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ANALYTIC ROOTS OF INVERTIBLE MATRIX FUNCTIONS*

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Abstract. Various conditions are developed that guarantee existence of analytic roots of a given analytic matrix function with invertible values defined on a simply connected domain.

Key words. Analytic matrix functions, Roots.

AMS subject classifications. 47A56.

1. Introduction. Let G be a simply connected domain in the complex plain $\mathbb{C}$, or an open interval of the real line. By an analytic, resp., a meromorphic, matrix function in G we mean a matrix whose entries are (scalar) analytic, resp., meromorphic functions in G. Unless specified otherwise, G will denote a fixed simply connected domain, or a fixed real interval.

Let $A(z)$, $z \in G$ be an $n \times n$ matrix function analytic and invertible in G. In this paper we study analytic mth roots of A, that is, matrix functions B which are analytic in G and satisfy the equation $B(z)^m = A(z)$ for all $z \in G$. Here $m \geq 2$ is positive integer. Of course, it is a well known fact from complex analysis that for $n = 1$ there are exactly m analytic mth root functions.

However, in the matrix case not much is known. See, for example, [2]. This is somewhat surprising, especially because the problem is a natural one.

To start with, consider an example.

EXAMPLE 1.1. Let G be a simply connected domain such that $-1, 0 \in G$ and $-\frac{1}{2} \notin G$. Consider the analytic matrix function

$$A(z) = \begin{bmatrix} (z+\frac{1}{2})^2 & 1 \\ 0 & \frac{1}{4} \end{bmatrix}, \quad z \in G.$$

Clearly $A(z)$ is invertible on G. An easy algebraic computation shows that there are only four meromorphic (in G) functions $B(z)$ such that $B(z)^2 = A(z)$, given by the formula

$$B(z) = \begin{bmatrix} \delta_1(z+\frac{1}{2}) & (\delta_1(z+\frac{1}{2}) + \frac{1}{2}\delta_2)^{-1} \\ 0 & \frac{1}{2}\delta_2 \end{bmatrix}, \quad z \in G, \tag{1.1}$$

where δ_1 and δ_2 are independent signs ± 1. None of the four functions $B(z)$ is analytic in G. □

One can use functional calculus to compute square roots of an analytic invertible $n \times n$ matrix function $A(z)$, as follows. Fix $z_0 \in G$. For $z \in G$ sufficiently close to z_0,

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use the formula

$$B(z) = \frac{1}{2\pi i}\int_{\Gamma}(\lambda I - A(z))^{-1}(\lambda)^{1/m}d\lambda, \tag{1.2}$$

where Γ is a simple closed rectifiable contour such that all eigenvalues of $A(z_0)$ are inside Γ, the origin is outside of Γ, and $(\lambda)^{1/m}$ is an analytic branch of the mth root function. Clearly, $B(z)^m = A(z)$. Formula (1.2) shows that locally, i.e., in a neighborhood of every given $z_0 \in G$, an analytic mth root of $A(z)$ always exists.

In the next section we treat the case of 2×2 analytic matrix functions and their analytic roots, using a direct approach. The general case of $n \times n$ matrix functions requires some preliminary results which are presented in Sections 3 and 4. Our main results are stated and proved in Section 5. Finally, in the last Section 6 we collect several corollaries concerning analytic roots of analytic matrix functions.

2. 2×2 matrix functions. We start with triangular 2×2 matrix functions. We say that the zeroes of a (scalar) analytic function $a(z)$, $z \in G$, are *majorized* by the zeroes of an analytic function $b(z)$, $z \in G$, if every zero z_0 of $a(z)$ is also a zero of $b(z)$, and the multiplicity of z_0 as a zero of $a(z)$ does not exceed the multiplicity of z_0 as a zero of $b(z)$, or in other words, if the quotient $b(z)/a(z)$ is analytic in G.

