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COMMUTING TRIPLES OF MATRICES*

YONGHO HAN†

Abstract. The variety $C(3,n)$ of commuting triples of $n \times n$ matrices over $\mathbb{C}$ is shown to be irreducible for $n = 7$. It had been proved that $C(3,n)$ is reducible for $n \geq 30$, but irreducible for $n \leq 6$. Guralnick and Omladič have conjectured that it is reducible for $n > 7$.

Key words. Irreducible variety of commuting triples, Approximation by generic matrices.

AMS subject classifications. 15A27, 15A30.

1. Introduction. Let F be an algebraically closed field. Let $C(3,n)$ denote the affine variety of commuting triples of elements in the ring $M_n(\mathrm{F})$ of $n \times n$ matrices over F. It was asked in [1] for which values of n is this variety irreducible. Guralnick [2] had proved that $C(3,n)$ is reducible for $n \geq 32$, but irreducible for $n \leq 3$. Using the results from [7], Guralnick and Sethuraman [3] had proved that $C(3,n)$ is irreducible when $n = 4$.

If the characteristic of F is zero, then we have a more specific answer for this question. From now on we assume that $char\mathrm{F} = 0$. Holbrook and Omladič [4] gave the answer to this question. In that paper [4] they focused on the problem of approximating 3-tuples of commuting $n \times n$ matrices over F by commuting m-tuples of generic matrices, i.e. matrices with distinct eigenvalues. Let $G(3,n)$ be the subset of $C(3,n)$ consisting of triples (A,B,C) such that A, B, and C are all generic. They asked whether $\overline{G(3,n)}$ equals $C(3,n)$. Here the overline means the closure of $G(3,n)$ with respect to the Zariski topology on F^{3n^2}. The problem is equivalent to that of whether the variety of $C(3,n)$ is irreducible. They had proved that $C(3,n)$ is reducible for $n \geq 30$, but irreducible for $n = 5$. Recently using this perturbation technique Omladič [8] gave the answer to this problem for the case $n = 6$; i.e., He proved that the variety $C(3,6)$ is irreducible if $char\mathrm{F} = 0$.

In this paper we consider the affine variety $C(3,7)$ and prove that $C(3,7)$ is also irreducible. Actually, the proofs of the theorems given in section 3, 4, 5, and 6 work also if the characteristic of the field F is nonzero. So it seems likely that some variations of the proofs will give the results in all characteristics. We have been informed that Omladič has independently obtained the same result.

2. Preliminaries. The following theorems summarized some known results.

THEOREM 2.1. *If the radical of the algebra generated by matrices (A,B,C) has square zero, then the triple belongs to $\overline{G(3,n)}$.*

Proof. It follows from Theorem 1 of [8]. ◻

THEOREM 2.2. *If in a 3-dimensional linear space $\mathfrak{L}$ of nilpotent commuting matrices there is a matrix of maximal possible rank with only one nonzero Jordan*

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†Department of Mathematics, University of Southern California, Los Angeles, CA 90089-2532, USA (yhh@usc.edu). Supported by a grant from NSF DMS-0140578.

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block, then any basis of this space belongs to $\overline{G(3,n)}$.

Proof. It follows from Theorem 2 of [8]. □

THEOREM 2.3. *If in a 3-dimensional linear space* $\mathfrak{L}$ *of nilpotent commuting matrices there is a matrix with only two Jordan blocks, then any basis of this space belongs to* $\overline{G(3,n)}$.

Proof. It follows from Corollary 10 of [7]. □

At first we give a result to be used in the sequel.

LEMMA 2.4. *Assume that* $C(3,m)$ *is irreducible for* $m < n$. *If* $(A,B,C) \in C(3,n)$ *and all three commute with an element that has at least two distinct eigenvalues, then* $(A,B,C) \in \overline{G(3,n)}$.

Proof. This implies we can assume that all three matrices are block diagonal with at least two blocks of sizes m and $n-m$ with $0 < m < n$. So by the assumption, $(A,B,C) \in \overline{G(3,m)} \times \overline{G(3,n-m)}$ and rather clearly $\overline{G(3,m)} \times \overline{G(3,n-m)} \subset \overline{G(3,n)}$. □

LEMMA 2.5. *Suppose that* $(A,B,C) \in \overline{G(3,n)}$ *and the algebra generated by* (A',B',C') *is contained in the algebra generated by* (A,B,C). *Then* $(A',B',C') \in \overline{G(3,n)}$.

Proof. We can find polynomials a, b and c in 3 variables such that $A' = a(A,B,C)$, $B' = b(A,B,C)$, and $C' = c(A,B,C)$. Consider the map from $C(3,n)$ to itself that sends $(X,Y,Z) \to (a(X,Y,Z), b(X,Y,Z), c(X,Y,Z))$. This is a morphism of varieties. Note that it sends $G(3,n)$ to $\overline{G(3,n)}$ (since it sends diagonalizable triples to diagonalizable triples) and so the same for $\overline{G(3,n)}$, whence the result. □

In particular, this shows that the property of being in $\overline{G(3,n)}$ depends only on the algebra generated by the triple.

COROLLARY 2.6. *Suppose that* $(A,B,C) \in C(3,n)$ *and* X *is a polynomial in* A, B. *Then* $(A,B,C) \in \overline{G(3,n)}$ *if and only if* $(A,B,C+X)$ *is.*

LEMMA 2.7. *Let* J *be a finite dimensional nil algebra generated by* $A_1, \ldots, A_m$. *Let* I *be the ideal generated by* A_iA_j *with* $1 \le i,j \le m$. *If* $B_1, \ldots, B_m \in I$, *then* J *is generated as an algebra by the* $A_i + B_i, 1 \le i \le m$.

Proof. Let R be the algebra generated by J and an identity element. Then J is the Jacobson radical of R. Now apply Nakayama's lemma. □

In particular, this implies the following.

COROLLARY 2.8. *Let* $(A,B,C) \in \overline{G(3,n)}$ *with* A, B *and* C *nilpotent. Let* X, Y *and* Z *be in the algebra generated by words of length two in* A, B *and* C *(i.e.,* $A^2, B^2, C^2, AB, AC, BC$*). Then* $(A,B,C) \in \overline{G(3,n)}$ *if and only if* $(A+X, B+Y, C+Z)$ *is in* $\overline{G(3,n)}$.

3. 4+2+1 case. In this section we give the result that a linear space of nilpotent commuting matrices of size 7×7 having a matrix of maximal rank with one Jordan block of order 4 and one Jordan block of order 2 can be perturbed by the generic triples.

THEOREM 3.1. *If in a 3-dimensional linear space* $\mathfrak{L}$ *of nilpotent commuting matrices of size* 7×7 *there is a matrix of maximal possible rank with one Jordan block of order* 4 *and one Jordan block of order* 2, *then any basis of this space belongs to* $\overline{G(3,7)}$.

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Proof. We can write $A \in \mathfrak{L}$ of maximal in some basis as

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

If $B \in \mathfrak{L}$, then the structure of B is well known. It is nilpotent and we may add to it a polynomial in A, so that it looks like

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & a & b & c \\ 0 & 0 & 0 & 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & d & e & 0 & f & g \\ 0 & 0 & 0 & d & 0 & 0 & 0 \\ 0 & 0 & 0 & h & 0 & i & 0 \end{bmatrix}.$$

Let C be a matrix in a $\mathfrak{L}$. Then C looks like

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 & a^{'} & b^{'} & c^{'} \\ 0 & 0 & 0 & 0 & 0 & a^{'} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & d^{'} & e^{'} & 0 & f^{'} & g^{'} \\ 0 & 0 & 0 & d^{'} & 0 & 0 & 0 \\ 0 & 0 & 0 & h^{'} & 0 & i^{'} & 0 \end{bmatrix}.$$

If $B^4 \neq 0$ then B has only one Jordan block and consequently we are done by Theorem 2.2. So we may assume that $B^4 = 0$. So we have $agdi = 0$. Now we may assume that at least one of a, g, d, i is 0. We will consider 15 separate cases.

Case 1. Assume that there is a matrix $B \in \mathfrak{L}$ such that $a \neq 0$, $d \neq 0$, and $g \neq 0$, so that for any $B \in \mathfrak{L}$ the corresponding entry $i = 0$. Without loss of generality we may assume that $a = 1$. As above we may assume that $i^{'} = 0$, but also $a^{'} = 0$. Since the algebra generated by A and B contains AB, by Corollary 2.8, we may assume that $b = 0$ and $b^{'} = 0$. The commutative relation of B and C implies that

$$ch^{'} + e^{'} = c^{'}h,\ fd^{'} + gh^{'} = f^{'}d + g^{'}h,\ d^{'} = f^{'} = g^{'} = 0.$$

Therefore we have that $h^{'} = 0$ and $e^{'} = c^{'}h$. Let

$$X = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},\ Y = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -c^{'} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

A straightforward computation reveals that $B + tX$ commutes with $C + tY$ for all $t \in \mathrm{F}$. Since for $t \neq 0$ the matrix $B + tX$ has more than one point in the its spectrum,

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we can use Lemma 2.4. Therefore the triple $(A, B + tX, C + tY)$ belongs to $\overline{G(3,7)}$ for all $t \in \mathrm{F}$, $t \neq 0$.

Case 2. Assume that there is a matrix $B \in \mathfrak{L}$ such that $a \neq 0$, $g \neq 0$, and $i \neq 0$, so that for any $B \in \mathfrak{L}$ the corresponding entry $d = 0$. Without loss of generality we may assume that $a = 1$. So we may assume that $a^{'} = d^{'} = 0$. Since the algebra generated by A and B contains AB, by Corollary 2.8, we may assume that $b = 0$ and $b^{'} = 0$. The commutative relation of B and C implies that

$$ch^{'} + e^{'} = c^{'}h,\ ci^{'} + f^{'} = c^{'}i,\ gh^{'} = g^{'}h,\ gi^{'} = g^{'}i,\ g^{'} = 0.$$

So we have that $h^{'} = 0$, $i^{'} = 0$, $e^{'} = c^{'}h$, and $f^{'} = c^{'}i$. In this case we may assume that $f^{'} \neq 0$. Indeed, if $f^{'} = 0$, then we could introduce the matrix

$$Z = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{i} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{h}{i} & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

that clearly commutes with both A and B, so that it suffices to prove our case for all triples $(A, B, C + tZ)$ for $t \in \mathrm{F}$, $t \neq 0$. Let

$$X = \begin{bmatrix} f^{'} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & f^{'} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & f^{'} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & f^{'} & 0 & 0 & 0 \\ 0 & 0 & -e^{'} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -e^{'} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & f^{'} \end{bmatrix}.$$

We can easily find that $XC = CX$ by the direct computation. So $B + tX$ commutes with C for all $t \in \mathrm{F}$. Moreover for $t \neq 0$, the matrix $B + tX$ has more than one point in the its spectrum. Therefore, by Lemma 2.4, the triple $(A, B + tX, C)$ belongs to $\overline{G(3,7)}$ for all $t \in \mathrm{F}$, $t \neq 0$.

Case 3. Assume that there is a matrix $B \in \mathfrak{L}$ such that $a \neq 0$, $d \neq 0$, and $i \neq 0$, so that for any $B \in \mathfrak{L}$ the corresponding entry $g = 0$. Now we may assume that $a = 1$. So we may assume that $a^{'} = g^{'} = 0$. Since the algebra generated by A and B contains AB, by Corollary 2.8, we may assume that $b = 0$ and $b^{'} = 0$. The commutative relation of B and C implies that

$$ch^{'} + e^{'} = c^{'}h,\ ci^{'} + f^{'} = c^{'}i,\ fd^{'} = f^{'}d,\ id^{'} = i^{'}d,\ d^{'} = 0.$$

So we have that $f^{'} = 0$, $i^{'} = 0$, $c^{'} = 0$, and $ch^{'} + e^{'} = 0$. If $h^{'} = 0$, then we could introduce the matrix

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$$Z = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

that clearly commutes with both A and B, so that it suffices to prove our case for all triples $(A, B, C + tZ)$ for $t \in \mathrm{F}$, $t \neq 0$. Now, we may assume that $h^{'} \neq 0$. Let

$$X = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & h^{'} & 0 & -e^{'} \\ 0 & 0 & 0 & 0 & 0 & h^{'} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then we can easily find that $XC = CX$. So $B + tX$ commutes with C for all $t \in \mathrm{F}$. Then for $t \neq 0$ the matrix $B + tX$ has more than one point in the its spectrum. By Lemma 2.4, the triple $(A, B + tX, C)$ belongs to $\overline{G(3, 7)}$ for all $t \in \mathrm{F}$, $t \neq 0$.

Case 4. Assume that there is a matrix $B \in \mathfrak{L}$ such that $d \neq 0$, $g \neq 0$, and $i \neq 0$, so that for any $B \in \mathfrak{L}$ the corresponding entry $a = 0$. Without loss of generality we may assume that $d = 1$. So we may assume that $a^{'} = d^{'} = 0$. Since the algebra generated by A and B contains AB, by Corollary 2.8, we may assume that $e = 0$ and $e^{'} = 0$. The commutative relation of B and C implies that

$$ch^{'} = c^{'}h + b^{'},\ ci^{'} = c^{'}i,\ gh^{'} = g^{'}h + f^{'},\ gi^{'} = g^{'}i,\ i^{'} = 0.$$

So we have that $c^{'} = 0$, $g^{'} = 0$, $ch^{'} = b^{'}$, and $gh^{'} = f^{'}$. If $f^{'} = 0$, then we could introduce the matrix

$$Z = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \frac{c}{g} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{g} & 0 & 0 & 0 \end{bmatrix}$$

that clearly commutes with both A and B, so that it suffices to prove our case for all triples $(A, B, C + tZ)$ for $t \in \mathrm{F}$, $t \neq 0$. So we may assume that $f^{'} \neq 0$. Let

$$X = \begin{bmatrix} f^{'} & 0 & 0 & 0 & -b^{'} & 0 & 0 \\ 0 & f^{'} & 0 & 0 & 0 & -b^{'} & 0 \\ 0 & 0 & f^{'} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & f^{'} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & f^{'} \end{bmatrix}.$$

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Then we can easily find that $XC = CX$. So $B + tX$ commutes with C for all $t \in \mathrm{F}$. Moreover for $t \neq 0$, the matrix $B + tX$ has more than one point in the its spectrum. By Lemma 2.4, the triple $(A, B + tX, C)$ belongs to $\overline{G(3, 7)}$ for all $t \in \mathrm{F}$, $t \neq 0$.

Case 5. Assume that there is a matrix $B \in \mathfrak{L}$ such that $a \neq 0$, $d \neq 0$. Then at least one of g and i is zero. If exactly one of g and i is zero, we are done by Case 1. and Case 3. So we may assume that there is a matrix $B \in \mathfrak{L}$ such that $a \neq 0$ and $d \neq 0$, so that for any $B \in \mathfrak{L}$ the corresponding entry $g = i = 0$. Without loss of generality we may assume that $a = 1$. So we may assume that $a^{'} = g^{'} = i^{'} = 0$. Since the algebra generated by A and B contains AB, by Corollary 2.8, we may assume that $b = 0$ and $b^{'} = 0$. The commutative relation of B and C implies that

$$ch^{'} + e^{'} = c^{'}h,\ d^{'} = 0,\ f^{'} = 0.$$

If $h^{'} = 0$, then we could introduce the matrix

$$Z = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

that clearly commutes with both A and B, so that it suffices to prove our case for all triples $(A, B, C + tZ)$ for $t \in \mathrm{F}$, $t \neq 0$. So we may assume that $h^{'} \neq 0$. Let

$$X = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & h^{'} & 0 & -e^{'} \\ 0 & 0 & 0 & 0 & 0 & h^{'} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then we can easily find that $XC = CX$. So $B + tX$ commutes with C for all $t \in \mathrm{F}$. Moreover for $t \neq 0$, the matrix $B + tX$ has more than one point in the its spectrum. We are done by Lemma 2.4.

Case 6. Assume that there is a matrix $B \in \mathfrak{L}$ such that $a \neq 0$, $i \neq 0$. Then at least one of d and g is zero. If only one of d and g is zero, we are done by Case 2. and 3. So we may assume that there is a matrix $B \in \mathfrak{L}$ such that $a \neq 0$ and $i \neq 0$, so that for any $B \in \mathfrak{L}$ the corresponding entry $d = g = 0$. Without loss of generality we may assume that $a = 1$. So we may assume that $a^{'} = g^{'} = d^{'} = 0$. Since the algebra generated by A and B contains AB, by Corollary 2.8, we may assume that $b = 0$ and $b^{'} = 0$. The commutative relation of B and C implies that

$$ch^{'} + e^{'} = c^{'}h, \qquad ci^{'} + f^{'} = c^{'}i.$$

If $c^{'} = 0$ and $i^{'} \neq 0$, then we could introduce the matrix

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$$Z = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & h & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

If $c^{'} \neq 0$ and $i^{'} = 0$, then we could introduce the matrix

$$Z = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -c & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

If $c^{'} = 0$ and $i^{'} = 0$, then we could introduce the matrix

$$Z = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & h & 0 & i-c & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

In all of these cases the matrix Z clearly commutes with both A and B, so that it suffices to prove our case for all triples $(A, B, C + tZ)$ for $t \in \mathrm{F}$, $t \neq 0$. So we may assume that $c^{'} \neq 0$ and $i^{'} \neq 0$. Let

$$X = \begin{bmatrix} x & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & x & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x & 0 & 0 & 0 \\ 0 & 0 & y & 0 & 0 & 0 & z \\ 0 & 0 & 0 & y & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, Y = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & w \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where $x = c^{'}i^{'}$, $y = -c^{'}h^{'}$, $z = -c^{'2}i$, and $w = -c^{'2}i^{'}$.

Then a straightforward computation reveals that $B + tX$ commutes with $C + tY$ for all $t \in \mathrm{F}$. Since for $t \neq 0$ the matrix $B + tX$ has more than one point in the its spectrum, by Lemma 2.4, the triple $(A, B + tX, C + tY)$ belongs to $\overline{G(3, 7)}$ for all $t \in \mathrm{F}$, $t \neq 0$.

