = E_1 \otimes E_2$ be the tensor product of E_1 and E_2 with the scalar product defined by

$$< y \otimes h, y_1 \otimes h_1 >_H \equiv < y, y_1 >_1 < h, h_1 >_2 \quad (y, y_1 \in E_1; \; h, h_1 \in E_2)$$

and the cross norm $\|.\|_H = \sqrt{< .,. >_H}$, cf. [9]. In addition, $I = I_H$ and I_l mean the unit operators in H and E_l, respectively. So $H = \mathbb{C}^n$ with $n = n_1 n_2$.

Recall that $M(E)$ is the set of all linear operators in a space E. In this section it is assumed that $A \in M(E_1)$ and $B \in M(E_2)$.

Again let Ω_A and Ω_B be open simple connected sets containing $\sigma(A)$ and $\sigma(B)$, respectively. Let f be a scalar function analytic on $\Omega_A \times \Omega_B$. Introduce the operator valued function

$$f(A,B) := -\frac{1}{4\pi^2} \int_{L_B} \int_{L_A} f(z,w) R_z(A) \otimes R_w(B) dw\, dz, \tag{3.1}$$

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where $L_A \subset \Omega_A, L_B \subset \Omega_B$ are closed contour surrounding $\sigma(A)$ and $\sigma(B)$, respectively. If the series

$$f(z,w) = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} c_{jk} z^j w^k$$

converges for $|z| \le r_s(A), |w| \le r_s(B)$, where $r_s(A)$ is the spectral radius of A, then (3.1) holds. Besides,

$$f(z,w) = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} c_{jk} A^j \otimes B^k.$$

THEOREM 3.1. *Let $A \in M(E_1)$ and $B \in M(E_2)$ and $f(z,w)$ be regular on a neighborhood of $co(A) \times co(B)$. Then*

$$\|f(A,B)\|_H \le \sum_{j=0}^{n_1-1}\sum_{k=0}^{n_2-1} \eta_j \eta_k g^j(A) g^k(B) \sup_{z\in co(A), w\in co(B)} |f^{(j,k)}(z,w)|.$$

Proof. Put $\tilde{A} = A \otimes I_2$, $\tilde{B} = I_1 \otimes B_2$. Now Lemma 2.2 implies

$$\|f(A,B)\|_H \le \sum_{j,k=0}^{j+k\le n-1} \sup_{z\in co(A), w\in co(B)} |f^{(j,k)}(z,w)| \frac{\| \, |V_{\tilde{A}}|^j \, |V_{\tilde{B}}|^k \|_H}{j!k!}$$

where $V_{\tilde{A}}, V_{\tilde{B}}$ are the nilpotent parts of $\tilde{A}$ and $\tilde{B}$, respectively. But

$$V_{\tilde{A}}^{n_1} = V_A^{n_1} \otimes I_2 = 0.$$

Similarly, $V_{\tilde{B}}^{n_2} = 0$. Thus,

$$\|f(A,B)\|_H \le \sum_{k=0}^{n_2}\sum_{j=0}^{n_1} \sup_{z\in co(A), w\in co(B)} |f^{(j,k)}(z,w)| \frac{\| \, |V_A|^j \|_1 \, \| \, |V_B|^k \|_2}{j!k!}$$

Now the required result follows from (2.7). ▯

Taking into account (1.3), we get

COROLLARY 3.2. *Under the hypothesis of* Theorem 3.1*, the estimate*

$$\|f(A,B)\|_H \le \sum_{k=0}^{n_2-1}\sum_{j=0}^{n_1-1} \frac{g^j(A) g^k(B)}{(j!k!)^{3/2}} \sup_{z\in co(A), w\in co(B)} |f^{(j,k)}(z,w)|$$

is true.

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