THEOREM 2.1. *A 2×2 analytic triangular matrix function*

$$A(z) = \begin{bmatrix} a_{11}(z) & a_{12}(z) \\ 0 & a_{22}(z) \end{bmatrix}, \quad z \in G \tag{2.1}$$

with diagonal entries non-zero everywhere in G admits an analytic square root if and only if at least one of the functions $\sqrt{a_{11}} + \sqrt{a_{22}}$ and $\sqrt{a_{11}} - \sqrt{a_{22}}$ has its zeroes in G majorized by the zeroes of a_{12}.

Here by $\sqrt{a_{11}}$ and $\sqrt{a_{22}}$ we understand either of the two branches of the analytic square root scalar function; the statement of the theorem does not depend on this choice.

Proof. The case when a_{12} is identically equal to zero is trivial: the condition on the majorization of zeroes holds automatically, and the square roots of A are delivered by the formula

$$B = \begin{bmatrix} \pm\sqrt{a_{11}} & 0 \\ 0 & \pm\sqrt{a_{22}} \end{bmatrix}$$

(this exhausts all the possibilities if the diagonal entries of A are not identical, and many more analytic square roots exist otherwise). So, it remains to consider the case of a_{12} not equal zero identically.

If $B(z) = [b_{ij}]_{i,j=1}^2$ is the analytic square root of $A(z)$ in G, then, in particular,

$$(b_{11} + b_{22})b_{12} = a_{12} \text{ and } (b_{11} + b_{22})b_{21} = 0. \tag{2.2}$$

Since a_{12} is not identically zero in G, neither is $b_{11} + b_{22}$. From the analyticity of functions involved it follows from the second equation in (2.2) that b_{21} is identically

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zero. In other words, the analytic quadratic roots of the triangular matrix under consideration are triangular as well.

But an upper triangular B is a square root of A if and only if

$$b_{11}^2 = a_{11},\ b_{22}^2 = a_{22},$$

and the first equation of (2.2) holds. The analytic square root of A therefore exists if and only if at least one of the equations

$$b_{12}(\sqrt{a_{11}} + \sqrt{a_{22}}) = a_{12}$$

and

$$b_{12}(\sqrt{a_{11}} - \sqrt{a_{22}}) = a_{12}$$

has an analytic in G solution b_{12}. This, in turn, is equivalent to the condition on the zeroes of the functions a_{12} and $\sqrt{a_{11}} \pm \sqrt{a_{22}}$ mentioned in the statement of the theorem. □

COROLLARY 2.2. *Let $A(z)$ be a 2×2 analytic matrix function on G. Assume that the eigenvalues $\lambda_1(z), \lambda_2(z)$ of $A(z)$, for every $z \in G$, can be enumerated so that $\lambda_1(z)$ and $\lambda_2(z)$ become analytic functions on G. If for at least one branch of the square root function we have*

$$\mathrm{tr}(A(z)) + 2\sqrt{\det(A(z))} \neq 0$$

for every $z \in G$, then there exists an analytic square root of $A(z)$.

For the proof use Theorem 2.1 and 4.1 (to be proved in a later section); the latter theorem allows us to assume that $A(z)$ is upper triangular.

3. Algebraic preliminaries.

THEOREM 3.1. *Let F be a field, and let $A = [a_{i,j}]_{i,j=1}^n$ be an $n \times n$ upper triangular matrix over the field. Assume that*

$$\mathrm{rank}(A - \lambda I) \geq n - 1 \qquad \forall \quad \lambda \in F. \tag{3.1}$$

Then every $n \times n$ matrix X over F such that $X^m = A$ for some positive integer m must be upper triangular.