Case 7. Assume that there is a matrix $B \in \mathfrak{L}$ such that $d \neq 0$, $i \neq 0$. Then at least one of a and g is zero. If only one of a and g is zero, we are done by Case 3. and 4. So we may assume that there is a matrix $B \in \mathfrak{L}$ such that $d \neq 0$ and $i \neq 0$, so that for any $B \in \mathfrak{L}$ the corresponding entry $a = g = 0$. Without loss of generality we may assume that $d = 1$. So we may assume that $a^{'} = g^{'} = d^{'} = 0$. Since the algebra

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generated by A and B contains AB, by Corollary 2.8, we may assume that $e = 0$ and $e^{'} = 0$. The commutative relation of B and C implies that

$$ch^{'} = b^{'} + c^{'}h, \; ci^{'} = c^{'}i, \; f^{'} = 0, \; i^{'} = 0.$$

Therefore $c^{'} = 0$ and $ch^{'} = b^{'}$. Let

$$X = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -i \end{bmatrix}, Y = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & h^{'}b & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & h^{'} & 0 & 0 & h^{'}f & 0 \\ 0 & 0 & 0 & h^{'} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

A straight computation reveals that $B + tX$ commutes with $C + tY$ for all $t \in \mathrm{F}$. Since for $t \neq 0$ the matrix $B+tX$ has more than one point in the its spectrum, we can use Lemma 2.4. So the triple $(A, B+tX, C+tY)$ belongs to $\overline{G(3,7)}$ for all $t \in \mathrm{F}$, $t \neq 0$.

Case 8. Assume that there is a matrix $B \in \mathfrak{L}$ such that $d \neq 0$, $g \neq 0$. Then at least one of a and i is zero. If only one of a and i is zero, we are done by Case 1. and 4. So we may assume that there is a matrix $B \in \mathfrak{L}$ such that $d \neq 0$ and $g \neq 0$, so that for any $B \in \mathfrak{L}$ the corresponding entry $a = i = 0$. Without loss of generality we may assume that $d = 1$. So we may assume that $a^{'} = i^{'} = d^{'} = 0$. Since the algebra generated by A and B contains AB, by Corollary 2.8, we may assume that $e = 0$ and $e^{'} = 0$. The commutative relation of B and C implies that

$$ch^{'} = b^{'} + c^{'}h, \qquad gh^{'} = f^{'} + g^{'}h.$$

If $g^{'} = 0$, then we could introduce the matrix

$$Z = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -h & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

that clearly commutes with both A and B, so that it suffices to prove our case for all triples $(A, B, C + tZ)$ for $t \in \mathrm{F}$, $t \neq 0$. So we may assume that $g^{'} \neq 0$. Let

$$X = \begin{bmatrix} 0 & 0 & 0 & 0 & c^{'} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c^{'} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & g^{'} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & g^{'} & 0 \\ 0 & 0 & 0 & 0 & 0 & gh^{'} & g^{'} \end{bmatrix}, Y = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & g^{'}h^{'} & 0 \end{bmatrix}.$$

A straightforward computation reveals that $B+tX$ commutes with $C+tY$ for all $t \in \mathrm{F}$. Moreover for $t \neq 0$, the matrix $B + tX$ has more than one point in the its spectrum. So, by Lemma 2.4, the triple $(A, B+tX, C+tY)$ belongs to $\overline{G(3,7)}$ for all $t \in \mathrm{F}$, $t \neq 0$.

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Case 9. Assume that there is a matrix $B \in \mathfrak{L}$ such that $g \neq 0$, $i \neq 0$. Then at least one of a and d is zero. If only one of a and d is zero, we are done by Case 2. and 4. So we may assume that there is a matrix $B \in \mathfrak{L}$ such that $g \neq 0$ and $i \neq 0$, so that for any $B \in \mathfrak{L}$ the corresponding entry $a = d = 0$. Without loss of generality we may assume that $g = 1$. So we may assume that $a^{'} = d^{'} = g^{'} = 0$. The commutative relation of B and C implies that

$$ch^{'} = c^{'}h,\ h^{'} = 0,\ i^{'} = 0,\ ci^{'} = c^{'}i.$$

Thus $c^{'} = 0$ and $ch^{'} = 0$. So the matrix C looks like

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & b^{'} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{'} & 0 & f^{'} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore there is a projection commuting with A and C.

Case 10. Assume that there is a matrix $B \in \mathfrak{L}$ such that $a \neq 0$, $g \neq 0$. Then at least one of d and i is zero. If only one of d and i is zero, we are done by Case 1. and 2. So we may assume that there is a matrix $B \in \mathfrak{L}$ such that $a \neq 0$ and $g \neq 0$, so that for any $B \in \mathfrak{L}$ the corresponding entry $d = i = 0$. Without loss of generality we may assume that $a = 1$. So we may assume that $d^{'} = i^{'} = a^{'} = 0$. Since the algebra generated by A and B contains AB, by Corollary 2.8, we may assume that $b = 0$ and $b^{'} = 0$. The commutative relation of B and C implies that

$$ch^{'} + e^{'} = c^{'}h,\ f^{'} = 0,\ g^{'} = 0,\ gh^{'} = 0.$$

Thus $h^{'} = 0$ and $e^{'} = c^{'}h$. Introduce matrices X and Y,

$$X = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, Y = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -c^{'} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

A direct computation reveals that $B + tX$ commutes with $C + tY$ for all $t \in \mathrm{F}$. Moreover for $t \neq 0$, the matrix $B + tX$ has more than one point in the its spectrum. So, by Lemma 2.4, the triple $(A, B+tX, C+tY)$ belongs to $\overline{G(3,7)}$ for all $t \in \mathrm{F}$, $t \neq 0$.

Case 11. Assume that there is a matrix $B \in \mathfrak{L}$ such that $a \neq 0$. Then at least one of d, i, and g is zero. If at least one of d, i, and g is nonzero, we are done by the above cases. So we may assume that there is a matrix $B \in \mathfrak{L}$ such that $a \neq 0$, so that for any $B \in \mathfrak{L}$ the corresponding entry $d = i = g = 0$. Without loss of generality we may assume that $a = 1$. So we may assume that $d^{'} = i^{'} = g^{'} = 0$ and $a^{'} = 0$. The commutative relation of B and C implies that

$$ch^{'} + e^{'} = c^{'}h, \qquad f^{'} = 0.$$

If $c^{'} = 0$ and $h^{'} \neq 0$, then we could introduce the matrix

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$$Z = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & h & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

If $c^{'} \neq 0$ and $h^{'} = 0$, then we could introduce the matrix

$$Z = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

If $c^{'} = 0$ and $h^{'} = 0$, then we could introduce the matrix

$$Z = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & h-c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

In all of these cases the matrix Z clearly commutes with both A and B, so that it suffices to prove our case for all triples $(A, B, C + tZ)$ for $t \in \mathrm{F}$, $t \neq 0$. So we may assume that $c^{'} \neq 0$ and $h^{'} \neq 0$. Let

$$X = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & h^{'} & 0 & -e^{'} \\ 0 & 0 & 0 & 0 & 0 & h^{'} & 0 \\ 0 & 0 & 0 & 0 & 0 & x & 0 \end{bmatrix}, \text{ where } x = -\frac{b^{'}h^{'}}{c^{'}}.$$

A straightforward computation reveals that $B + tX$ commutes with C for all $t \in \mathrm{F}$. Moreover for $t \neq 0$, the matrix $B + tX$ has more than one point in the its spectrum. So, by Lemma 2.4, the triple $(A, B + tX, C)$ belongs to $\overline{G(3, 7)}$ for all $t \in \mathrm{F}$, $t \neq 0$.

Case 12. Assume that there is a matrix $B \in \mathfrak{L}$ such that $g \neq 0$. Then at least one of a, d, and i is zero. If at least one of a, d, and i is nonzero, we are done by the above cases. So we may assume that there is a matrix $B \in \mathfrak{L}$ such that $g \neq 0$, so that for any $B \in \mathfrak{L}$ the corresponding entry $a = d = i = 0$. Without loss of generality we may assume that $g = 1$. So we may assume that $a^{'} = d^{'} = i^{'} = 0$ and $g^{'} = 0$. The commutative relation of B and C implies that

$$ch^{'} = c^{'}h, \qquad h^{'} = 0.$$

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So we have $c^{'}h = 0$. If $c^{'} = 0$ then we are done as we can find a projection P commuting with A and C. If $c^{'} \neq 0$ then $h = 0$. If $f^{'} = 0$, then we could introduce the matrix

$$Z = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

that clearly commutes with both A and B, so that it suffices to prove our case for all triples $(A, B, C + tZ)$ for $t \in \mathrm{F}$, $t \neq 0$. So we may assume that $f^{'} \neq 0$. Let

$$X = \begin{bmatrix} c^{'} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & c^{'} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c^{'} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c^{'} & 0 & 0 & 0 \\ 0 & 0 & x & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x & 0 & 0 & 0 \\ 0 & 0 & 0 & y & 0 & b^{'} & c^{'} \end{bmatrix}, \text{ where } x = -\frac{e^{'}c^{'}}{f^{'}} \text{ and } y = \frac{e^{'}b^{'}}{f^{'}}.$$

Then we can easily find that $XC = CX$. So $B + tX$ commutes with C for all $t \in \mathrm{F}$. Thus for $t \neq 0$ the matrix $B + tX$ has more than one point in the its spectrum. So, by Lemma 2.4, the triple $(A, B + tX, C)$ belongs to $\overline{G(3, 7)}$ for all $t \in \mathrm{F}$, $t \neq 0$.

Case 13. Assume that there is a matrix $B \in \mathfrak{L}$ such that $d \neq 0$. Then at least one of a, g, and i is zero. If at least one of a, g, and i is nonzero, we are done by the above cases. So we may assume that there is a matrix $B \in \mathfrak{L}$ such that $d \neq 0$, so that for any $B \in \mathfrak{L}$ the corresponding entry $a = g = i = 0$. Without loss of generality we may assume that $d = 1$. So we may assume that $a^{'} = g^{'} = i^{'} = 0$ and $d^{'} = 0$. The commutative relation of B and C implies that

$$ch^{'} = b^{'} + c^{'}h, \qquad f^{'} = 0.$$

If $c^{'} = 0$ and $h^{'} \neq 0$, then we could introduce the matrix

$$Z = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -h & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

If $c^{'} \neq 0$ and $h^{'} = 0$, then we could introduce the matrix

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$$Z = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & c & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

If $c^{'} = h^{'} = 0$, then we could introduce the matrix

$$Z = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & c-h & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

In all of these three cases the matrix Z clearly commutes with both A and B, so that it suffices to prove our case for all triples $(A, B, C + tZ)$ for $t \in \mathrm{F}$, $t \neq 0$. So we may assume that $c^{'} \neq 0$ and $h^{'} \neq 0$. Let

$$X = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & h^{'} & 0 & -e^{'} \\ 0 & 0 & 0 & 0 & 0 & h^{'} & 0 \\ 0 & 0 & 0 & 0 & 0 & x & 0 \end{bmatrix}, \text{ where } x = -\frac{b^{'}h^{'}}{c^{'}} .$$

By a straight computation we have $XC = CX$. Since $XC = CX$, $B + tX$ commutes with C for all $t \in \mathrm{F}$. Moreover, for $t \neq 0$ the matrix $B + tX$ has more than one point in the its spectrum. Thus, by Lemma 2.4, the triple $(A, B+tX, C)$ belongs to $\overline{G(3,7)}$ for all $t \in \mathrm{F}$, $t \neq 0$.

Case 14. Assume that there is a matrix $B \in \mathfrak{L}$ such that $i \neq 0$. Then at least one of a, d, and g is zero. If at least one of a, d, and g is nonzero, we are done by the above cases. So we may assume that there is a matrix $B \in \mathfrak{L}$ such that $i \neq 0$, so that for any $B \in \mathfrak{L}$ the corresponding entry $a = d = g = 0$. Without loss of generality we may assume that $i = 1$. So we may assume that $a^{'} = g^{'} = d^{'} = 0$ and $i^{'} = 0$. The commutativity condition of B and C implies that

$$ch^{'} = c^{'}h, \qquad c^{'} = 0.$$

So we have $ch^{'} = 0$. If $h^{'} = 0$ then we are done as we can find a projection P commuting with A and C. If $h^{'} \neq 0$ then $c = 0$. Let

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$$X = \begin{bmatrix} 0 & 0 & 0 & 0 & b' & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b' & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & e' & 0 & f' & 0 & 0 \\ 0 & 0 & 0 & e' & 0 & f' & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then we can find that $XC = CX$. So $B + tX$ commutes with C for all $t \in$ F. Moreover we may assume that $f' \neq 0$. Indeed, if $f' = 0$, then we can introduce the matrix

$$Z = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

that clearly commutes with A and B, so that it suffices to prove our case for all triples $(A, B, C + tZ)$ for $t \in$ F, $t \neq 0$. So for $t \neq 0$ the matrix $B + tX$ has more than one point in the its spectrum. Thus, by Lemma 2.4, the triple $(A, B + tX, C)$ belongs to $\overline{G(3,7)}$ for all $t \in$ F, $t \neq 0$.

Case 15. Assume that there is a matrix $B \in \mathfrak{L}$ such that a, d, g, and i are zero. If at least one of a', d', g', and i' is not zero, then we can change a role of B and C. So we are done by above cases. Thus we may assume that $a' = g' = d' = 0$ and $i' = 0$. The commutativity condition of B and C implies that

$$ch' = c'h.$$

It means that (c, h) and (c', h') are linearly dependent, so that we may choose one of them, say the first one, to be zero. It follows that the projection

$$P = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

commutes with A and B, and we are done. ◻

4. 3+3+1 case.

In this section we give the result that a linear space of nilpotent commuting matrices of size 7×7 having a matrix of maximal rank with two Jordan blocks of order 3 can be perturbed by the generic triples.

THEOREM 4.1. *If in a 3-dimensional linear space $\mathfrak{L}$ of nilpotent commuting matrices of size 7×7 there is a matrix of maximal possible rank with two Jordan block of order 3, then any basis of this space belongs to $\overline{G(3,7)}$.*

Proof. We can write $A \in \mathfrak{L}$ of maximal in some basis as

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$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

If $B \in \mathfrak{L}$, then the structure of B is well known. It is nilpotent and we may add to it a polynomial in A, so that it looks like

$$B = \begin{bmatrix} 0 & 0 & 0 & a & b & c & d \\ 0 & 0 & 0 & 0 & a & b & 0 \\ 0 & 0 & 0 & 0 & 0 & a & 0 \\ e & f & g & 0 & h & i & j \\ 0 & e & f & 0 & 0 & h & 0 \\ 0 & 0 & e & 0 & 0 & 0 & 0 \\ 0 & 0 & k & 0 & 0 & l & 0 \end{bmatrix}.$$

Let C be a matrix in a $\mathfrak{L}$. Then C looks like

$$C = \begin{bmatrix} 0 & 0 & 0 & a^{'} & b^{'} & c^{'} & d^{'} \\ 0 & 0 & 0 & 0 & a^{'} & b^{'} & 0 \\ 0 & 0 & 0 & 0 & 0 & a^{'} & 0 \\ e^{'} & f^{'} & g^{'} & 0 & h^{'} & i^{'} & j^{'} \\ 0 & e^{'} & f^{'} & 0 & 0 & h^{'} & 0 \\ 0 & 0 & e^{'} & 0 & 0 & 0 & 0 \\ 0 & 0 & k^{'} & 0 & 0 & l^{'} & 0 \end{bmatrix}.$$

Since the matrix B is nilpotent, we find a relation $ae = 0$. Now we may assume that at least one of a and e is 0. We will consider 2 separate cases.

Case 1. Assume that there is a matrix $B \in \mathfrak{L}$ such that $a \neq 0$, so that for any $B \in \mathfrak{L}$ the corresponding entry $e = 0$. Without loss of generality we may assume that $a = 1$. So we may assume that $a^{'} = 0$ and $e^{'} = 0$. If $B^4 \neq 0$ then B has only one Jordan block and consequently we are done by Theorem 2.2. So we may assume that $B^4 = 0$. Since $B^4 = 0$, we have that $f = 0$ and $jk = 0$. By commutativity condition of B and C we have $f^{'} = 0$. Now we may assume that at least one of j and k is zero. So there are 3 possibilities.

(i) Assume that there is a matrix $B \in \mathfrak{L}$ such that $j \neq 0$, so that for any $B \in \mathfrak{L}$ the corresponding entry $k = 0$. Then we may assume that $k^{'} = 0$. The commutative relation of B and C implies that

$$g^{'} = h^{'} = j^{'} = 0, \, dl^{'} + i^{'} = b^{'}h + d^{'}l, \, jl^{'} = 0.$$

So we have $l^{'} = 0$ and $i^{'} = b^{'}h + d^{'}l$. If $b^{'} = 0$, then we could introduce the matrix

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$$Z = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & h & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

that clearly commutes with both A and B, so that it suffices to prove our case for all triples $(A, B, C + tZ)$ for $t \in \mathrm{F}$, $t \neq 0$. So we may assume that $b^{'} \neq 0$. Let

$$X = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x & y & 0 & 0 \\ 0 & 0 & 0 & 0 & x & y & 0 \\ 0 & 0 & 0 & 0 & 0 & x & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, Y = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & z & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & z & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where $x = b^{'}$, $y = bb^{'} - c^{'}$, and $z = b^{'2}$.

A straightforward computation reveals that $B+tX$ commutes with $C+tY$ for all $t \in \mathrm{F}$. Moreover for $t \neq 0$, the matrix $B + tX$ has more than one point in the its spectrum. So, by Lemma 2.4, the triple $(A, B+tX, C+tY)$ belongs to $\overline{G(3, 7)}$ for all $t \in \mathrm{F}$, $t \neq 0$.

(ii) Assume that there is a matrix $B \in \mathfrak{L}$ such that $k \neq 0$, so that for any $B \in \mathfrak{L}$ the corresponding entry $j = 0$. Then we may assume that $j^{'} = 0$. The commutative relation of B and C implies that

$$d^{'} = g^{'} = h^{'} = k^{'} = 0, \; dl^{'} + i^{'} = b^{'}h.$$

Let

$$X = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & b & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & b & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, Y = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b^{'} & c^{'} & 0 \\ 0 & 0 & 0 & 0 & 0 & b^{'} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

By the straightforward computation we find that $B + tX$ commutes with $C + tY$ for all $t \in \mathrm{F}$. Moreover for $t \neq 0$, the matrix $B + tX$ has more than one point in the its spectrum. So, by Lemma 2.4, the triple $(A, B + tX, C + tY)$ belongs to $\overline{G(3, 7)}$ for all $t \in \mathrm{F}$, $t \neq 0$.

(iii) Assume that there is a matrix $B \in \mathfrak{L}$ such that j and k are zero. If at least one of $j^{'}$ and $k^{'}$ is nonzero, then we can change a role of B and C. So we are done by above Cases. Thus we may assume that $j^{'} = k^{'} = 0$. Then B and C look like

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$$B = \begin{bmatrix} 0 & 0 & 0 & 1 & b & c & d \\ 0 & 0 & 0 & 0 & 1 & b & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & g & 0 & h & i & 0 \\ 0 & 0 & 0 & 0 & 0 & h & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & l & 0 \end{bmatrix}, C = \begin{bmatrix} 0 & 0 & 0 & 0 & b^{'} & c^{'} & d^{'} \\ 0 & 0 & 0 & 0 & 0 & b^{'} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & g^{'} & 0 & h^{'} & i^{'} & 0 \\ 0 & 0 & 0 & 0 & 0 & h^{'} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & l^{'} & 0 \end{bmatrix}.$$

The commutative relation of B and C implies that

$$g^{'} = 0, \; h^{'} = 0, \; dl^{'} + i^{'} = b^{'}h + d^{'}l.$$

If $l^{'} = 0$, then we could introduce the matrix

$$Z = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -d & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

that clearly commutes with both A and B, so that it suffices to prove our case for all triples $(A, B, C + tZ)$ for $t \in \mathrm{F}$, $t \neq 0$. So we may assume that $l^{'} \neq 0$. Let

$$X = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & l^{'} & bl^{'} & 0 & d^{'}l \\ 0 & 0 & 0 & 0 & l^{'} & bl^{'} & 0 \\ 0 & 0 & 0 & 0 & 0 & l^{'} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & l^{'} \end{bmatrix}, Y = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b^{'}l^{'} & c^{'}l^{'} & d^{'}l^{'} \\ 0 & 0 & 0 & 0 & 0 & b^{'}l^{'} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

A straightforward computation reveals that $B+tX$ commutes with $C+tY$ for all $t \in \mathrm{F}$. Moreover for $t \neq 0$, the matrix $B + tX$ has more than one point in the its spectrum. So, by Lemma 2.4, the triple $(A, B+tX, C+tY)$ belongs to $\overline{G(3, 7)}$ for all $t \in \mathrm{F}$, $t \neq 0$.

Case 2. Assume that there is a matrix $B \in \mathfrak{L}$ such that $e \neq 0$, so that for any $B \in \mathfrak{L}$ the corresponding entry $a = 0$. Without loss of generality we may assume that $e = 1$. So we may assume that $a^{'} = 0$ and $e^{'} = 0$. If $B^4 \neq 0$ then B has only one Jordan block and consequently we are done by Theorem 2.2. So we may assume that $B^4 = 0$. Since $B^4 = 0$, we have that $b = 0$ and $dl = 0$. By commutativity condition of B and C we have $b^{'} = 0$. Now we may assume that at least one of d and l is zero. So there are 3 possibilities.

(i) Assume that there is a matrix $B \in \mathfrak{L}$ such that $d \neq 0$, so that for any $B \in \mathfrak{L}$ the corresponding entry $l = 0$. Then we may assume that $l^{'} = 0$. The commutative relation of B and C implies that

$$c^{'} = d^{'} = h^{'} = k^{'} = 0, \; hf^{'} = j^{'}k + i^{'}.$$

Let

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$$X = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -f & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -f & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, Y = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -f^{'} & -g^{'} & 0 \\ 0 & 0 & 0 & 0 & 0 & -f^{'} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

A direct computation reveals that $B + tX$ commutes with $C + tY$ for all $t \in \mathrm{F}$. Moreover for $t \neq 0$, the matrix $B + tX$ has more than one point in the its spectrum. So, by Lemma 2.4, the triple $(A, B+tX, C+tY)$ belongs to $\overline{G(3,7)}$ for all $t \in \mathrm{F}$, $t \neq 0$.

(ii) Assume that there is a matrix $B \in \mathfrak{L}$ such that $l \neq 0$, so that for any $B \in \mathfrak{L}$ the corresponding entry $d = 0$. Then we may assume that $d^{'} = 0$. The commutative relation of B and C implies that

$$c^{'} = j^{'} = h^{'} = l^{'} = 0,\ hf^{'} + jk^{'} = i^{'}.$$

Let

$$X = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & f & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & f & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, Y = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & f^{'} & g^{'} & 0 \\ 0 & 0 & 0 & 0 & 0 & f^{'} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

A straightforward computation reveals that $B+tX$ commutes with $C+tY$ for all $t \in \mathrm{F}$. Moreover for $t \neq 0$, the matrix $B + tX$ has more than one point in the its spectrum. So, by Lemma 2.4, the triple $(A, B+tX, C+tY)$ belongs to $\overline{G(3,7)}$ for all $t \in \mathrm{F}$, $t \neq 0$.

(iii) Assume that there is a matrix $B \in \mathfrak{L}$ such that d and l are zero. If at least one of $d^{'}$ and $l^{'}$ is nonzero, then we can change a role of B and C. So we are done by above cases. Thus we may assume that $d^{'} = l^{'} = 0$. The commutative relation of B and C implies that

$$c^{'} = h^{'} = 0,\ hf^{'} + jk^{'} = j^{'}k + i^{'}.$$

If $j^{'} = 0$, then we could introduce the matrix

$$Z = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -k & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

that clearly commutes with both A and B, so that it suffices to prove our case for all triples $(A, B, C + tZ)$ for $t \in \mathrm{F}$, $t \neq 0$. So we may assume that $j^{'} \neq 0$, say $j^{'} = 1$. Now we may assume that $j = 0$. Let

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$$X = \begin{bmatrix} 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&1&f&0&0\\ 0&0&0&0&1&f&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1 \end{bmatrix}, Y = \begin{bmatrix} 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&f^{'}&g^{'}&0\\ 0&0&0&0&0&f^{'}&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&k^{'}&0 \end{bmatrix}.$$

A direct computation reveals that $B + tX$ commutes with $C + tY$ for all $t \in \mathrm{F}$. Moreover for $t \neq 0$, the matrix $B + tX$ has more than one point in the its spectrum. So, by Lemma 2.4, the triple $(A, B+tX, C+tY)$ belongs to $\overline{G(3,7)}$ for all $t \in \mathrm{F}$, $t \neq 0$.

Case 3. Assume that there is a matrix $B \in \mathfrak{L}$ such that a and e are zero. If at least one of $a^{'}$ and $e^{'}$ is nonzero, then we can change a role of B and C. So we are done by Case 1 and Case 2. Thus we may assume that $a^{'} = e^{'} = 0$. The commutativity relation of B and C implies that

$$bf^{'} + dk^{'} = b^{'}f + d^{'}k, \qquad bh^{'} + dl^{'} = b^{'}h + d^{'}l,$$

$$hf^{'} + jk^{'} = h^{'}f + j^{'}k, \qquad fb^{'} + jl^{'} = f^{'}b + j^{'}l.$$

Now we will try to consider 2 cases.

(i) Assume that (k, l) and $(k^{'}, l^{'})$ are linearly independent. Without loss of generality we may assume that $(k, l) = (1, 0)$ and $(k^{'}, l^{'}) = (0, 1)$. By the commutativity relation, we have a relation $b(j^{'}h^{'}) - f(dh^{'}) = b^{'}(j^{'}h) - f^{'}(dh)$. Let

$$X = \begin{bmatrix} 0&0&0&x^{'}&0&0&y^{'}\\ 0&0&0&0&x^{'}&0&0\\ 0&0&0&0&0&x^{'}&0\\ z^{'}&0&0&0&0&0&w^{'}\\ 0&z^{'}&0&0&0&0&0\\ 0&0&z^{'}&0&0&0&0\\ 0&0&0&0&0&0&\alpha^{'} \end{bmatrix}, Y = \begin{bmatrix} 0&0&0&x&0&0&y\\ 0&0&0&0&x&0&0\\ 0&0&0&0&0&x&0\\ z&0&0&0&0&0&w\\ 0&z&0&0&0&0&0\\ 0&0&z&0&0&0&0\\ 0&0&0&0&0&0&\alpha \end{bmatrix},$$

where $\alpha^{'} = -h^{'}d$, $\alpha = -hj^{'}$, $x^{'} = hd$, $x = h^{'}d$, $z^{'} = hj^{'}$, $z = h^{'}j^{'}$, $y^{'} = xi - x^{'}i^{'}$, $y = cz - gx - c^{'}z^{'} + g^{'}x^{'}$, $w^{'} = g^{'}x^{'} - c^{'}z^{'} + cz - gx$, and $w = iz - i^{'}z^{'}$. A direct computation reveals that $B + tX$ commutes with $C + tY$ for all $t \in \mathrm{F}$. If at least one of $h^{'}d$ and $hj^{'}$ is nonzero then one of $B + tX$ and $C + tY$ has more than one point in the its spectrum. So, by Lemma 2.4, the triple $(A, B+tX, C+tY)$ belongs to $\overline{G(3,7)}$ for all $t \in \mathrm{F}$, $t \neq 0$.

Now we will consider the case that $h^{'}d = 0$ and $hj^{'} = 0$.

(1) Assume that $d = h = h^{'} = j^{'} = 0$. By the commutativity relation, we have that $j = bf^{'} - fb^{'}$ and $d^{'} = j$. If $f^{'} \neq 0$, then we can take X and Y as follows.

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$$X=\begin{bmatrix} 0 & 0 & 0 & \alpha & 0 & 0 & x \\ 0 & 0 & 0 & 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & y \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & f^{'} \end{bmatrix}, Y=\begin{bmatrix} 0 & 0 & 0 & \beta & 0 & 0 & y \\ 0 & 0 & 0 & 0 & \beta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where $\alpha=-f$, $\beta=-f^{'}$, $x=fi^{'}-f^{'}i$, and $y=f^{'}g-fg^{'}$. By a straight computation we can find that $B+tX$ commutes with $C+tY$ for all $t\in \mathrm{F}$. Moreover for $t\neq 0$, the matrix $B+tX$ has more than one point in the its spectrum. So, by Lemma 2.4, the triple $(A,B+tX,C+tY)$ belongs to $\overline{G(3,7)}$ for all $t\in\mathrm{F}$, $t\neq 0$.

If $f^{'}=0$. By commutativity condition of B and C we have $j=d^{'}=-fb^{'}$. Then we can take X and Y as follows.

$$X=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -c^{'} \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, Y=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -g^{'} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

if $j=0$ and $f\neq 0$.

$$X=\begin{bmatrix} 0 & 0 & 0 & g & 0 & 0 & x \\ 0 & 0 & 0 & 0 & g & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & g & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -g^{'} \end{bmatrix}, Y=\begin{bmatrix} 0 & 0 & 0 & g^{'} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & g^{'} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & g^{'} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where $x=g^{'}i-gi^{'}$, if $j=0$ and $f=0$.

$$X=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & x \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & y \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & b \end{bmatrix}, Y=\begin{bmatrix} 0 & 0 & 0 & \alpha & 0 & 0 & z \\ 0 & 0 & 0 & 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha & 0 \\ \beta & 0 & 0 & 0 & 0 & 0 & w \\ 0 & \beta & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where $\alpha=-b$, $\beta=-f$, $x=-bi$, $y=gb-fc$, $z=bg-cf$, and $w=-fi$, if $j\neq 0$.

In each case a straightforward computation reveals that $B+tX$ commutes with $C+tY$ for all $t\in\mathrm{F}$. In the second case we may assume that $g^{'}\neq 0$. In the third case we also assume that $b\neq 0$. Moreover for $t\neq 0$, the matrix $B+tX$ has more than one point in the its spectrum. So, by Lemma 2.4, the triple $(A,B+tX,C+tY)$ belongs to $\overline{G(3,7)}$ for all $t\in\mathrm{F}$, $t\neq 0$.

(2) Assume that $d=h=h^{'}=0$, and $j^{'}\neq 0$. By the commutativity condi-

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tion we have $j^{'} = hf^{'} - h^{'}f$. But this relation means $j^{'} = 0$. So this case can not occur.

(3) Assume that $d = h^{'} = j^{'} = 0$, and $h \neq 0$. Then by the commutativity condition we have

$$j = d^{'} = b^{'} = f^{'} = 0.$$

Let

$$X = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -c^{'} \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, Y = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -g^{'} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

A straightforward computation reveals that $B+tX$ commutes with $C+tY$ for all $t \in \mathrm{F}$. Moreover for $t \neq 0$, the matrix $B + tX$ has more than one point in the its spectrum. So, by Lemma 2.4, the triple $(A, B+tX, C+tY)$ belongs to $\overline{G(3,7)}$ for all $t \in \mathrm{F}$, $t \neq 0$.

(4) Assume that $h = h^{'} = j^{'} = 0$, and $d \neq 0$. By the commutativity condition we have $d = hb^{'} - h^{'}b$. But this relation means $d = 0$. So this case can not occur.

(5) Assume that $d = h = j^{'} = 0$, and $h^{'} \neq 0$. The commutativity relation of B and C implies that

$$j = f = b = d^{'} = 0.$$

If $c = 0$, then we could introduce the matrix

$$Z = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

that clearly commutes with both A and C, so that it suffices to prove our case for all triples $(A, B+tZ, C)$ for $t \in \mathrm{F}$, $t \neq 0$. Now we may assume that $c \neq 0$. Let

$$X = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -ci \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & c \end{bmatrix}, Y = \begin{bmatrix} 0 & 0 & 0 & x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x & 0 \\ y & 0 & 0 & 0 & 0 & 0 & z \\ 0 & y & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & y & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where $x = -c$, $y = -g$, and $z = -gi$.

A direct computation reveals that $B + tX$ commutes with $C + tY$ for all $t \in \mathrm{F}$. Moreover for $t \neq 0$, the matrix $B + tX$ has more than one point in the its spectrum. So, by Lemma 2.4, the triple $(A, B+tX, C+tY)$ belongs to $\overline{G(3,7)}$ for all $t \in \mathrm{F}$, $t \neq 0$.

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(6) Assume that $h = h^{'} = 0$, $d \neq 0$, and $j^{'} \neq 0$. By the commutativity condition we have $d = hb^{'} - h^{'}b$. But this relation means $d = 0$. So this case can not be happened.

(7) Assume that $h^{'} = j^{'} = 0$, $d \neq 0$, and $h \neq 0$. The commutative relation of B and C implies that

$$j = d^{'} = -b^{'}f, \qquad d = b^{'}h.$$

If $b = 0$, then we could introduce the matrix

$$Z = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

that clearly commutes with both A and C, so that it suffices to prove our case for all triples $(A, B + tZ, C)$ for $t \in \mathrm{F}$, $t \neq 0$. Now we may assume that $b \neq 0$. Let

$$X = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & x \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & y \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & b \end{bmatrix}, Y = \begin{bmatrix} 0 & 0 & 0 & z & 0 & 0 & y \\ 0 & 0 & 0 & 0 & z & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & z & 0 \\ \alpha & 0 & 0 & \beta & 0 & 0 & \gamma \\ 0 & \alpha & 0 & 0 & \beta & 0 & 0 \\ 0 & 0 & \alpha & 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where $x = ch - bi$, $y = gb - cf$, $z = -b$, $\alpha = -f$, $\beta = -h$, and $\gamma = gh - if$.

Then a straightforward computation reveals that $B + tX$ commutes with $C + tY$ for all $t \in \mathrm{F}$. Moreover for $t \neq 0$, the matrix $C + tY$ has more than one point in the its spectrum. So, by Lemma 2.4, the triple $(A, B + tX, C + tY)$ belongs to $\overline{G(3, 7)}$ for all $t \in \mathrm{F}$, $t \neq 0$.

(8) Assume that $d = h = 0$, $j^{'} \neq 0$, and $h^{'} \neq 0$. The commutativity relation of B and C implies that

$$b = 0,\ j = d^{'} = -b^{'}f,\ j^{'} = -h^{'}f.$$

If $f^{'} = 0$, then we could introduce the matrix

$$Z = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

that clearly commutes with both A and B, so that it suffices to prove our case for all triples $(A, B, C + tZ)$ for $t \in \mathrm{F}$, $t \neq 0$. So we may assume that $f^{'} \neq 0$. Let

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$$X=\begin{bmatrix} x & 0 & 0 & y & 0 & 0 & z\\ 0 & x & 0 & 0 & y & 0 & 0\\ 0 & 0 & x & 0 & 0 & y & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & w\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & b^{'} \end{bmatrix},\ Y=\begin{bmatrix} 0 & 0 & 0 & \alpha & 0 & 0 & w\\ 0 & 0 & 0 & 0 & \alpha & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & \alpha & 0\\ 0 & 0 & 0 & \beta & 0 & 0 & \gamma\\ 0 & 0 & 0 & 0 & \beta & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & \beta & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where $x=-\frac{j^{'}}{f'}$, $y=\frac{j}{f'}$, $z=-b^{'}i+ch^{'}+\frac{c^{'}j^{'}}{f'}-\frac{i^{'}j}{f'}$, $w=gb^{'}+\frac{g^{'}j}{f'}$ $\alpha=-b^{'}$, $\beta=-h^{'}$, and $\gamma=gh^{'}+\frac{j^{'}g^{'}}{f'}$.

Then a straightforward computation reveals that $B+tX$ commutes with $C+tY$ for all $t\in \mathrm{F}$. Moreover for $t\neq 0$, the matrix $C+tY$ has more than one point in the its spectrum. So, by Lemma 2.4, the triple $(A,B+tX,C+tY)$ belongs to $\overline{G(3,7)}$ for all $t\in\mathrm{F}$, $t\neq 0$.

(9) Assume that $d=j^{'}=0$, $h\neq 0$, and $h^{'}\neq 0$. The commutativity relation of B and C implies that

$$bh^{'}=b^{'}h,\ j=d^{'}=bf^{'}-b^{'}f,\ hf^{'}=h^{'}f.$$

Let

$$X=\begin{bmatrix} 0 & 0 & 0 & x & 0 & 0 & y\\ 0 & 0 & 0 & 0 & x & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & x & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & z\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & h^{'} \end{bmatrix},\ Y=\begin{bmatrix} 0 & 0 & 0 & w & 0 & 0 & z\\ 0 & 0 & 0 & 0 & w & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & w & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where $x=-h$, $y=hi^{'}-h^{'}i$, $z=gh^{'}-g^{'}h$, $w=-h^{'}$.

Then a straightforward computation reveals that $B+tX$ commutes with $C+tY$ for all $t\in\mathrm{F}$. Moreover for $t\neq 0$, the matrix $B+tX$ has more than one point in the its spectrum. So, by Lemma 2.4, the triple $(A,B+tX,C+tY)$ belongs to $\overline{G(3,7)}$ for all $t\in\mathrm{F}$, $t\neq 0$.