Proof. Passing to the algebraic closure of F, we may assume that F is algebraically closed. Let X be such that $X^m = A$. By the spectral mapping theorem (which is easily seen using the Jordan form of X)

$$\{\lambda^m \ :\ \lambda \in \sigma(X)\} = \sigma(A).$$

In particular, there exists $\lambda_0 \in \sigma(X)$ such that $\lambda_0^m = a_{1,1}$. If x is a corresponding eigenvector of X, then

$$Ax = X^m x = \lambda_0^m x = a_{1,1}x.$$

Thus, x is an eigenvector of A corresponding to the eigenvalue $a_{1,1}$. But the condition (3.1) implies that A is nonderogatory: Only one eigenvector (up to a nonzero scalar

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multiple) for every eigenvalue. So x is a scalar multiple of $[1\ 0\ \ldots\ 0]^T$, and therefore the first column of X (except possibly for the top entry) consists of zeroes. Now use induction on n to complete the proof. $\square$

The same result holds for any polynomial equation

$$\sum_{j=1}^{m} c_j X^j = A,$$

rather than $X^m = A$.

Theorem 3.1 says nothing about existence of mth roots X of A. A necessary condition (under the hypotheses of Theorem 3.1) is obviously that mth roots of the diagonal elements of A exist in F. From now on we assume that F has characteristic zero (or more precisely that the characteristic of F does not divide m). It turns out that under this assumption, the necessary condition for existence of the mth roots of invertible nonderogatory matrices is also sufficient.

We proceed by induction on the size n of matrices. Let $X = [x_{i,j}]_{i,j=1}^n$ be an $n \times n$ upper triangular matrix: $x_{i,j} = 0$ if $i > j$. Then (using induction on m for example) one verifies that the $(1, n)$ entry of X^m has the form

$$[X^m]_{1,n} = \left(\sum_{j=0}^{m-1} x_{1,1}^j x_{n,n}^{m-1-j} \right) x_{1,n} \\ + P_{m,n}(x_{1,1}, x_{1,2}, \ldots, x_{1,n-1}, x_{2,2}, \ldots, x_{2,n}, \ldots x_{n,n}),$$

where

$$P_{m,n} = P_{m,n}(x_{1,1}, x_{1,2}, \ldots, x_{1,n-1}, x_{2,2}, \ldots, x_{2,n}, \ldots x_{n,n})$$

is a certain homogeneous polynomial of degree m with integer coefficients of the $(n^2 + n - 2)/2$ variables $x_{i,j}$, $1 \le i \le j \le n$, $(i, j) \ne (1, n)$. For example,

$$P_{3,3} = (x_{1,1}x_{1,2} + x_{1,2}x_{2,2})x_{2,3} + x_{1,2}x_{2,3}x_{3,3}.$$

Theorem 3.2. *Let $A = [a_{i,j}]_{i,j=1}^n$, $a_{i,j} \in F$, be an upper triangular invertible matrix, and assume that there exist mth roots $\sqrt[m]{a_{j,j}} \in F$, for $j = 1, 2, \ldots, n$. Then there exist at least m distinct mth roots of A with entries in F.*

Proof. Select $\sqrt[m]{a_{j,j}}$ so that

$$a_{i,i} = a_{j,j} \quad \Longrightarrow \quad \sqrt[m]{a_{i,i}} = \sqrt[m]{a_{j,j}}. \tag{3.2}$$

Construct the elements of the upper triangular $n \times n$ matrix $X = [b_{i,j}]_{i,j=1}^n$ such that $X^m = A$ by induction on $j - i$. For the base of induction, let

$$b_{i,i} = \sqrt[m]{a_{i,i}} \ne 0, \quad i = 1, 2, \ldots, n. \tag{3.3}$$

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If $b_{i,j}$ with $j - i < k$ are already constructed, we let $b_{1,k+1}, b_{2,k+2}, \ldots, b_{n-k,n}$ be defined by the equalities

$$a_{1,k+1} = \left(\sum_{j=0}^{m-1} b_{1,1}^{j} b_{k+1,k+1}^{m-1-j} \right) b_{1,k+1} + P_{m,k+1}(b_{1,1}, b_{1,2}, \ldots, b_{1,k}, b_{2,2}, \ldots, b_{2,k+1}, \ldots b_{k+1,k+1}), \tag{3.4}$$