(ii) Assume that (k,l) and $(k^{'},l^{'})$ are linearly dependent. Without loss of generality we may assume that $(k^{'},l^{'})=(0,0)$. We will consider 4 possibilities.

(1) Assume that $d^{'}=j^{'}=0$. Then there is a projection P commuting with A and C.

(2) Assume that $d^{'}\neq 0$, and $j^{'}=0$. Without loss of generality we may assume that $d^{'}=1$. So we may assume that $d=0$. The commutative relation of B and C implies that

$$bf^{'}=b^{'}f+k,\ bh^{'}=b^{'}h+l,\ hf^{'}=h^{'}f,\ bf^{'}=b^{'}f.$$

So we have $k=0$. If $j=0$, then we could introduce the matrix

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$$Z = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

that clearly commutes with both A and C, so that it suffices to prove our case for all triples $(A, B + tZ, C)$ for $t \in \mathrm{F}$, $t \neq 0$. Now we may assume that $j \neq 0$. Now we will consider 2 cases.

(2-a) Assume that $h^{'} \neq 0$. Then we can find X and Y as follows.

$$X = \begin{bmatrix} 0 & 0 & 0 & h & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & h & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & h & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x & 0 & 0 & y & z \end{bmatrix}, Y = \begin{bmatrix} 0 & 0 & 0 & h^{'} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & h^{'} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & h^{'} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & w & 0 \end{bmatrix},$$

where $x = hg^{'} - h^{'}g$, $y = hi^{'} - h^{'}i$, $z = -jh^{'}$, $w = \frac{hg^{'} - h^{'}g}{j}$.

A straightforward computation reveals that $B+tX$ commutes with $C+tY$ for all $t \in \mathrm{F}$. Moreover for $t \neq 0$, the matrix $B + tX$ has more than one point in the its spectrum. So, by Lemma 2.4, the triple $(A, B+tX, C+tY)$ belongs to $\overline{G(3, 7)}$ for all $t \in \mathrm{F}$, $t \neq 0$.

(2-b) Assume that $h^{'} = 0$. If at least one of f and $f^{'}$ is nonzero. Let

$$X = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & f & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & f & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & f & 0 \\ 0 & 0 & 0 & 0 & 0 & x & 0 \end{bmatrix}, Y = \begin{bmatrix} -f^{'} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -f^{'} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -f^{'} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & y & 0 & 0 & 0 & 0 \end{bmatrix},$$

where $x = cf^{'} - c^{'}f$, $y = \frac{gf^{'} - g^{'}f}{j}$.

A straightforward computation reveals that $B + tX$ commutes with $C + tY$ for all $t \in \mathrm{F}$. Moreover for $t \neq 0$, at least one of $B + tX$ and $C + tY$ has more than one point in the its spectrum. So, by Lemma 2.4, the triple $(A, B + tX, C + tY)$ belongs to $\overline{G(3, 7)}$ for all $t \in \mathrm{F}$, $t \neq 0$.

If $f = f^{'} = 0$. If $b = 0$, then we could introduce the matrix

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$$Z = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

that clearly commutes with both A and C, so that it suffices to prove our case for all triples $(A, B + tZ, C)$ for $t \in \mathrm{F}$, $t \neq 0$. Now we may assume that $b \neq 0$. Let

$$X = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 & 0 & x & 0 \end{bmatrix}, Y = \begin{bmatrix} -b^{'} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -b^{'} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -b^{'} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & y & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\text{where } x = cb^{'} - c^{'}b,\ y = \tfrac{gb^{'} - g^{'}b}{j}.$$

Then a straightforward computation reveals that $B + tX$ commutes with $C + tY$ for all $t \in \mathrm{F}$. Moreover for $t \neq 0$, $B + tX$ has more than one point in the its spectrum. So, by Lemma 2.4, the triple $(A, B+tX, C+tY)$ belongs to $\overline{G(3, 7)}$ for all $t \in \mathrm{F}$, $t \neq 0$.

(3) Assume that $j^{'} \neq 0$, and $d^{'} = 0$. Without loss of generality we may assume that $j^{'} = 1$. So we may assume that $j = 0$. The commutative relation of B and C implies that

$$bf^{'} = b^{'}f,\ bh^{'} = b^{'}h,\ hf^{'} = h^{'}f + k,\ b^{'}f = bf^{'} + l.$$

So we have $l = 0$. If $d = 0$, then we could introduce the matrix

$$Z = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

that clearly commutes with both A and C, so that it suffices to prove our case for all triples $(A, B + tZ, C)$ for $t \in \mathrm{F}$, $t \neq 0$. Now we may assume that $d \neq 0$. Now we will consider 2 cases.

(3-a) Assume that $h^{'} \neq 0$. Then we can find X and Y as follows.

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$$X = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ h & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & h & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & h & 0 & 0 & 0 & 0 \\ 0 & 0 & x & 0 & 0 & y & z \end{bmatrix}, Y = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ h^{'} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & h^{'} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & h^{'} & 0 & 0 & 0 & 0 \\ 0 & 0 & w & 0 & 0 & 0 & 0 \end{bmatrix},$$

where $x = ih^{'} - i^{'}h$, $y = hc^{'} - h^{'}c$, $z = -dh^{'}$, $w = \frac{hc^{'} - h^{'}c}{d}$.

Then a straightforward computation reveals that $B + tX$ commutes with $C + tY$ for all $t \in \mathrm{F}$. Moreover for $t \neq 0$, the matrix $B + tX$ has more than one point in the its spectrum. So, by Lemma 2.4, the triple $(A, B + tX, C + tY)$ belongs to $\overline{G(3,7)}$ for all $t \in \mathrm{F}$, $t \neq 0$.

(3-b) Assume that $h^{'} = 0$. If at least one of f and $f^{'}$ is nonzero. Let

$$X = \begin{bmatrix} f & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & f & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & f & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x & 0 & 0 & 0 & 0 \end{bmatrix}, Y = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -f^{'} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -f^{'} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -f^{'} & 0 \\ 0 & 0 & 0 & 0 & 0 & y & 0 \end{bmatrix},$$

where $x = gf^{'} - g^{'}f$, $y = \frac{cf^{'} - c^{'}f}{d}$.

Then a straightforward computation reveals that $B + tX$ commutes with $C + tY$ for all $t \in \mathrm{F}$. Moreover for $t \neq 0$, at least one of $B + tX$ and $C + tY$ has more than one point in the its spectrum. So, by Lemma 2.4, the triple $(A, B + tX, C + tY)$ belongs to $\overline{G(3,7)}$ for all $t \in \mathrm{F}$, $t \neq 0$.

If $f = f^{'} = 0$. If $b = 0$, then we could introduce the matrix

$$Z = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

that clearly commutes with both A and C, so that it suffices to prove our case for all triples $(A, B + tZ, C)$ for $t \in \mathrm{F}$, $t \neq 0$. Now we may assume that $b \neq 0$. Let

$$X = \begin{bmatrix} b & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x & 0 & 0 & 0 & 0 \end{bmatrix}, Y = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -b^{'} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -b^{'} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -b^{'} & 0 \\ 0 & 0 & 0 & 0 & 0 & y & 0 \end{bmatrix},$$

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where $x = gb' - g'b$, $y = \frac{cb'-c'b}{d}$.

Then a straightforward computation reveals that $B + tX$ commutes with $C + tY$ for all $t \in \mathrm{F}$. Moreover for $t \neq 0$, $B + tX$ has more than one point in the its spectrum. So, by Lemma 2.4, the triple $(A, B+tX, C+tY)$ belongs to $\overline{G(3,7)}$ for all $t \in \mathrm{F}$, $t \neq 0$.

(4) Assume that $j' \neq 0$, and $d' \neq 0$. Without loss of generality we may assume that $j' = 1$. So we may assume that $j = 0$. The commutative relation of B and C implies that

$$bf' = b'f + kd', \tag{4.1}$$

$$bh' = b'h + d'l, \tag{4.2}$$

$$hf' = h'f + k, \tag{4.3}$$

$$fb' = f'b + l. \tag{4.4}$$

By the equation (4.1) and (4.4) we have

$$l + kd' = 0. \tag{4.5}$$

Now we will consider 2 cases.

(4-a) Assume that $l \neq 0$. Then due to equation (4.5) we have $k \neq 0$. If $d = 0$, then we could introduce the matrix

$$Z = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

that clearly commutes with both A and C, so that it suffices to prove our case for all triples $(A, B + tZ, C)$ for $t \in \mathrm{F}$, $t \neq 0$. Now we may assume that $d \neq 0$. Let

$$X = \begin{bmatrix} x & 0 & 0 & y & 0 & 0 & 0 \\ 0 & x & 0 & 0 & y & 0 & 0 \\ 0 & 0 & x & 0 & 0 & y & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & z & 0 & 0 & w & 0 \end{bmatrix}, Y = \begin{bmatrix} 0 & 0 & 0 & \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \beta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & 0 & 0 & \gamma & 0 \end{bmatrix},$$

where $x = h - \frac{b}{d'}$, $\beta = -h' + \frac{b'}{d'}$, $y = b - d'h$, $\alpha = b' - d'h'$, $z = -g\beta - g'x$,

$w = g\alpha - g'y$, and $\gamma = \frac{-c\beta - c'x + \alpha i - yi' + wd'}{d}$.

Then a direct computation reveals that $B + tX$ commutes with $C + tY$ for all $t \in \mathrm{F}$. If $x = \beta = 0$, then $b = d'h$ and $b' = d'h'$. Using the equation (4.2), we have $d'l = bh' - b'h = (d'h)h' - (d'h')h = 0$. Since $d'l \neq 0$, it is a contradiction. Therefore at least one of x and β is nonzero. Thus for $t \neq 0$ at least one of $B + tX$ and

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$C+tY$ has more than one point in the its spectrum. So, by Lemma 2.4, the triple $(A, B+tX, C+tY)$ belongs to $\overline{G(3,7)}$ for all $t \in \mathrm{F}$, $t \neq 0$.

(4-b) Assume that $l = 0$. Then due to equation (5) we have $k = 0$. Then B and C look like

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & b & c & d \\ 0 & 0 & 0 & 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & f & g & 0 & h & i & 0 \\ 0 & 0 & f & 0 & 0 & h & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, C = \begin{bmatrix} 0 & 0 & 0 & 0 & b^{'} & c^{'} & d^{'} \\ 0 & 0 & 0 & 0 & 0 & b^{'} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & f^{'} & g^{'} & 0 & h^{'} & i^{'} & 1 \\ 0 & 0 & f^{'} & 0 & 0 & h^{'} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

In this case we may assume that $d \neq 0$, say $d = 1$. Then we may assume that $d^{'} = 0$. If at least one of $h^{'}$ and h is nonzero. Let

$$X = \begin{bmatrix} 0 & 0 & 0 & h & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & h & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & h & 0 \\ h & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & h & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & h & 0 & 0 & 0 & 0 \\ 0 & 0 & x & 0 & 0 & y & -h^{'} \end{bmatrix}, Y = \begin{bmatrix} 0 & 0 & 0 & z & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & z & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & z & 0 \\ z & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & z & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & z & 0 & 0 & 0 & 0 \\ 0 & 0 & y & 0 & 0 & x & -h \end{bmatrix},$$

where $x = ih^{'} - i^{'}h$, $y = gh^{'} - ch^{'} - g^{'}h + c^{'}h$, and $z = h^{'}$.

Then a direct computation reveals that $B+tX$ commutes with $C+tY$. Moreover for $t \neq 0$, at least one of $B+tX$ and $C+tY$ has more than one point in the its spectrum. So, by Lemma 2.4, the triple $(A, B+tX, C+tY)$ belongs to $\overline{G(3,7)}$ for all $t \in \mathrm{F}$, $t \neq 0$.

If $h = h^{'} = 0$. Let

$$X = \begin{bmatrix} 0 & 0 & 0 & b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b & 0 \\ f & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & f & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & f & 0 & 0 & 0 & 0 \\ 0 & 0 & x & 0 & 0 & 0 & -f^{'} \end{bmatrix}, Y = \begin{bmatrix} 0 & 0 & 0 & y & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & y & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & y & 0 \\ z & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & z & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & z & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & w & -b \end{bmatrix},$$

where $x = if^{'} - i^{'}f$, $y = b^{'}$, $z = f^{'}$, and $w = b^{'}i - bi^{'}$,
if at least one of b and $f^{'}$ is not zero.

Let

$$X = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 & c & 1 \end{bmatrix}, Y = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & c & 0 & 0 & 0 & 0 \end{bmatrix},$$

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$$\text{if } b = f' = 0.$$

In both cases a straightforward computation reveals that $B + tX$ commutes with $C + tY$. In the first case for $t \neq 0$ at least one of $B + tX$ and $C + tY$ has more than one point in the its spectrum. So, by Lemma 2.4, the triple $(A, B + tX, C + tY)$ belongs to $\overline{G(3,7)}$ for all $t \in \mathrm{F}$, $t \neq 0$. In the second case $B + tX$ has more than one point in the its spectrum. So the triple $(A, B + tX, C + tY)$ belongs to $\overline{G(3,7)}$ for all $t \in \mathrm{F}$, $t \neq 0$. ▯

5. 3+2+2 case. In this section we give the result that a linear space of nilpotent commuting matrices of size 7×7 having a matrix of maximal rank with one Jordan block of order 3 and two Jordan blocks of order 2 can be perturbed by the generic triples.

THEOREM 5.1. *If in a 3-dimensional linear space $\mathfrak{L}$ of nilpotent commuting matrices of size 7×7 there is a matrix of maximal possible rank with one Jordan block of order 3 and two Jordan blocks of order 2, then any basis of this space belongs to $\overline{G(3,7)}$.*

Proof. We can write $A \in \mathfrak{L}$ of maximal in some basis as

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

If $B \in \mathfrak{L}$, then the structure of B is well known. It is nilpotent and we may add to it a polynomial in A, so that it looks like

$$B = \begin{bmatrix} 0 & 0 & 0 & a & b & c & d \\ 0 & 0 & 0 & 0 & a & 0 & c \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & e & f & 0 & g & h & i \\ 0 & 0 & e & 0 & 0 & 0 & h \\ 0 & j & k & l & m & 0 & n \\ 0 & 0 & j & 0 & l & 0 & 0 \end{bmatrix}.$$

Let C be a matrix in a $\mathfrak{L}$. Then C looks like

$$C = \begin{bmatrix} 0 & 0 & 0 & a' & b' & c' & d' \\ 0 & 0 & 0 & 0 & a' & 0 & c' \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & e' & f' & 0 & g' & h' & i' \\ 0 & 0 & e' & 0 & 0 & 0 & h' \\ 0 & j' & k' & l' & m' & 0 & n' \\ 0 & 0 & j' & 0 & l' & 0 & 0 \end{bmatrix}.$$

Since B is nilpotent, $hl = 0$. Now we may assume that at least one of h and l is 0. We will consider 2 separate cases.

Case 1. Assume that there is a matrix $B \in \mathfrak{L}$ such that $l \neq 0$, so that for

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any $B \in \mathfrak{L}$ the corresponding entry $h = 0$. In this case we may assume that $l = 1$. So we may assume that $l^{'} = 0$ and $h^{'} = 0$. Since the algebra generated by A and B contains AB, by Corollary 2.8, we may assume that $m = 0$ and $m^{'} = 0$. By the rank condition of B, we have $ce = 0$. So we will consider 3 subcases.

(i) Assume that there is a matrix $B \in \mathfrak{L}$ such that $e \neq 0$, so that for any $B \in \mathfrak{L}$ the corresponding entry $c = 0$. By assumption we may assume that $c^{'} = 0$. The commutative relation of B and C implies that $a^{'} = e^{'} = i^{'} = 0$. Since rank of B is less than or equal to 4, we have the following equations:

$$\det \begin{bmatrix} 0 & 0 & a & b & d \\ e & f & 0 & g & i \\ 0 & e & 0 & 0 & 0 \\ j & k & 1 & 0 & n \\ 0 & j & 0 & 1 & 0 \end{bmatrix} = e(e(d-an)+aij) = 0, \tag{5.1}$$

$$\det \begin{bmatrix} 0 & 0 & a & b & d \\ 0 & 0 & 0 & a & 0 \\ e & f & 0 & g & i \\ j & k & 1 & 0 & n \\ 0 & j & 0 & 1 & 0 \end{bmatrix} = aj(e(d-an)+aij) = 0, \tag{5.2}$$

$$\det \begin{bmatrix} 0 & 0 & a & b & d \\ 0 & 0 & 0 & a & 0 \\ e & f & 0 & g & i \\ 0 & e & 0 & 0 & 0 \\ j & k & 1 & 0 & n \end{bmatrix} = -ae(e(d-an)+aij) = 0. \tag{5.3}$$

We will consider 2 cases.

(i)-(1) Assume that $a = 0$. Then, by (5.1), we have $d = 0$. The commutativity relation of B and C implies that

$$d^{'} = b^{'} = 0,\ ij^{'} = g^{'}e,\ f^{'} + j^{'}n = jn^{'},\ g^{'} = n^{'}.$$

Let

$$X = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & e & 0 & 1 & 0 & 0 & n \\ 0 & 0 & e & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

A straightforward computation reveals that $B + tX$ commutes with C for all $t \in \mathrm{F}$. Moreover for $t \neq 0$, $B + tX$ has more than one point in the its spectrum. So, by Lemma 2.4, the triple $(A, B + tX, C)$ belongs to $\overline{G(3,7)}$ for all $t \in \mathrm{F}$, $t \neq 0$.

(i)-(2) Assume that $a \neq 0$. The commutative relation of B and C implies that

$$af^{'} + dj^{'} = b^{'}e + d^{'}j,\ ag^{'} = d^{'},\ ij^{'} = g^{'}e,\ f^{'} + j^{'}n = jn^{'}, g^{'} = n^{'} + aj^{'}.$$

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If $j^{'} = 0$, then we could introduce the matrix

$$Z = \begin{bmatrix} 0 & 0 & 0 & 0 & x & 0 & y \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & z & 0 & w & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & v \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where $x = \frac{d-a^2j-an}{e}$, $y = \frac{ai}{e}$, $z = \frac{ij}{e} - aj - n$, $w = \frac{i}{e}$, $v = \frac{i}{e} - a$

that clearly commutes with both A and B, so that it suffices to prove our case for all triples $(A, B, C + tZ)$ for $t \in \mathrm{F}$, $t \neq 0$. So we may assume that $j^{'} \neq 0$, say $j^{'} = 1$. Then we may assume that $j = 0$. So we have $af^{'} + d = b^{'}e$ and $f^{'} + n = 0$. From (5.1) we have $d - an = 0$. Therefore we have $b^{'} = 0$. Therefore C looks like

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & d^{'} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & f^{'} & 0 & g^{'} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & k^{'} & 0 & 0 & 0 & n^{'} \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Let

$$X = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & 0 & 0 & d \\ 0 & 0 & 0 & 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

A straightforward computation reveals that $B + tX$ commutes with C for all $t \in \mathrm{F}$. Moreover for $t \neq 0$, $B + tX$ has more than one point in the its spectrum. So, by Lemma 2.4, the triple $(A, B + tX, C)$ belongs to $\overline{G(3, 7)}$ for all $t \in \mathrm{F}$, $t \neq 0$.

(ii) Assume that there is a matrix $B \in \mathfrak{L}$ such that $c \neq 0$, so that for any $B \in \mathfrak{L}$ the corresponding entry $e = 0$. By assumption we may assume that $e^{'} = 0$. The commutativity relation of B and C implies that $c^{'} = i^{'} = j^{'} = 0$. Since rank of B is less than or equal to 4, we have the following equations:

$$\det \begin{bmatrix} 0 & a & b & c & d \\ 0 & 0 & a & 0 & c \\ f & 0 & g & 0 & i \\ k & 1 & 0 & 0 & n \\ j & 0 & 1 & 0 & 0 \end{bmatrix} = -(c(f - gj) + aij) = 0, \tag{5.4}$$

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$$\det \begin{bmatrix} 0 & 0 & b & c & d \\ 0 & 0 & a & 0 & c \\ 0 & f & g & 0 & i \\ j & k & 0 & 0 & n \\ 0 & j & 1 & 0 & 0 \end{bmatrix} = j(c(f-gj)+aij) = 0, \tag{5.5}$$

$$\det \begin{bmatrix} 0 & 0 & a & b & d \\ 0 & 0 & 0 & a & c \\ 0 & f & 0 & g & i \\ j & k & 1 & 0 & n \\ 0 & j & 0 & 1 & 0 \end{bmatrix} = ja(c(f-gj)+aij) = 0. \tag{5.6}$$

We will consider 2 cases.

(ii)-(1) Assume that $j = 0$. Then, by (5.4), we have $f = 0$. The commutativity relation of B and C implies that

$$f' = k' = 0,\ ag' = a'g + d',\ cn' = a'i,\ g' = n'.$$

Let

$$X = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & g \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

A straightforward computation reveals that $B + tX$ commutes with C for all $t \in \mathrm{F}$. Moreover for $t \neq 0$, $B + tX$ has more than one point in the its spectrum. So, by Lemma 2.4, the triple $(A, B + tX, C)$ belongs to $\overline{G(3, 7)}$ for all $t \in \mathrm{F}$, $t \neq 0$.

(ii)-(2) Assume that $j \neq 0$. The commutativity relation of B and C implies that

$$af' + ck' = a'f + d'j,\ ag' = a'g + d',\ cn' = a'i,\ f' = jn', ja' + g' = n'.$$

If $a' = 0$, then we could introduce the matrix

$$Z = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & x \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & y & 0 & z & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & w & 0 & 0 & 0 & v \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where $x = \frac{ai}{c} - aj - g$, $y = \frac{ij}{c}$, $z = \frac{i}{c} - j$, $w = \frac{f - aj^2 - gj}{c}$, $v = \frac{i}{c}$ that clearly commutes with both A and B, so that it suffices to prove our case for all triples $(A, B, C + tZ)$ for $t \in \mathrm{F}$, $t \neq 0$. So we may assume that $a' \neq 0$, say $a' = 1$. Then we may assume that $a = 0$. So we have $ck' = f + d'j$ and $d' + g = 0$. From (5.4) we have $f - gj = 0$. Therefore we have $k' = 0$. Let

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$$X = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & f \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & j & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & j \end{bmatrix}.$$

A straightforward computation reveals that $B + tX$ commutes with C for all $t \in \mathrm{F}$. Moreover for $t \neq 0$, $B + tX$ has more than one point in the its spectrum. So, by Lemma 2.4, the triple $(A, B + tX, C)$ belongs to $\overline{G(3,7)}$ for all $t \in \mathrm{F}$, $t \neq 0$.

(iii) Assume that there is a matrix $B \in \mathfrak{L}$ such that c and e are zero. If at least one of $c^{'}$ and $e^{'}$ is nonzero, then we can change a role of B and C. So we are done by (i) and (ii). Thus we may assume that $c^{'} = e^{'} = 0$. Since rank of B is less than or equal to 4,

$$\det \begin{bmatrix} 0 & 0 & a & b & d \\ 0 & 0 & 0 & a & 0 \\ 0 & f & 0 & g & i \\ j & k & 1 & 0 & n \\ 0 & j & 0 & 1 & 0 \end{bmatrix} = a^2 i j^2 = 0.$$

So at least one of a, i and j is zero. Now we will consider 7 possibilities.

(iii)-(1) Assume that there is a matrix $B \in \mathfrak{L}$ such that $i \neq 0$ and $j \neq 0$, so that for any $B \in \mathfrak{L}$ the corresponding entry $a = 0$. By assumption we may assume that $a^{'} = 0$. The commutativity relation of B and C implies that

$$d^{'} = i^{'} = j^{'} = 0,\ g^{'} = n^{'},\ f^{'} = jn^{'}.$$

If $g^{'} = 0$, then we could introduce the matrix

$$Z = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & j & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

that clearly commutes with both A and B, so that it suffices to prove our case for all triples $(A, B, C + tZ)$ for $t \in \mathrm{F}$, $t \neq 0$. So we may assume that $g^{'} \neq 0$. Let

$$X = \begin{bmatrix} g^{'} & 0 & 0 & -b^{'} & 0 & 0 & 0 \\ 0 & g^{'} & 0 & 0 & -b^{'} & 0 & 0 \\ 0 & 0 & g^{'} & 0 & 0 & 0 & 0 \\ 0 & -f^{'} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -f^{'} & 0 & 0 & 0 & 0 \\ 0 & -k^{'} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -k^{'} & 0 & 0 & 0 & 0 \end{bmatrix}.$$

A direct computation reveals that $B + tX$ commutes with C for all $t \in \mathrm{F}$. Moreover for $t \neq 0$ $B + tX$ has more than one point in the its spectrum. So, by Lemma 2.4,

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the triple $(A, B+tX, C)$ belongs to $\overline{G(3,7)}$ for all $t \in \mathrm{F}$, $t \neq 0$.

(iii)-(2) Assume that there is a matrix $B \in \mathfrak{L}$ such that $a \neq 0$ and $j \neq 0$, so that for any $B \in \mathfrak{L}$ the corresponding entry $i = 0$. By assumption we may assume that $i^{'} = 0$. The commutativity relation of B and C implies that

$$dj^{'} + af^{'} = d^{'}j + a^{'}f,\ ag^{'} = a^{'}g + d^{'},\ jn^{'} = f^{'} + j^{'}n,\ aj^{'} + n^{'} = g^{'} + a^{'}j.$$

Let

$$X = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -a & -b \\ 0 & 0 & 0 & 0 & 0 & 0 & -a \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & x \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix},\ Y = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & j^{'} & k^{'} & 0 & 0 & 0 & -aj^{'} \\ 0 & 0 & j^{'} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where $x = -g - aj$.

Then a straightforward computation reveals that $B + tX$ commutes with $C + tY$ for all $t \in \mathrm{F}$. Moreover for $t \neq 0$, $B + tX$ has more than one point in the its spectrum. So, by Lemma 2.4, the triple $(A, B+tX, C+tY)$ belongs to $\overline{G(3,7)}$ for all $t \in \mathrm{F}$, $t \neq 0$.

(iii)-(3) Assume that there is a matrix $B \in \mathfrak{L}$ such that $a \neq 0$ and $i \neq 0$, so that for any $B \in \mathfrak{L}$ the corresponding entry $j = 0$. By assumption we may assume that $j^{'} = 0$. The commutativity relation of B and C implies that

$$a^{'} = f^{'} = i^{'} = 0,\ g^{'} = n^{'},\ ag^{'} = d^{'}.$$

If $g^{'} = 0$, then we could introduce the matrix

$$Z = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

that clearly commutes with both A and B, so that it suffices to prove our case for all triples $(A, B, C+tZ)$ for $t \in \mathrm{F}$, $t \neq 0$. So we may assume that $g^{'} \neq 0$. Let

$$X = \begin{bmatrix} g^{'} & 0 & 0 & -b^{'} & 0 & -d^{'} & 0 \\ 0 & g^{'} & 0 & 0 & -b^{'} & 0 & -d^{'} \\ 0 & 0 & g^{'} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -k^{'} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -k^{'} & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then a straightforward computation reveals that $B + tX$ commutes with C for all $t \in \mathrm{F}$. Moreover for $t \neq 0$, $B + tX$ has more than one point in the its spectrum. So, by Lemma 2.4, the triple $(A, B+tX, C)$ belongs to $\overline{G(3,7)}$ for all $t \in \mathrm{F}$, $t \neq 0$.

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(iii)-(4) Assume that there is a matrix $B \in \mathfrak{L}$ such that $j \neq 0$. Then at least one of a and i is zero. If only one of a and i is zero, we are done by above subcases. So we may assume that there is a matrix $B \in \mathfrak{L}$ such that $j \neq 0$, so that for any $B \in \mathfrak{L}$ the corresponding entry $a = i = 0$. So we may assume that $a^{'} = i^{'} = 0$. The commutative relation of B and C implies that

$$d^{'} = 0,\ g^{'} = n^{'},\ dj^{'} = 0, \text{ and } f^{'} + nj^{'} = n^{'}j.$$

If $j^{'} = 0$. Now we may assume that $g^{'} \neq 0$. Indeed, if $g^{'} = 0$, then we could introduce the matrix

$$Z = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & j & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

that clearly commutes with both A and B, so that it suffices to prove our case for all triples $(A, B, C + tZ)$ for $t \in \mathrm{F}$, $t \neq 0$. Let

$$X = \begin{bmatrix} g^{'} & 0 & 0 & -b^{'} & 0 & 0 & 0 \\ 0 & g^{'} & 0 & 0 & -b^{'} & 0 & 0 \\ 0 & 0 & g^{'} & 0 & 0 & 0 & 0 \\ 0 & -f^{'} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -f^{'} & 0 & 0 & 0 & 0 \\ 0 & -k^{'} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -k^{'} & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then a straightforward computation reveals that $B + tX$ commutes with C for all $t \in \mathrm{F}$. Moreover for $t \neq 0$, $B + tX$ has more than one point in the its spectrum. So, by Lemma 2.4, the triple $(A, B + tX, C)$ belongs to $\overline{G(3, 7)}$ for all $t \in \mathrm{F}$, $t \neq 0$.

If $j^{'} \neq 0$, say $j^{'} = 1$. Then we may assume that $j = 0$. By the commutative condition, we have $d = 0$. Let

$$X = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -f^{'} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, Y = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -b^{'} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then a straightforward computation reveals that $B + tX$ commutes with $C + tY$ for all $t \in \mathrm{F}$. Moreover for $t \neq 0$, $B + tX$ has more than one point in the its spectrum. So, by Lemma 2.4, the triple $(A, B+tX, C+tY)$ belongs to $\overline{G(3, 7)}$ for all $t \in \mathrm{F}$, $t \neq 0$.

(iii)-(5) Assume that there is a matrix $B \in \mathfrak{L}$ such that $i \neq 0$. Then at least one of a and j is zero. If only one of a and j is zero, we are done by above subcases. So we may assume that there is a matrix $B \in \mathfrak{L}$ such that $i \neq 0$, so that for any $B \in \mathfrak{L}$ the corresponding entry $a = j = 0$. So we may assume that $a^{'} = j^{'} = 0$. The

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commutativity relation of B and C implies that

$$g^{'} = n^{'},\, d^{'} = i^{'} = f^{'} = 0.$$

If $g^{'} = 0$, then we could introduce the matrix

$$Z = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

that clearly commutes with both A and B, so that it suffices to prove our case for all triples $(A, B, C + tZ)$ for $t \in \mathrm{F}$, $t \neq 0$. So we may assume that $g^{'} \neq 0$. Let

$$X = \begin{bmatrix} g^{'} & 0 & 0 & -b^{'} & 0 & 0 & 0 \\ 0 & g^{'} & 0 & 0 & -b^{'} & 0 & 0 \\ 0 & 0 & g^{'} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -k^{'} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -k^{'} & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then a straightforward computation reveals that $B + tX$ commutes with C for all $t \in \mathrm{F}$. Moreover for $t \neq 0$, $B + tX$ has more than one point in the its spectrum. So, by Lemma 2.4, the triple $(A, B + tX, C)$ belongs to $\overline{G(3, 7)}$ for all $t \in \mathrm{F}$, $t \neq 0$.

(iii)-(6) Assume that there is a matrix $B \in \mathfrak{L}$ such that $a \neq 0$. Then at least one of i and j is zero. If only one of i and j is zero, we are done by above subcases. So we may assume that there is a matrix $B \in \mathfrak{L}$ such that $a \neq 0$, so that for any $B \in \mathfrak{L}$ the corresponding entry $i = j = 0$. So we may assume that $i^{'} = j^{'} = 0$. The commutativity relation of B and C implies that

$$f^{'} = 0,\, g^{'} = n^{'},\, a^{'} f = 0,\, a g^{'} = a^{'} g + d^{'}.$$

If $a^{'} = 0$. Now we may assume that $g^{'} \neq 0$. Indeed, if $g^{'} = 0$, then we could introduce the matrix

$$Z = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

that clearly commutes with both A and B, so that it suffices to prove our case for all triples $(A, B, C + tZ)$ for $t \in \mathrm{F}$, $t \neq 0$. Let

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$$X = \begin{bmatrix} g' & 0 & 0 & -b' & 0 & -d' & 0 \\ 0 & g' & 0 & 0 & -b' & 0 & -d' \\ 0 & 0 & g' & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -k' & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -k' & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then a straightforward computation reveals that $B + tX$ commutes with C for all $t \in \mathrm{F}$. Moreover for $t \neq 0$, $B + tX$ has more than one point in the its spectrum. So, by Lemma 2.4, the triple $(A, B + tX, C)$ belongs to $\overline{G(3, 7)}$ for all $t \in \mathrm{F}$, $t \neq 0$.

If $a' \neq 0$, say $a' = 1$. Then we may assume that $a = 0$. By the commutative condition, we have $f = 0$. Let

$$X = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -d' \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, Y = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -k' & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then a direct computation reveals that $B + tX$ commutes with $C + tY$ for all $t \in \mathrm{F}$. Moreover for $t \neq 0$, $B + tX$ has more than one point in the its spectrum. So, by Lemma 2.4, the triple $(A, B + tX, C + tY)$ belongs to $\overline{G(3, 7)}$ for all $t \in \mathrm{F}$, $t \neq 0$.

(iii)-(7) Assume that there is a matrix $B \in \mathfrak{L}$ such that a, i, and j are zero. If at least one of a', i' and j' is nonzero, then we can change the role of B and C. So we are done by above Cases. Thus we may assume that $a' = i' = j' = 0$. The commutativity relation of B and C implies that

$$d' = f' = 0, \; g' = n'.$$

If $g' = 0$, then we could introduce the matrix

$$Z = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

that clearly commutes with both A and B, so that it suffices to prove our case for all triples $(A, B, C + tZ)$ for $t \in \mathrm{F}$, $t \neq 0$. So we may assume that $g' \neq 0$. Let

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$$X = \begin{bmatrix} g' & 0 & 0 & -b' & 0 & 0 & 0 \\ 0 & g' & 0 & 0 & -b' & 0 & 0 \\ 0 & 0 & g' & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -k' & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -k' & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then a straightforward computation reveals that $B + tX$ commutes with C for all $t \in \mathrm{F}$. Moreover for $t \neq 0$, $B + tX$ has more than one point in the its spectrum. So, by Lemma 2.4, the triple $(A, B + tX, C)$ belongs to $\overline{G(3, 7)}$ for all $t \in \mathrm{F}$, $t \neq 0$.

Case 2. Assume that there is a matrix $B \in \mathfrak{L}$ such that $h \neq 0$, so that for any $B \in \mathfrak{L}$ the corresponding entry $l = 0$. So we may assume that $h = 1$. Thus we may assume that $h' = 0$ and $l' = 0$. Since the algebra generated by A and B contains AB, by Corollary 2.8, we may assume that $i = 0$ and $i' = 0$. By the rank condition of B, we have $aj = 0$. Thus at least one of a and j is zero. Now we will consider 3 subcases.

(i) Assume that there is a matrix $B \in \mathfrak{L}$ such that $j \neq 0$, so that for any $B \in \mathfrak{L}$ the corresponding entry $a = 0$. By assumption we may assume that $a' = 0$. The commutativity relation of B and C implies that $c' = m' = j' = 0$. Since rank of B is less than or equal to 4,

$$\det \begin{bmatrix} 0 & 0 & b & c & d \\ e & f & g & 1 & 0 \\ 0 & e & 0 & 0 & 1 \\ j & k & m & 0 & n \\ 0 & j & 0 & 0 & 0 \end{bmatrix} = j(j(b - gc) + ecm) = 0, \tag{5.7}$$

$$\det \begin{bmatrix} 0 & 0 & b & c & d \\ 0 & 0 & 0 & 0 & c \\ e & f & g & 1 & 0 \\ j & k & m & 0 & n \\ 0 & j & 0 & 0 & 0 \end{bmatrix} = -jc(j(b - ac) + ecm) = 0, \tag{5.8}$$

$$\det \begin{bmatrix} 0 & 0 & b & c & d \\ 0 & 0 & 0 & 0 & c \\ e & f & g & 1 & 0 \\ 0 & e & 0 & 0 & 1 \\ j & k & m & 0 & n \end{bmatrix} = ce(j(b - gc) + ecm) = 0. \tag{5.9}$$

We will consider 2 cases.

(i)-(1) Assume that $c = 0$. Then, by (5.7), we have $b = 0$. The commutativity relation of B and C implies that

$$d' = b' = 0,\ me' = n'j,\ k' + e'g = eg',\ g' = n'.$$

Let

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$$X = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & e & 0 & 0 & g & 1 & 0 \\ 0 & 0 & e & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then a straightforward computation reveals that $B + tX$ commutes with C for all $t \in \mathrm{F}$. Moreover for $t \neq 0$, $B + tX$ has more than one point in the its spectrum. So, by Lemma 2.4, the triple $(A, B + tX, C)$ belongs to $\overline{G(3,7)}$ for all $t \in \mathrm{F}$, $t \neq 0$.