and so on, the last equality being

$$a_{n-k,n} = \left(\sum_{j=0}^{m-1} b_{n-k,n-k}^{j} b_{n,n}^{m-1-j} \right) b_{n-k,n} + P_{m,k+1}(b_{n-k,n-k}, b_{n-k,n-k+1}, \ldots, b_{n-k,n-1}, b_{n-k+1,n-k+1}, \ldots, b_{n-k+1,n}, \ldots b_{n,n}). \tag{3.5}$$

Condition (3.2) guarantees that

$$\sum_{j=0}^{m-1} b_{i,i}^{j} b_{j,j}^{m-1-j} \neq 0, \quad i, j = 1, \ldots, n,$$

and therefore equalities (3.4)–(3.5) can be uniquely solved for $b_{1,k+1}, \ldots, b_{n-k,n}$. The proof is completed. □

COROLLARY 3.3. *Let $A = [a_{i,j}]_{i,j=1}^{n}$, $a_{i,j} \in F$, be an upper triangular invertible matrix. Assume that there exist mth roots $\sqrt[m]{a_{j,j}} \in F$, for $j = 1, 2, \ldots, n$. Assume furthermore that the condition (3.1) is satisfied. Then there exist not more than m^n distinct mth roots of A with entries in F.*

For the proof combine Theorem 3.1 and (the proof of) Theorem 3.2.

4. Analytic preliminaries. It is well-known that eigenvalues of an analytic matrix function need not be analytic. More precisely, there need not exist an enumeration of the eigenvalues at each point that yields analytic functions on the whole domain, or even locally.

Under some additional hypotheses, the analyticity of eigenvalues can be guaranteed. For example, the well-known Rellich's theorem [6], [7], also [4, Chapter S6]), asserts that if G is a real interval and $A(z)$, $z \in G$, is a Hermitian valued analytic matrix function, then the eigenvalues of $A(z)$ are analytic. Results on triangularization of analytic matrix functions under certain additional conditions, see for example [3], also yield as a by-product analyticity of eigenvalues.

We formulate a general result on analyticity of eigenvalues. The result may be known, but we did not find a comparable statement in the literature. The simple connectedness of G is not needed here.

THEOREM 4.1. *Let G be a domain in $\mathbb{C}$, or an interval in $\mathbb{R}$. Let $A(z)$ be an analytic $n \times n$ matrix function on G. The following statements are equivalent:*

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(a) *The eigenvalues* $\lambda_1(z), \ldots, \lambda_n(z)$ *of* $A(z)$ *can be enumerated so that, for every* $z \in G$, $\lambda_1(z), \ldots, \lambda_n(z)$ *are analytic functions of* $z \in G$.
(b) *There exists an invertible analytic matrix function* $S(z)$ *such that* $S(z)^{-1}A(z)S(z)$ *is upper triangular for every* $z \in G$.

Proof. The implication (b) $\Longrightarrow$ (a) is obvious. Assume (a) holds. Let $\lambda_1(z)$ be an analytic eigenvalue of $A(z)$. Consider the analytic matrix function $B(z) := A(z) - \lambda_1(z)I$.

At this point we use the property (proved in [9]) that the ring $\mathcal{A}(G)$ of analytic functions on G is a *Smith domain.* Recall that a Smith domain is a commutative unital ring $\mathcal{R}$ without divisors of zero such that every matrix X with entries in $\mathcal{R}$ can be transformed via $X \longrightarrow EXF$, where E and F are invertible matrices over $\mathcal{R}$, to a diagonal form $\mathrm{diag}(x_1, \ldots, x_p)$ (possibly bordered by zero rows and/or columns), with $x_1, \ldots, x_p \in \mathcal{R} \setminus \{0\}$ such that x_j is divisible by x_{j+1} in $\mathcal{R}$, for $j = 1, 2, \ldots, p-1$. Therefore we have a representation of $B(z)$ in the form

$$B(z) = E(z)(\mathrm{diag}(x_1, \ldots, x_n))F(z),$$

where $E(z)$ and $F(z)$ are invertible analytic (in G) $n \times n$ matrix functions, and $x_1, \ldots, x_n$ are analytic scalar functions. Since $\det B(z) = 0$ for all $z \in G$, we must have that at least one of the functions $x_1, \ldots, x_n$ is identically zero. Say, $x_1 \equiv 0$. Then

$$F(z)A(z)F(z)^{-1} = F(z)B(z)F(z)^{-1} + \lambda_1(z)I = \begin{bmatrix} \lambda_1(z) & * \\ 0 & A_1(z) \end{bmatrix},$$

where $A_1(z)$ is an $(n-1) \times (n-1)$ analytic matrix function. It is easy to see that the statement (a) holds for $A_1(z)$ (since it holds for $A(z)$). Now we complete the proof by using induction on n, and by applying the induction hypothesis to $A_1(z)$. $\square$

A sufficient condition for analyticity of eigenvalues is that the eigenvalues are contained in a simple differentiable curve, such as the real line (if for example the matrix function is Hermitian valued) or the unit circle (if for example the matrix function is unitary valued). We quote a statement from [8] (Theorem 3.3 there).

Proposition 4.2. *Let* $A(z)$ *be an* $n \times n$ *matrix function, analytic on a real interval* G. *If for every* $z \in G$, *the eigenvalues* $\lambda_1(z), \ldots, \lambda_n(z)$ *of* $A(z)$ *belong to a fixed differentiable curve, then* $\lambda_1(z), \ldots, \lambda_n(z)$ *can be enumerated so that they become analytic functions of* $z \in G$.

5. Main results. We state the main results of the paper.

Theorem 5.1. *Let* G *be a domain in* $\mathbb{C}$, *and let* $A(z)$, $z \in G$, *be an invertible analytic matrix function. Assume that the eigenvalues of* $A(z)$ *can be enumerated so that they form analytic functions in* G. *Fix an integer* $m \geq 2$. *Then:*
(a) *There exist at least* m *distinct meromorphic* $n \times n$ *matrix functions* $B_j(z)$, $j = 1, 2, \ldots, m$, *such that* $B_j(z)^m = A(z)$.
(b) *If in addition, for every analytic function* $\lambda(z)$ *at least one of the* $(n-1) \times (n-1)$ *subdeterminants of* $A(z) - \lambda(z)I$ *is not identically zero, then there exist at most* m^n *distinct matrix functions* $B_j(z)$ *as in* (a).

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Proof. Applying Theorem 4.1 if necessary, we may reduce the general case to that of a triangular matrix function A. Letting the field of scalar meromorphic functions play the role of F, we then derive part (a) from Theorem 3.2. Now observe that for matrix functions A as in (b) condition (3.1) holds. Indeed, suppose that λ is a meromorphic on G function for which $A(z) - \lambda(z)I$ has rank smaller than $n-1$ everywhere on G. Then all the $(n-1)\times(n-1)$ subdeterminants of $A(z)-\lambda(z)I$ vanish on G, and $\lambda(z)$ (as an eigenvalue of $A(z)$) must be bounded together with A (and therefore analytic) on all domains lying strictly inside G. Thus, λ is analytic on G, which is not allowed by (b). It remains to invoke Corollary 3.3. $\square$

The hypothesis in (b) is satisfied, for example, if $A(z)$ is a lower Hessenberg matrix with no identically zero elements on the superdiagonal.