(i)-(2) Assume that $c \neq 0$. The commutativity relation of B and C implies that

$$be^{'} + ck^{'} = b^{'}e + d^{'}j,\ cn^{'} = b^{'},\ me^{'} = n^{'}j,\ k^{'} + e^{'}g = eg^{'}, n^{'} = g^{'} + ce^{'}.$$

If $e^{'} = 0$, then we could introduce the matrix

$$Z = \begin{bmatrix} 0 & 0 & 0 & 0 & x & 0 & y \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & z & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & w & 0 & 0 & 0 & v \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where $x = \frac{mc}{j}$, $y = \frac{b - ec^2 - gc}{j}$, $z = \frac{m}{j} - c$, $w = \frac{me}{j} - ec - g$, $v = \frac{m}{j}$

that clearly commutes with both A and B, so that it suffices to prove our case for all triples $(A, B, C + tZ)$ for $t \in \mathrm{F}$, $t \neq 0$. So we may assume that $e^{'} \neq 0$, say $e^{'} = 1$. Then we may assume that $e = 0$. So we have $ck^{'} + b = d^{'}j$ and $k^{'} + g = 0$. From (5.7) we have $b - gc = 0$. Therefore we have $d^{'} = 0$. Let

$$X = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b & c & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & c \end{bmatrix}.$$

Then a direct computation reveals that $B + tX$ commutes with C for all $t \in \mathrm{F}$. Moreover for $t \neq 0$, $B + tX$ has more than one point in the its spectrum. So, by Lemma 2.4, the triple $(A, B + tX, C)$ belongs to $\overline{G(3,7)}$ for all $t \in \mathrm{F}$, $t \neq 0$.

(ii) Assume that there is a matrix $B \in \mathfrak{L}$ such that $a \neq 0$, so that for any $B \in \mathfrak{L}$ the corresponding entry $j = 0$. By assumption we may assume that $j^{'} = 0$. The commutativity relation of B and C implies that $a^{'} = e^{'} = m^{'} = 0$. Since rank of

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B is less than or equal to 4,

$$\det \begin{bmatrix} 0 & a & b & c & d \\ 0 & 0 & a & 0 & c \\ f & 0 & g & 1 & 0 \\ e & 0 & 0 & 0 & 1 \\ k & 0 & m & 0 & n \end{bmatrix} = a(a(k-en)+ecm) = 0, \tag{5.10}$$

$$\det \begin{bmatrix} 0 & 0 & b & c & d \\ 0 & 0 & a & 0 & c \\ e & f & g & 1 & 0 \\ 0 & e & 0 & 0 & 1 \\ 0 & k & m & 0 & n \end{bmatrix} = ec(a(k-en)+ecm) = 0, \tag{5.11}$$

$$\det \begin{bmatrix} 0 & 0 & a & b & d \\ 0 & 0 & 0 & a & c \\ e & f & 0 & g & 0 \\ 0 & e & 0 & 0 & 1 \\ 0 & k & 0 & m & n \end{bmatrix} = -ea(a(k-en)+ecm) = 0. \tag{5.12}$$

We will consider 2 cases.

(ii)-(1) Assume that $e = 0$. Then, by (5.10), we have $k = 0$. The commutativity relation of B and C implies that

$$f^{'} = k^{'} = 0,\ ag^{'} = c^{'}m,\ cn^{'} = b^{'} + c^{'}n,\ g^{'} = n^{'}.$$

Let

$$X = \begin{bmatrix} 0 & 0 & 0 & 0 & c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & n & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then a straightforward computation reveals that $B + tX$ commutes with C for all $t \in \mathrm{F}$. Moreover for $t \neq 0$, $B + tX$ has more than one point in the its spectrum. So, by Lemma 2.4, the triple $(A, B + tX, C)$ belongs to $\overline{G(3,7)}$ for all $t \in \mathrm{F}$, $t \neq 0$.

(ii)-(2) Assume that $e \neq 0$. The commutativity relation of B and C implies that

$$af^{'} + ck^{'} = b^{'}e + c^{'}k,\ ag^{'} = c^{'}m,\ cn^{'} = b^{'} + c^{'}n,\ k^{'} = eg^{'},\ ec^{'} + n^{'} = g^{'}.$$

If $c^{'} = 0$, then we could introduce the matrix

$$Z = \begin{bmatrix} 0 & 0 & 0 & 0 & x & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & y & 0 & z & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & w & 0 & 0 & 0 & v \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

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$$\text{where } x = \tfrac{mc}{a} - ec - n,\ y = \tfrac{k-e^2c-en}{a},\ z = \tfrac{m}{a},\ w = \tfrac{me}{a},\ v = \tfrac{m}{a} - e$$

that clearly commutes with both A and B, so that it suffices to prove our case for all triples $(A, B, C + tZ)$ for $t \in \mathrm{F}$, $t \neq 0$. So we may assume that $c^{'} \neq 0$, say $c^{'} = 1$. Then we may assume that $c = 0$. So we have $af^{'} = k + b^{'}e$ and $b^{'} + n = 0$. From (5.10) we have $k - en = 0$. Therefore we have $f^{'} = 0$. Let

$$X = \begin{bmatrix} 0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0 \\ 0&0&0&e&0&0&0 \\ 0&0&0&0&e&0&0 \\ 0&0&0&0&k&0&0 \\ 0&0&0&0&0&0&0 \end{bmatrix}.$$

Then a straightforward computation reveals that $B + tX$ commutes with C for all $t \in \mathrm{F}$. Moreover for $t \neq 0$, $B + tX$ has more than one point in the its spectrum. So, by Lemma 2.4, the triple $(A, B + tX, C)$ belongs to $\overline{G(3, 7)}$ for all $t \in \mathrm{F}$, $t \neq 0$.

(iii) Assume that there is a matrix $B \in \mathfrak{L}$ such that a and j are zero. If at least one of $a^{'}$ and $j^{'}$ is nonzero, then we can change the role of B and C. So we are done by (i) and (ii). Thus we may assume that $a^{'} = j^{'} = 0$. Since rank of B is less than or equal to 4,

$$\det \begin{bmatrix} 0&0&b&c&d \\ 0&0&0&0&c \\ e&f&g&1&0 \\ 0&e&0&0&1 \\ j&k&m&0&n \end{bmatrix} = c^2e^2m = 0.$$

Thus at least one of c, e and m is zero. So there are 7 possibilities.

(iii)-(1) Assume that there is a matrix $B \in \mathfrak{L}$ such that $e \neq 0$ and $m \neq 0$, so that for any $B \in \mathfrak{L}$ the corresponding entry $c = 0$. By assumption we may assume that $c^{'} = 0$. The commutativity relation of B and C implies that

$$b^{'} = e^{'} = m^{'} = 0,\ g^{'} = n^{'},\ k^{'} = eg^{'}.$$

If $g^{'} = 0$, then we could introduce the matrix

$$Z = \begin{bmatrix} 0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0 \\ 0&0&e&0&0&0&1 \\ 0&0&0&0&0&0&0 \end{bmatrix}$$

that clearly commutes with both A and B, so that it suffices to prove our case for all triples $(A, B, C + tZ)$ for $t \in \mathrm{F}$, $t \neq 0$. So we may assume that $g^{'} \neq 0$. Let

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$$X = \begin{bmatrix} g' & 0 & 0 & 0 & 0 & -d' & 0 \\ 0 & g' & 0 & 0 & 0 & 0 & -d' \\ 0 & 0 & g' & 0 & 0 & 0 & 0 \\ 0 & -f' & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -f' & 0 & 0 & 0 & 0 \\ 0 & -k' & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -k' & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then a straightforward computation reveals that $B + tX$ commutes with C for all $t \in \mathrm{F}$. Moreover for $t \neq 0$, $B + tX$ has more than one point in the its spectrum. So, by Lemma 2.4, the triple $(A, B + tX, C)$ belongs to $\overline{G(3, 7)}$ for all $t \in \mathrm{F}$, $t \neq 0$.

(iii)-(2) Assume that there is a matrix $B \in \mathfrak{L}$ such that $c \neq 0$ and $e \neq 0$, so that for any $B \in \mathfrak{L}$ the corresponding entry $m = 0$. By assumption we may assume that $m' = 0$. The commutativity relation of B and C implies that

$$be' + ck' = b'e + c'k,\ cn' = c'n + b',\ eg' = k' + e'g,\ ec' + n' = g' + e'c.$$

Let

$$X = \begin{bmatrix} 0 & 0 & 0 & -c & -d & 0 & 0 \\ 0 & 0 & 0 & 0 & -c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & x & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},\ Y = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & e' & f' & 0 & -e'c & 0 & 0 \\ 0 & 0 & e' & 0 & 0 & 0 & 0 \end{bmatrix},$$

where $x = -n - ce$.

Then a direct computation reveals that $B + tX$ commutes with $C + tY$ for all $t \in \mathrm{F}$. Moreover for $t \neq 0$, $B + tX$ has more than one point in the its spectrum. So, by Lemma 2.4, the triple $(A, B + tX, C + tY)$ belongs to $\overline{G(3, 7)}$ for all $t \in \mathrm{F}$, $t \neq 0$.

(iii)-(3) Assume that there is a matrix $B \in \mathfrak{L}$ such that $c \neq 0$ and $m \neq 0$, so that for any $B \in \mathfrak{L}$ the corresponding entry $e = 0$. By assumption we may assume that $e' = 0$. The commutativity relation of B and C implies that

$$c' = k' = m' = 0,\ g' = n',\ cn' = b'.$$

If $g' = 0$, then we could introduce the matrix

$$Z = \begin{bmatrix} 0 & 0 & 0 & 0 & c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

that clearly commutes with both A and B, so that it suffices to prove our case for all triples $(A, B, C + tZ)$ for $t \in \mathrm{F}$, $t \neq 0$. So we may assume that $g' \neq 0$. Let

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$$X = \begin{bmatrix} g^{'} & 0 & 0 & -b^{'} & 0 & -d^{'} & 0 \\ 0 & g^{'} & 0 & 0 & -b^{'} & 0 & -d^{'} \\ 0 & 0 & g^{'} & 0 & 0 & 0 & 0 \\ 0 & -f^{'} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -f^{'} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then a straightforward computation reveals that $B + tX$ commutes with C for all $t \in \mathrm{F}$. Moreover for $t \neq 0$, $B + tX$ has more than one point in the its spectrum. So, by Lemma 2.4, the triple $(A, B + tX, C)$ belongs to $\overline{G(3,7)}$ for all $t \in \mathrm{F}$, $t \neq 0$.

(iii)-(4) Assume that there is a matrix $B \in \mathfrak{L}$ such that $e \neq 0$. Then at least one of c and m is zero. If only one of c and m is zero, we are done by above subcases. So we may assume that there is a matrix $B \in \mathfrak{L}$ such that $e \neq 0$, so that for any $B \in \mathfrak{L}$ the corresponding entry $c = m = 0$. So we may assume that $c^{'} = m^{'} = 0$. The commutativity relation of B and C implies that

$$b^{'} = 0,\ g^{'} = n^{'},\ be^{'} = 0, \text{ and } k^{'} + ge^{'} = g^{'}e.$$

If $e^{'} = 0$. Now we may assume that $g^{'} \neq 0$. Indeed, if $g^{'} = 0$, then we could introduce the matrix

$$Z = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & e & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

that clearly commutes with both A and B, so that it suffices to prove our case for all triples $(A, B, C + tZ)$ for $t \in \mathrm{F}$, $t \neq 0$. Let

$$X = \begin{bmatrix} g^{'} & 0 & 0 & 0 & 0 & -d^{'} & 0 \\ 0 & g^{'} & 0 & 0 & 0 & 0 & -d^{'} \\ 0 & 0 & g^{'} & 0 & 0 & 0 & 0 \\ 0 & -f^{'} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -f^{'} & 0 & 0 & 0 & 0 \\ 0 & -k^{'} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -k^{'} & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then a straightforward computation reveals that $B + tX$ commutes with C for all $t \in \mathrm{F}$. Moreover for $t \neq 0$, $B + tX$ has more than one point in the its spectrum. So, by Lemma 2.4, the triple $(A, B + tX, C)$ belongs to $\overline{G(3,7)}$ for all $t \in \mathrm{F}$, $t \neq 0$. If $e^{'} \neq 0$, say $e^{'} = 1$. Then we may assume that $e = 0$. By the commutative condition, we have $b = 0$. Let

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$$X = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -k^{'} & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, Y = \begin{bmatrix} 0 & 0 & 0 & 0 & -d^{'} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then a direct computation reveals that $B + tX$ commutes with $C + tY$ for all $t \in \mathrm{F}$. Moreover for $t \neq 0$, $B + tX$ has more than one point in the its spectrum. So, by Lemma 2.4, the triple $(A, B + tX, C + tY)$ belongs to $\overline{G(3,7)}$ for all $t \in \mathrm{F}$, $t \neq 0$.

(iii)-(5) Assume that there is a matrix $B \in \mathfrak{L}$ such that $m \neq 0$. Then at least one of c and e is zero. If only one of c and e is zero, we are done by above subcases. So we may assume that there is a matrix $B \in \mathfrak{L}$ such that $m \neq 0$, so that for any $B \in \mathfrak{L}$ the corresponding entry $c = e = 0$. So we may assume that $c^{'} = e^{'} = 0$. The commutative relation of B and C implies that

$$g^{'} = n^{'},\ b^{'} = k^{'} = m^{'} = 0.$$

If $g^{'} = 0$, then we could introduce the matrix

$$Z = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

that clearly commutes with both A and B, so that it suffices to prove our case for all triples $(A, B, C + tZ)$ for $t \in \mathrm{F}$, $t \neq 0$. So we may assume that $g^{'} \neq 0$. Let

$$X = \begin{bmatrix} g^{'} & 0 & 0 & 0 & 0 & -d^{'} & 0 \\ 0 & g^{'} & 0 & 0 & 0 & 0 & -d^{'} \\ 0 & 0 & g^{'} & 0 & 0 & 0 & 0 \\ 0 & -f^{'} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -f^{'} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then a straightforward computation reveals that $B + tX$ commutes with C for all $t \in \mathrm{F}$. Moreover for $t \neq 0$, $B + tX$ has more than one point in the its spectrum. So, by Lemma 2.4, the triple $(A, B + tX, C)$ belongs to $\overline{G(3,7)}$ for all $t \in \mathrm{F}$, $t \neq 0$.

(iii)-(6) Assume that there is a matrix $B \in \mathfrak{L}$ such that $c \neq 0$. Then at least one of e and m is zero. If only one of e and m is zero, we are done by above subcases. So we may assume that there is a matrix $B \in \mathfrak{L}$ such that $c \neq 0$, so that for any $B \in \mathfrak{L}$ the corresponding entry $e = m = 0$. So we may assume that $e^{'} = m^{'} = 0$. The commutativity relation of B and C implies that

$$k^{'} = 0,\ g^{'} = n^{'},\ c^{'}k = 0,\ cn^{'} = c^{'}n + b^{'}.$$

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If $c^{'} = 0$. Now we may assume that $g^{'} \neq 0$. Indeed, if $g^{'} = 0$, then we could introduce the matrix

$$Z = \begin{bmatrix} 0 & 0 & 0 & 0 & c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

that clearly commutes with both A and B, so that it suffices to prove our case for all triples $(A, B, C + tZ)$ for $t \in \mathrm{F}$, $t \neq 0$. Let

$$X = \begin{bmatrix} g^{'} & 0 & 0 & -b^{'} & 0 & -d^{'} & 0 \\ 0 & g^{'} & 0 & 0 & -b^{'} & 0 & -d^{'} \\ 0 & 0 & g^{'} & 0 & 0 & 0 & 0 \\ 0 & -f^{'} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -f^{'} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then a direct computation reveals that $B + tX$ commutes with C for all $t \in \mathrm{F}$. Moreover for $t \neq 0$, $B + tX$ has more than one point in the its spectrum. So, by Lemma 2.4, the triple $(A, B + tX, C)$ belongs to $\overline{G(3,7)}$ for all $t \in \mathrm{F}$, $t \neq 0$.

If $c^{'} \neq 0$, say $c^{'} = 1$. Then we may assume that $c = 0$. By the commutative condition, we have $k = 0$. Let

$$X = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -b^{'} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, Y = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & f^{'} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then a straightforward computation reveals that $B + tX$ commutes with $C + tY$ for all $t \in \mathrm{F}$. Moreover for $t \neq 0$, $B + tX$ has more than one point in the its spectrum. So, by Lemma 2.4, the triple $(A, B + tX, C + tY)$ belongs to $\overline{G(3,7)}$ for all $t \in \mathrm{F}$, $t \neq 0$.

(iii)-(7) Assume that there is a matrix $B \in \mathfrak{L}$ such that c, e, and m are zero. If at least one of $c^{'}$, $e^{'}$, and $m^{'}$ is nonzero, then we can change the role of B and C. So we are done by above cases. Thus we may assume that $c^{'} = e^{'} = m^{'} = 0$. The commutativity relation of B and C implies that

$$b^{'} = k^{'} = 0, \, g^{'} = n^{'}.$$

If $g^{'} = 0$, then we could introduce the matrix

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$$Z = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

that clearly commutes with both A and B, so that it suffices to prove our case for all triples $(A, B, C + tZ)$ for $t \in \mathrm{F}$, $t \neq 0$. So we may assume that $g^{'} \neq 0$. Let

$$X = \begin{bmatrix} g^{'} & 0 & 0 & 0 & 0 & -d^{'} & 0 \\ 0 & g^{'} & 0 & 0 & 0 & 0 & -d^{'} \\ 0 & 0 & g^{'} & 0 & 0 & 0 & 0 \\ 0 & -f^{'} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -f^{'} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then a straightforward computation reveals that $B + tX$ commutes with C for all $t \in \mathrm{F}$. Moreover for $t \neq 0$, $B + tX$ has more than one point in the its spectrum. So, by Lemma 2.4, the triple $(A, B + tX, C)$ belongs to $\overline{G(3, 7)}$ for all $t \in \mathrm{F}$, $t \neq 0$.

Case 3. Assume that there is a matrix $B \in \mathfrak{L}$ such that h and l are zero. If at least one of $h^{'}$ and $l^{'}$ is nonzero, then we can change the role of B and C. So we are done by Case 1. and Case 2. Thus we may assume that $h^{'} = l^{'} = 0$. In this case we will consider 16 subcases.