We now turn our attention to analytic mth roots. First of all note that for every invertible $n\times n$ analytic matrix function the existence of an mth analytic root in neighborhood of every point $z_0 \in G$ is guaranteed (cf. Example 1.1). For the existence of analytic (in G) mth roots, it is clear from Theorem 5.1 that, assuming analyticity of eigenvalues, a sufficient condition would involve majorization relations between zeroes of certain analytic functions. In general, these relations are not very transparent; they are implicitly given in the proof of Theorem 3.2. We provide a full description of these relations for 2×2 matrix functions, and an inductive construction for the general $n\times n$ case.

THEOREM 5.2. *Let $A(z) = [a_{i,j}(z)]_{i,j=1}^n$, $a_{i,j}(z) = 0$ if $i > j$, be an upper triangular invertible matrix function, analytic in G. Define inductively the analytic functions $b_{i,j}(z)$, $1\le i\le j\le n$, by the properties* (3.2)−(3.4), *under the assumptions that the zeroes of*

$$\sum_{j=0}^{m-1} b_{q,q}^j b_{q+k,q+k}^{m-1-j}$$

are majorized by the zeroes of

$$a_{q,q+k} - P_{m,k+1}(b_{q,q}, b_{q,q+1}, \ldots, b_{q,q+k-1}, b_{q+1,q+1}, \ldots, b_{q+1,q+k}, \ldots b_{q+k,q+k}),$$

for $q = 1,2,\ldots,n-k$, and for $k = 1,2,\ldots,n-1$. Then the matrix $B(z) := [b_{i,j}(z)]_{i,j=1}^n$, where $b_{i,j}(z) = 0$ if $i > j$, is an analytic mth root of $A(z)$.

In particular:

COROLLARY 5.3. *Let $A(z) = [a_{i,j}(z)]_{i,j=1}^n$, $a_{i,j}(z) = 0$ if $i > j$, be an upper triangular invertible matrix function, analytic in G. If the analytic mth roots $b_{j,j}(z) = \sqrt[m]{a}_{j,j}(z)$, $(j = 1,2,\ldots,n)$ can be chosen so that*

$$a_{i,i} = a_{j,j} \quad \Longrightarrow \quad \sqrt[m]{a_{i,i}} = \sqrt[m]{a_{j,j}} \tag{5.1}$$

and

$$\sum_{q=0}^{m-1} b_{i,i}^q(z) b_{j,j}^{m-1-q}(z) \neq 0 \quad \text{for all } z\in G, \quad 1\le i<j\le n, \tag{5.2}$$

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then $A(z)$ has an analytic mth root.

For the proof, simply observe that (5.2) guarantees that the majorization of zeroes as required in Theorem 5.2 is satisfied trivially.

For 2×2 matrices, the result of Theorem 5.2 simplifies considerably, and yields the following generalization of Theorem 2.1:

THEOREM 5.4. *Let*

$$A(z) = \begin{bmatrix} a_{1,1}(z) & a_{1,2}(z) \\ 0 & a_{2,2}(z) \end{bmatrix}, \quad z \in G \tag{5.3}$$

be an analytic 2×2 upper triangular matrix function with diagonal entries non-zero everywhere in G. Then $A(z)$ has an analytic mth root if and only if at least one of the m^2 functions

$$\sum_{j=0}^{m-1} \left(\sqrt[m]{a_{1,1}} \right)^j \left(\sqrt[m]{a_{2,2}} \right)^{m-1-j},$$

where any of the m values of the mth root may be chosen independently for $\sqrt[m]{a_{1,1}}$ and for $\sqrt[m]{a_{2,2}}$, has its zeroes in G majorized by the zeroes of $a_{1,2}$.

6. Deductions from the main theorems. We collect in this section several results that can be proved using the main theorems of Section 5.