(i) Assume that there is a matrix $B \in \mathfrak{L}$ such that $a \neq 0$, $c \neq 0$, $e \neq 0$, and $j \neq 0$. Now we may assume that $a = 1$. Since the algebra generated by A and B contains AB, by Corollary 2.8, we may assume that $b = 0$ and $b^{'} = 0$. Moreover we may assume that $a^{'} = 0$. The commutativity relation of B and C implies that

$$f^{'} + ck^{'} = d^{'}j,\ g^{'} + cm^{'} = 0,\ i^{'} + cn^{'} = 0,\ g^{'}e + i^{'}j = 0,\ m^{'}e + n^{'}j = 0,$$

and $c^{'} = e^{'} = j^{'} = 0$. If $d^{'} = 0$ and $m^{'} \neq 0$, then we could introduce the matrix

$$Z = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & j & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

If $d^{'} \neq 0$ and $m^{'} = 0$, then we could introduce the matrix

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$$Z = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -c & 0 & \frac{ce}{j} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -\frac{e}{j} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

If $d^{'} = 0$ and $m^{'} = 0$, then we could introduce the matrix

$$Z = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & j & 0 & -c & 0 & \frac{ce}{j} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -\frac{e}{j} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

In each case the matrix Z clearly commutes with both A and B, so that it suffices to prove our case for all triples $(A, B, C + tZ)$ for $t \in \mathrm{F}$, $t \neq 0$. So we may assume that $d^{'} \neq 0$, say $d^{'} = 1$. We can also assume that $m^{'} \neq 0$. So we may assume that $d = 0$. So B and C look like

$$B = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & c & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & c \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & e & f & 0 & g & 0 & i \\ 0 & 0 & e & 0 & 0 & 0 & 0 \\ 0 & j & k & 0 & m & 0 & n \\ 0 & 0 & j & 0 & 0 & 0 & 0 \end{bmatrix}, C = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & f^{'} & 0 & g^{'} & 0 & i^{'} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & k^{'} & 0 & m^{'} & 0 & n^{'} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Let

$$X = \begin{bmatrix} m^{'} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & m^{'} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & m^{'} & 0 & 0 & 0 & 0 \\ 0 & -k^{'} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -k^{'} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, Y = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & x \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where $x = \frac{g^{'}k^{'} - f^{'}m^{'}}{j}$.

Then a straightforward computation reveals that $B + tX$ commutes with $C + tY$ for all $t \in \mathrm{F}$. Moreover for $t \neq 0$, $B + tX$ has more than one point in the its spectrum. So, by Lemma 2.4, the triple $(A, B+tX, C+tY)$ belongs to $\overline{G(3,7)}$ for all $t \in \mathrm{F}$, $t \neq 0$.

(ii) Assume that there is a matrix $B \in \mathfrak{L}$ such that $a \neq 0$, $e \neq 0$, $j \neq 0$, and $c = 0$. If $B^4 \neq 0$ then B has only one Jordan block and consequently we are done by Theorem 2.2. So we may assume that $B^4 = 0$. So we have $ae = -cj$. So we have a relation $ae = 0$. Since $a \neq 0$ and $e \neq 0$, it is a contradiction. Therefore this case can not occur.

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(iii) Assume that there is a matrix $B \in \mathfrak{L}$ such that $a \neq 0$, $c \neq 0$, $e \neq 0$, and $j = 0$. By same reason in (ii), it can not occur.

(iv) Assume that there is a matrix $B \in \mathfrak{L}$ such that $a \neq 0$, $c \neq 0$, $j \neq 0$, and $e = 0$. It also can not occur.

(v) Assume that there is a matrix $B \in \mathfrak{L}$ such that $c \neq 0$, $e \neq 0$, $j \neq 0$, and $a = 0$. It also can not occur.

(vi) Assume that there is a matrix $B \in \mathfrak{L}$ such that $a \neq 0$, $e \neq 0$, $c = 0$, and $j = 0$. It also can not occur.

(vii) Assume that there is a matrix $B \in \mathfrak{L}$ such that $a \neq 0$, $j \neq 0$, $c = 0$, and $e = 0$. Now we may assume that $a = 1$. Since the algebra generated by A and B contains AB, by Corollary 2.8, we may assume that $b = 0$ and $b^{'} = 0$. Moreover we may assume that $a^{'} = 0$. The commutativity relation of B and C implies that

$$c^{'} = e^{'} = g^{'} = i^{'} = j^{'} = n^{'} = 0,\ f^{'} = d^{'}j.$$

Let

$$X = \begin{bmatrix} j & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & j & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & j & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},\ Y = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -jd^{'} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -k^{'} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then a straightforward computation reveals that $B + tX$ commutes with $C + tY$ for all $t \in \mathrm{F}$. Moreover for $t \neq 0$, $B + tX$ has more than one point in the its spectrum. So, by Lemma 2.4, the triple $(A, B+tX, C+tY)$ belongs to $\overline{G(3,7)}$ for all $t \in \mathrm{F}$, $t \neq 0$.

(viii) Assume that there is a matrix $B \in \mathfrak{L}$ such that $a \neq 0$, $c \neq 0$, $e = 0$, and $j = 0$. Now we may assume that $a = 1$. Since the algebra generated by A and B contains AB, by Corollary 2.8, we may assume that $b = 0$ and $b^{'} = 0$. Moreover we may assume that $a^{'} = 0$. The commutativity relation of B and C implies that

$$f^{'} + ck^{'} = c^{'}k,\ g^{'} + cm^{'} = c^{'}m,\ i^{'} + cn^{'} = c^{'}n,\ e^{'} = j^{'} = 0.$$

If $c^{'} = 0$, then we could introduce the matrix

$$Z = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & k & 0 & m & 0 & n \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

that clearly commutes with both A and B, so that it suffices to prove our case for all triples $(A, B, C + tZ)$ for $t \in \mathrm{F}$, $t \neq 0$. So we may assume that $c^{'} \neq 0$, say $c^{'} = 1$. We may assume that $c = 0$. Since the algebra generated by A and C contains AC, by Corollary 2.8, we may assume that $d^{'} = 0$ and $d = 0$. Let

$$\alpha = n^{'}i - mi - n^2 + ng + f,\ \beta = -m^2 + k^{'} - m^{'}n + mn^{'} + m^{'}g,\ x = (n - g)\alpha,$$

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$$y = -m\alpha + i\beta,\ z = (n-g)\beta.$$

If $\alpha \neq 0$, then we can find X and Y as follows.

$$X = \begin{bmatrix} -\alpha & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\alpha & 0 & 0 & 0 & 0 \\ 0 & x & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x & 0 & 0 & 0 & 0 \\ 0 & y & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & y & 0 & 0 & 0 & 0 \end{bmatrix},\ Y = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & y & 0 & 0 & 0 & \alpha & 0 \\ 0 & 0 & y & 0 & 0 & 0 & \alpha \\ 0 & z & 0 & \beta & 0 & 0 & 0 \\ 0 & 0 & z & 0 & \beta & 0 & 0 \end{bmatrix}.$$

Then a straightforward computation reveals that $B + tX$ commutes with $C + tY$ for all $t \in$ F. Moreover for $t \neq 0$, $B + tX$ has more than one point in the its spectrum. So, by Lemma 2.4, the triple $(A, B + tX, C + tY)$ belongs to $\overline{G(3,7)}$ for all $t \in$ F, $t \neq 0$.

If $\alpha = 0$. Let

$$X = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 & 0 & 0 \end{bmatrix},\ Y = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 & 0 & 0 \\ 0 & n-g & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & n-g & 0 & 1 & 0 & 0 \end{bmatrix}.$$

In this case we may assume that $i \neq 0$. Indeed, if $i = 0$, then we could introduce the matrix

$$Z = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

that clearly commutes with both A and C, so that it suffices to prove our case for all triples $(A, B + tZ, C)$ for $t \in$ F, $t \neq 0$. Then a straightforward computation reveals that $B + tX$ commutes with $C + tY$ for all $t \in$ F. Moreover the rank of $B + tX$ is greater than or equal to 5. We are done by Theorem 2.2. and Theorem 2.3.

(ix) Assume that there is a matrix $B \in \mathfrak{L}$ such that $e \neq 0$, $j \neq 0$, $a = 0$, and $c = 0$. Now we may assume that $e = 1$. Since the algebra generated by A and B contains AB, by Corollary 2.8, we may assume that $f = 0$ and $f^{'} = 0$. Moreover we may assume that $e^{'} = 0$. The commutativity relation of B and C implies that

$$b^{'} + jd^{'} = j^{'}d,\ g^{'} + ji^{'} = j^{'}i,\ m^{'} + jn^{'} = j^{'}n,\ a^{'} = c^{'} = 0.$$

If $j^{'} = 0$, then we could introduce the matrix

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$$Z = \begin{bmatrix} 0 & 0 & 0 & 0 & d & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & n & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

that clearly commutes with both A and B, so that it suffices to prove our case for all triples $(A, B, C + tZ)$ for $t \in \mathrm{F}$, $t \neq 0$. Thus we may assume that $j^{'} \neq 0$, say $j^{'} = 1$. We may assume that $j = 0$. Since the algebra generated by A and C contains AC, by Corollary 2.8, we may assume that $k^{'} = 0$ and $k = 0$. Let

$$\alpha = -b + n^2 - ng + im - n^{'}m,\ \beta = i^2 - d^{'} + i^{'}n - i^{'}g - in^{'},\ x = (n-g)\alpha,$$
$$y = m\beta - i\alpha,\ z = (n-g)\beta.$$

If $\alpha \neq 0$, then we can find X and Y as follows.

$$X = \begin{bmatrix} -\alpha & 0 & 0 & x & 0 & y & 0 \\ 0 & -\alpha & 0 & 0 & x & 0 & y \\ 0 & 0 & -\alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},\ Y = \begin{bmatrix} 0 & 0 & 0 & y & 0 & z & 0 \\ 0 & 0 & 0 & 0 & y & 0 & z \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \beta \\ 0 & 0 & 0 & \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha & 0 & 0 \end{bmatrix}.$$

Then a straightforward computation reveals that $B + tX$ commutes with $C + tY$ for all $t \in \mathrm{F}$. Moreover for $t \neq 0$, $B + tX$ has more than one point in the its spectrum. So, by Lemma 2.4, the triple $(A, B + tX, C + tY)$ belongs to $\overline{G(3,7)}$ for all $t \in \mathrm{F}$, $t \neq 0$.

If $\alpha = 0$. Let

$$X = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & m & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & m \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},\ Y = \begin{bmatrix} 0 & 0 & 0 & m & 0 & w & 0 \\ 0 & 0 & 0 & 0 & m & 0 & w \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where $w = n - g$.

In this case we may assume that $m \neq 0$. Indeed, if $m = 0$, then we could introduce the matrix

$$Z = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

that clearly commutes with both A and C, so that it suffices to prove our case for all triples $(A, B + tZ, C)$ for $t \in \mathrm{F}$, $t \neq 0$. Then a straightforward computation reveals

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that $B + tX$ commutes with $C + tY$ for all $t \in \mathrm{F}$. Moreover the rank of $B + tX$ is greater than or equal to 5. We are done by Theorem 2.2 and Theorem 2.3.

(x) Assume that there is a matrix $B \in \mathfrak{L}$ such that $e \neq 0$, $c \neq 0$, $a = 0$, and $j = 0$. Now we may assume that $c = 1$. Since the algebra generated by A and B contains AB, by Corollary 2.8, we may assume that $d = 0$ and $d^{'} = 0$. Moreover we may assume that $c^{'} = 0$. The commutativity relation of B and C implies that

$$a^{'} = e^{'} = h^{'} = j^{'} = m^{'} = n^{'} = 0,\ k^{'} = b^{'}e.$$

Let

$$X = \begin{bmatrix} e & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & e & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & e & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, Y = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -f^{'} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -eb^{'} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then a straight forward computation reveals that $B + tX$ commutes with $C + tY$ for all $t \in \mathrm{F}$. Moreover for $t \neq 0$, $B + tX$ has more than one point in the its spectrum. So, by Lemma 2.4, the triple $(A, B+tX, C+tY)$ belongs to $\overline{G(3,7)}$ for all $t \in \mathrm{F}$, $t \neq 0$.

(xi) Assume that there is a matrix $B \in \mathfrak{L}$ such that $c \neq 0$, $j \neq 0$, $a = 0$, and $e = 0$. It can not occur.

(xii) Assume that there is a matrix $B \in \mathfrak{L}$ such that $a \neq 0$, $c = 0$, $e = 0$, and $j = 0$. Now we may assume that $a = 1$. Since the algebra generated by A and B contains AB, we may assume that $b = 0$ and $b^{'} = 0$. Moreover we may assume that $a^{'} = 0$. The commutativity relation of B and C implies that

$$f^{'} = c^{'}k,\ g^{'} = c^{'}m,\ i^{'} = c^{'}n,\ e^{'} = j^{'} = 0.$$

If $c^{'} = 0$, then we could introduce the matrix

$$Z = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & k & 0 & m & 0 & n \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

that clearly commutes with both A and B, so that it suffices to prove our case for all triples $(A, B, C+tZ)$ for $t \in \mathrm{F}$, $t \neq 0$. So we may assume that $c^{'} \neq 0$, say $c^{'} = 1$. Since the algebra generated by A and C contains AC, by Corollary 2.8, we may assume that $d^{'} = 0$ and $d = 0$. Then B and C are exactly same matrices in the subcase (viii). Therefore we are done by (viii).

(xiii) Assume that there is a matrix $B \in \mathfrak{L}$ such that $e \neq 0$, $a = 0$, $c = 0$, and $j = 0$. Now we may assume that $e = 1$. Since the algebra generated by A and B contains AB, by Corollary 2.8, we may assume that $f = 0$ and $f^{'} = 0$. Moreover we

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may assume that $e^{'} = 0$. The commutativity relation of B and C implies that

$$b^{'} = dj^{'},\ g^{'} = hj^{'},\ m^{'} = nj^{'},\ a^{'} = c^{'} = 0.$$

If $j^{'} = 0$, then we could introduce the matrix

$$Z = \begin{bmatrix} 0 & 0 & 0 & 0 & d & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & n & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

that clearly commutes with both A and B, so that it suffices to prove our case for all triples $(A, B, C + tZ)$ for $t \in \mathrm{F}$, $t \neq 0$. Thus we may assume that $j^{'} \neq 0$, say $j^{'} = 1$. Since the algebra generated by A and C contains AC, by Corollary 2.8, we may assume that $k^{'} = 0$ and $k = 0$. Then the matrices B and C are exactly same matrices in the subcase (ix). Therefore we are done by (ix).

(xiv) Assume that there is a matrix $B \in \mathfrak{L}$ such that $j \neq 0$, $a = 0$, $c = 0$, and $e = 0$. Now we may assume that $j = 1$. Since the algebra generated by A and B contains AB, by Corollary 2.8, we may assume that $k = 0$ and $k^{'} = 0$. Moreover we may assume that $j^{'} = 0$. The commutativity relation of B and C implies that

$$d^{'} = be^{'},\ i^{'} = ge^{'},\ n^{'} = me^{'},\ a^{'} = c^{'} = 0.$$

If $e^{'} = 0$, then we could introduce the matrix

$$Z = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & b \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & g \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & m \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

that clearly commutes with both A and B, so that it suffices to prove our case for all triples $(A, B, C + tZ)$ for $t \in \mathrm{F}$, $t \neq 0$. Thus we may assume that $e^{'} \neq 0$, say $e^{'} = 1$. Since the algebra generated by A and C contains AC, by Corollary 2.8, we may assume that $f^{'} = 0$ and $f = 0$. If we change the role of B and C, then this case is same as the case (xiii). So we are done by (xiii).

(xv) Assume that there is a matrix $B \in \mathfrak{L}$ such that $c \neq 0$, $a = 0$, $e = 0$, and $j = 0$. Now we may assume that $c = 1$. Since the algebra generated by A and B contains AB, by Corollary 2.8, we may assume that $d = 0$ and $d^{'} = 0$. Moreover we may assume that $c^{'} = 0$. The commutativity relation of B and C implies that

$$k^{'} = a^{'}f,\ m^{'} = a^{'}g,\ n^{'} = a^{'}i,\ e^{'} = j^{'} = 0.$$

If $a^{'} = 0$, then we could introduce the matrix

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$$Z = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & f & 0 & g & 0 & i \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

that clearly commutes with both A and B, so that it suffices to prove our case for all triples $(A, B, C + tZ)$ for $t \in \mathrm{F}$, $t \neq 0$. So we may assume that $a^{'} \neq 0$, say $a^{'} = 1$. Since the algebra generated by A and C contains AC, by Corollary 2.8, we may assume that $b^{'} = 0$ and $b = 0$. If we change the role of B and C, then this case is same as the case (xii). So we are done by (xii).

(xvi) Assume that there is a matrix $B \in \mathfrak{L}$ such that $a = c = e = j = 0$. If at least one of $a^{'}$, $c^{'}$, $e^{'}$, and $j^{'}$ is nonzero, then we can change the role of B and C. So we are done by above cases. Thus we may assume that $a^{'} = c^{'} = e^{'} = j^{'} = 0$. If $i = 0$, then we could introduce the matrix

$$Z = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

that clearly commutes with both A and C, so that it suffices to prove our case for all triples $(A, B + tZ, C)$ for $t \in \mathrm{F}$, $t \neq 0$. So we may assume that $i \neq 0$. So we may assume that $i^{'} = 0$. Moreover we may assume that $g^{'} \neq 0$ and $n^{'} \neq 0$. Let

$$X = \begin{bmatrix} x & 0 & 0 & y & 0 & z & 0 \\ 0 & x & 0 & 0 & y & 0 & z \\ 0 & 0 & x & 0 & 0 & 0 & 0 \\ 0 & v & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & v & 0 & 0 & 0 & 0 \\ 0 & w & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & w & 0 & 0 & 0 & 0 \end{bmatrix},$$

where $x = g^{'}n^{'}$, $y = m^{'}d^{'} - n^{'}b^{'}$, $z = -d^{'}g^{'}$, $v = -n^{'}f^{'}$, and $w = m^{'}f^{'} - k^{'}g^{'}$. Then a straightforward computation reveals that $B + tX$ commutes with C for all $t \in \mathrm{F}$. Moreover for $t \neq 0$, $B + tX$ has more than one point in the its spectrum. So, by Lemma 2.4, the triple $(A, B + tX, C)$ belongs to $\overline{G(3, 7)}$ for all $t \in \mathrm{F}$, $t \neq 0$. $\square$

6. 3+2+1+1 case. In this section we give the result that a linear space of nilpotent commuting matrices of size 7×7 having a matrix of maximal rank with one Jordan block of order 3 and one Jordan block of order 2 can be perturbed by the generic triples.