THEOREM 6.1. *Let $A(z)$, $z \in G$, be an invertible analytic $n \times n$ matrix, and assume that the number r of distinct eigenvalues of the matrix $A(z)$ is independent of $z \in G$. Then $A(z)$ has at least m^r distinct analytic mth roots.*

Proof. Since the number of distinct eigenvalues is constant, the eigenvalues may be enumerated so that they become analytic functions (indeed, at a branch point several generally distinct eigenvalues must coalesce). By Theorem 4.1 we may assume that $A(z)$ is upper triangular: $A(z) = [a_{i,j}(z)]_{i,j=1}^n$, where $a_{i,j}(z) = 0$ if $i > j$. Moreover, the proof of Theorem 4.1 shows that the diagonal elements $a_{j,j}$ (actually, the eigenvalues of $A(z)$) may be arranged in clusters:

$$\begin{gathered} a_{1,1}(z) \equiv a_{2,2}(z) \equiv \cdots \equiv a_{k_1,k_1}(z), \ldots, \\ \vdots \\ a_{k_{r-1}+1,k_{r-1}+1}(z) \equiv a_{k_{r-1}+2,k_{r-1}+2}(z) \equiv \cdots \equiv a_{k_r,k_r}(z). \end{gathered}$$

Here $k_1 < k_2 < \cdots < k_r = n$, and $a_{k_j,k_j}(z) \neq a_{k_p,k_p}(z)$ for $j \neq p$ and for all $z \in G$.

Next, we use the well-known property that a linear transformation of a rectangular matrix X defined by $X \mapsto QX - XR$, is invertible provided the square size matrices Q and R have no eigenvalues in common. We apply a suitable transformation of the form

$$A(z) \mapsto S(z)^{-1} A(z) S(z), \tag{6.1}$$

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where

$$S(z) = \begin{bmatrix} I_{k_1 \times k_1} & S_{1,2}(z) & S_{1,3}(z) & \cdots & S_{1,r}(z) \\ 0 & I_{(k_2-k_1)\times(k_2-k_1)} & S_{2,3}(z) & \cdots & S_{2,r}(z) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I_{(k_r-k_{r-1})\times(k_r-k_{r-1})} \end{bmatrix},$$

with a $(k_j - k_{j-1}) \times (k_q - k_{q-1})$ analytic matrix function $S_{j,q}(z)$ $(1 \le j < q \le r)$. (We formally put $k_0 = 0$.) A transformation (6.1) can be chosen in such a way that the resulting analytic matrix function

$$B(z) := S(z)^{-1}A(z)S(z)$$

is block diagonal:

$$B(z) = \operatorname{diag}(B_1(z), B_2(z), \ldots, B_r(z)),$$

where $B_j(z)$ is a $(k_j - k_{j-1}) \times (k_j - k_{j-1})$ analytic upper triangular matrix function with $a_{k_{j-1}+1}(z), \ldots, a_{k_j}(z)$ on the main diagonal. Since the functions on the main diagonal of $B_j(z)$ are identical, Corollary 5.3 is applicable, and the result follows. $\square$

It is well-known that if $A(z)$ is an analytic invertible matrix function on a real interval G, then $A(z)$ has an analytic polar decomposition and an analytic singular value decomposition (see, for example, [1], [5]). Both properties follow easily from the fact (which can be deduced from Rellich's theorem, [6], [7]) that the positive definite analytic matrix function $A(z)^*A(z)$ has a positive definite analytic square root. Using Theorem 5.2, a more general statement can be obtained:

THEOREM 6.2. *Let $A(z)$ be $n \times n$ analytic invertible matrix function on a real interval G. If the eigenvalues of $A(z)$ are positive for every $z \in G$, then $A(z)$ has analytic mth roots for every positive integer m.*

Proof. By Proposition 4.2, the eigenvalues of $A(z)$ can be chosen analytic. By Theorem 4.1, we may assume that $A(z)$ is upper triangular. Taking positive mth roots of the diagonal entries of $A(z)$, we see that the hypotheses of Corollary 5.3 are satisfied. An application of Corollary 5.3 completes the proof. $\square$

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