THEOREM 6.1. *If in a 3-dimensional linear space $\mathfrak{L}$ of nilpotent commuting matrices of size 7×7 there is a matrix of maximal possible rank with one Jordan*

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block of order 3 *and one Jordan blocks of order* 2*, then any basis of this space belongs to* $\overline{G(3,7)}$*.*

Proof. We can write $A \in \mathfrak{L}$ of maximal in some basis as

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

If $B \in \mathfrak{L}$, then the structure of B is well known. It is nilpotent and we may add to it a polynomial in A, so that it looks like

$$B = \begin{bmatrix} 0 & 0 & 0 & a & b & c & d \\ 0 & 0 & 0 & 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & e & f & 0 & g & h & i \\ 0 & 0 & e & 0 & 0 & 0 & 0 \\ 0 & 0 & j & 0 & k & x & y \\ 0 & 0 & m & 0 & n & z & w \end{bmatrix}.$$

Since B is nilpotent we may assume that

$$\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} 0 & l \\ 0 & 0 \end{bmatrix}.$$

Therefore B looks like

$$B = \begin{bmatrix} 0 & 0 & 0 & a & b & c & d \\ 0 & 0 & 0 & 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & e & f & 0 & g & h & i \\ 0 & 0 & e & 0 & 0 & 0 & 0 \\ 0 & 0 & j & 0 & k & 0 & l \\ 0 & 0 & m & 0 & n & 0 & 0 \end{bmatrix}.$$

Let C be a matrix in a $\mathfrak{L}$. Then C looks like

$$C = \begin{bmatrix} 0 & 0 & 0 & a' & b' & c' & d' \\ 0 & 0 & 0 & 0 & a' & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & e' & f' & 0 & g' & h' & i' \\ 0 & 0 & e' & 0 & 0 & 0 & 0 \\ 0 & 0 & j' & 0 & k' & 0 & l' \\ 0 & 0 & m' & 0 & n' & 0 & 0 \end{bmatrix}.$$

Since the rank of B is less than or equal to 3,

$$\det \begin{bmatrix} 0 & 0 & a & b \\ 0 & 0 & 0 & a \\ e & f & 0 & g \\ 0 & e & 0 & 0 \end{bmatrix} = ea = 0.$$

Thus at least one of a and e is zero. Now we will consider 3 cases.

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Case 1. Assume that there is a matrix $B \in \mathfrak{L}$ such that $a \neq 0$, so that for any $B \in \mathfrak{L}$ the corresponding entry $e = 0$. By assumption we may assume that $e^{'} = 0$. We may assume that $a = 1$. So we may assume that $a^{'} = 0$. Since the algebra generated by A and B contains AB, by Corollary 2.8, we may assume that $b = 0$ and $b^{'} = 0$. By the commutativity condition of B and C we have $h^{'} = 0$. Since $B^3 = 0$,

$$hj + im + clm = hk + in + cln = hl = hlm = hln = 0.$$

So in this case at least one of h and l is zero. Now we will consider 3 possibilities.

(i) Assume that there is a matrix $B \in \mathfrak{L}$ such that $h \neq 0$, so that for any $B \in \mathfrak{L}$ the corresponding entry $l = 0$. By assumption we may assume that $l^{'} = 0$. Since $rank(B) \leq 3$, we have $hj = hm = 0$. So we have $j = m = 0$. The commutativity relation of B and C implies that

$i^{'} = 0$, $g^{'} + ck^{'} + dn^{'} = c^{'}k + d^{'}n$, $f^{'} + cj^{'} + dm^{'} = 0$, $hk^{'} + in^{'} = 0$, $hj^{'} + im^{'} = 0$.

Let

$$X = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \; Y = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c^{'} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then a straightforward computation reveals that $B + tX$ commutes with $C + tY$ for all $t \in \mathrm{F}$. Moreover for $t \neq 0$, $rank(B + tX) \geq 4$. We are done by Theorem 3.1, 4.1, and 5.1.

(ii) Assume that there is a matrix $B \in \mathfrak{L}$ such that $l \neq 0$, so that for any $B \in \mathfrak{L}$ the corresponding entry $h = 0$. By assumption we may assume that $h^{'} = 0$. Since $rank(B) \leq 3$, we have $lm = 0$. So we have $m = 0$. The commutativity condition implies that $m^{'} = 0$.

(ii)-(1) If there is a matrix $B \in \mathfrak{L}$ such that $n \neq 0$. Let

$$X = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & n \end{bmatrix}, \; Y = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & x & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & y & 0 & z & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where $x = \frac{dn^{'} - d^{'}n}{l}$, $y = xj$, $z = xk$.

Then a direct computation reveals that $B + tX$ commutes with $C + tY$ for all $t \in \mathrm{F}$. Moreover for $t \neq 0$, $B + tX$ has more than one point in the its spectrum. So, by Lemma 2.4, the triple $(A, B + tX, C + tY)$ belongs to $\overline{G(3,7)}$ for all $t \in \mathrm{F}$, $t \neq 0$.

(ii)-(2) If there is a matrix $B \in \mathfrak{L}$ such that $n = 0$. Then, by commutativity condition, we have $n^{'} = 0$. If $i^{'} = 0$, then we could introduce the matrix

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$$Z = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \frac{1}{l} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{j}{l} & 0 & \frac{k}{l} & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

that clearly commutes with both A and B, so that it suffices to prove our case for all triples $(A, B, C + tZ)$ for $t \in$ F, $t \neq 0$. So we may assume that $i' \neq 0$, say $i' = 1$. Then we may assume that $i = 0$. By the rank condition, $lf = 0$. So we have $f = 0$. Let

$$X = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then a straightforward computation reveals that $B + tX$ commutes with C for all $t \in$ F. Moreover for $t \neq 0$, $rank(B + tX) \geq 4$. So, by Theorem 3.1, 4.1, and 6.1, the triple $(A, B + tX, C)$ belongs to $\overline{G(3, 7)}$ for all $t \in$ F, $t \neq 0$.

(iii) Assume that there is a matrix $B \in \mathfrak{L}$ such that $h = l = 0$. If l' is not zero, then we can change the role of B and C. So we are done by above cases. Thus we may assume that $l' = 0$. Since $B^3 = 0$, $in = im = 0$. So we will consider 2 possibilities.

(iii)-(1) Assume that there is a matrix $B \in \mathfrak{L}$ such that $i \neq 0$. Then we have $m = n = 0$. Since $rank(B) \leq 3$, $ij = 0$. So we have $j = 0$. Let

$$X = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, Y = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c' & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then a straightforward computation reveals that $B + tX$ commutes with $C + tY$ for all $t \in$ F. Moreover for $t \neq 0$, $rank(B + tX) \geq 4$. Thus, by Theorem 3.1, 4.1, and 5.1, the triple $(A, B + tX, C + tY)$ belongs to $\overline{G(3, 7)}$ for all $t \in$ F, $t \neq 0$.

(iii)-(2) Assume that there is a matrix $B \in \mathfrak{L}$ such that $i = 0$. Then the commutativity condition implies that $i' = 0$. The commutative relation of B and C implies that

$$f' + cj' + dm' = c'j + d'm, \; g' + ck' + dn' = c'k + d'n.$$

If $d' = 0$, then we could introduce the matrix

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$$Z = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & m & 0 & n & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

that clearly commutes with both A and B, so that it suffices to prove our case for all triples $(A, B, C + tZ)$ for $t \in \mathrm{F}$, $t \neq 0$. So we may assume that $d^{'} \neq 0$, say $d^{'} = 1$. Then we may assume that $d = 0$. Now the commutativity relation of B and C implies that

$$f^{'} + cj^{'} = c^{'}j + m,\ g^{'} + ck^{'} = c^{'}k + n.$$

If $n^{'} = 0$, then we could introduce the matrix

$$Z = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

that clearly commutes with both A and B, so that it suffices to prove our case for all triples $(A, B, C + tZ)$ for $t \in \mathrm{F}$, $t \neq 0$. So we may assume that $n^{'} \neq 0$. Finally we also assume that $(j^{'}, k^{'})$ and $(m^{'}, n^{'})$ are linearly dependent. If not for each $t \in \mathrm{F}$, $t \neq 0$ we could introduce the matrix

$$Z = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{j^{'}c}{t} & 0 & \frac{k^{'}c}{t} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{j^{'}}{t} & 0 & -\frac{k^{'}}{t} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

that clearly commutes with both A and B, so that it suffices to prove our case for all triples $(A, B, C + tZ)$ for $t \in \mathrm{F}$, $t \neq 0$. Let

$$X = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & \frac{m^{'}}{n^{'}} & 0 & 0 & 0 & c & \frac{g^{'}}{n^{'}} \\ 0 & 0 & \frac{m^{'}}{n^{'}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},\ Y = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c^{'} & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then a direct computation reveals that $B + tX$ commutes with $C + tY$ for all $t \in \mathrm{F}$. Moreover for $t \neq 0$, $B + tX$ has more than one point in the its spectrum. So, by Lemma 2.4, the triple $(A, B + tX, C + tY)$ belongs to $\overline{G(3, 7)}$ for all $t \in \mathrm{F}$, $t \neq 0$.

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Case 2. Assume that there is a matrix $B \in \mathfrak{L}$ such that $e \neq 0$, so that for any $B \in \mathfrak{L}$ the corresponding entry $a = 0$. We may assume that $e = 1$. So we may assume that $e^{'} = 0$ and $a^{'} = 0$. Since the algebra generated by A and B contains AB, by Corollary 2.8, we may assume that $f = 0$ and $f^{'} = 0$. By the commutative condition of B and C we have $n^{'} = 0$. Since $B^3 = 0$,

$$ck + dn + clm = hk + in + hlm = ln = cln = hln = 0.$$

Now we will consider 3 possibilities.

(i) Assume that there is a matrix $B \in \mathfrak{L}$ such that $n \neq 0$, so that for any $B \in \mathfrak{L}$ the corresponding entry $l = 0$. By assumption we may assume that $l^{'} = 0$. Since $rank(B) \leq 3$, we have $dn = cn = 0$. So we have $d = c = 0$. Let

$$X = \begin{bmatrix} 0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0 \end{bmatrix}, Y = \begin{bmatrix} 0&0&0&0&m^{'}&0&0 \\ 0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0 \end{bmatrix}.$$

Then a straightforward computation reveals that $B + tX$ commutes with $C + tY$ for all $t \in \mathrm{F}$. Since for $t \neq 0$ $rank(B + tX) \geq 4$, the triple $(A, B + tX, C + tY)$ belongs to $\overline{G(3, 7)}$ for all $t \in \mathrm{F}$, $t \neq 0$.

(ii) Assume that there is a matrix $B \in \mathfrak{L}$ such that $l \neq 0$, so that for any $B \in \mathfrak{L}$ the corresponding entry $n = 0$. By assumption we may assume that $n^{'} = 0$. Since $rank(B) \leq 3$, we have $lc = 0$. So we have $c = 0$. The commutative condition implies that $c^{'} = 0$.

(ii)-(1) If there is a matrix $B \in \mathfrak{L}$ such that $h \neq 0$. Let

$$X = \begin{bmatrix} 0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0 \\ 0&0&0&0&0&h&0 \\ 0&0&0&0&0&0&0 \end{bmatrix}, Y = \begin{bmatrix} 0&0&0&0&x&0&0 \\ 0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0 \\ 0&0&0&0&y&0&0 \\ 0&0&0&0&0&0&0 \\ 0&0&0&0&0&h^{'}&0 \\ 0&0&z&0&0&0&0 \end{bmatrix},$$

where $x = zd$, $y = zi$, $z = \frac{jh^{'} - j^{'}h}{l}$.

Then a straightforward computation reveals that $B + tX$ commutes with $C + tY$ for all $t \in \mathrm{F}$. Since for $t \neq 0$ $B + tX$ has more than one point in the its spectrum, the triple $(A, B + tX, C + tY)$ belongs to $\overline{G(3, 7)}$ for all $t \in \mathrm{F}$, $t \neq 0$.

(ii)-(2) If there is a matrix $B \in \mathfrak{L}$ such that $h = 0$. Then, by commutative condition, we have $h^{'} = 0$. In this case we may assume that $k^{'} \neq 0$, say $k^{'} = 1$. Then we may assume that $k = 0$. By the rank condition, $lb = 0$. So we have $b = 0$. Let

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$$X = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then a straightforward computation reveals that $B + tX$ commutes with C for all $t \in \mathrm{F}$. Since for $t \neq 0$ $rank(B + tX) \geq 4$, the triple $(A, B + tX, C)$ belongs to $\overline{G(3, 7)}$ for all $t \in \mathrm{F}$, $t \neq 0$.

(iii) Assume that there is a matrix $B \in \mathfrak{L}$ such that $l = n = 0$. If $l^{'}$ is not zero, then we can change the role of B and C. So we are done by above cases. Thus we may assume that $l^{'} = 0$. Since $B^3 = 0$, $ck = hk = 0$.

(iii)-(1) Assume that there is a matrix $B \in \mathfrak{L}$ such that $k \neq 0$. Then we have $c = h = 0$. Since $rank(B) \leq 3$, $ij = 0$. So we have $j = 0$. Let

$$X = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, Y = \begin{bmatrix} 0 & 0 & 0 & 0 & m^{'} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then a direct computation reveals that $B + tX$ commutes with $C + tY$ for all $t \in \mathrm{F}$. Since for $t \neq 0$ $rank(B + tX) \geq 4$, the triple $(A, B + tX, C + tY)$ belongs to $\overline{G(3, 7)}$ for all $t \in \mathrm{F}$, $t \neq 0$.

(iii)-(2) Assume that there is a matrix $B \in \mathfrak{L}$ such that $k = 0$. So the commutative condition implies that $k^{'} = 0$. The commutative relation of B and C implies that

$$b^{'} + c^{'}j + d^{'}m = cj^{'} + dm^{'},\ g^{'} + h^{'}j + i^{'}m = hj^{'} + im^{'}.$$

If $m^{'} = 0$, then we could introduce the matrix

$$Z = \begin{bmatrix} 0 & 0 & 0 & 0 & d & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

that clearly commutes with both A and B, so that it suffices to prove our case for all triples $(A, B, C + tZ)$ for $t \in \mathrm{F}$, $t \neq 0$. So we may assume that $m^{'} \neq 0$, say $m^{'} = 1$. Then we may assume that $m = 0$. Now the commutativity relation of B and C implies that

$$b^{'} + c^{'}j = cj^{'} + d,\ g^{'} + h^{'}j = hj^{'} + i.$$

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If $i^{'} = 0$, then we could introduce the matrix

$$Z = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

that clearly commutes with both A and B, so that it suffices to prove our case for all triples $(A, B, C + tZ)$ for $t \in \mathrm{F}$, $t \neq 0$. So we may assume that $i^{'} \neq 0$. Finally we also assume that $(c^{'}, h^{'})$ and $(d^{'}, i^{'})$ are linearly dependent. If not for each $t \in \mathrm{F}$, $t \neq 0$ we could introduce the matrix

$$Z = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{c^{'}j}{t} & -\frac{c^{'}}{t} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{h^{'}j}{t} & -\frac{h^{'}}{t} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

that clearly commutes with both A and B, so that it suffices to prove our case for all triples $(A, B, C + tZ)$ for $t \in \mathrm{F}$, $t \neq 0$. Let

$$X = \begin{bmatrix} -1 & 0 & 0 & \frac{d^{'}}{i^{'}} & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & \frac{d^{'}}{i^{'}} & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & j & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{g^{'}}{i^{'}} & 0 & 0 \end{bmatrix}, Y = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & j^{'} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

Then a direct computation reveals that $B + tX$ commutes with $C + tY$ for all $t \in \mathrm{F}$. Moreover for $t \neq 0$, $B + tX$ has more than one point in the its spectrum. So, by Lemma 2.4, the triple $(A, B + tX, C + tY)$ belongs to $\overline{G(3, 7)}$ for all $t \in \mathrm{F}$, $t \neq 0$.

Case 3. Assume that there is a matrix $B \in \mathfrak{L}$ such that $a = e = 0$. If one of $a^{'}$ and $e^{'}$ is nonzero, then we can change the role of B and C. So we are done by above Case 1 and Case 2. So we may assume that $a^{'} = e^{'} = 0$. If $B^3 \neq 0$ then B has only one Jordan block and consequently we are done by Theorem 2.2. So we may assume that $B^3 = 0$. So we have $clm = cln = hlm = hln = 0$. We will consider 2 subcases.

(i) Assume that there is a matrix $B \in \mathfrak{L}$ such that $l \neq 0$. So we may assume that $l = 1$ and $l^{'} = 0$. Since $B^3 = 0$ and $l \neq 0$, we have $cm = cn = hm = hn = 0$. We will consider 4 subcases.

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(i)-(1) Assume that there is a matrix $B \in \mathfrak{L}$ such that $c \neq 0$ and $h \neq 0$, so that for any $B \in \mathfrak{L}$ the corresponding entry $m = n = 0$. So we may assume that $m' = n' = 0$. The commutative relation of B and C implies that

$$c' = h' = j' = k' = 0.$$

Let

$$X = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then a straightforward computation reveals that $B + tX$ commutes with C for all $t \in \mathrm{F}$. Moreover for $t \neq 0$, $B + tX$ has more than one point in the its spectrum. So, by Lemma 2.4, the triple $(A, B + tX, C)$ belongs to $\overline{G(3, 7)}$ for all $t \in \mathrm{F}$, $t \neq 0$.

(i)-(2) Assume that there is a matrix $B \in \mathfrak{L}$ such that $c = 0$ and $h \neq 0$, so that for any $B \in \mathfrak{L}$ the corresponding entry $m = n = 0$. So we may assume that $m' = n' = 0$. By commutative condition we have $c' = 0$. The commutativity relation of B and C implies that

$$h' = j' = k' = 0.$$

Let

$$X = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then a straightforward computation reveals that $B + tX$ commutes with C for all $t \in \mathrm{F}$. Moreover for $t \neq 0$, $B + tX$ has more than one point in the its spectrum. So, by Lemma 2.4, the triple $(A, B + tX, C)$ belongs to $\overline{G(3, 7)}$ for all $t \in \mathrm{F}$, $t \neq 0$.

(i)-(3) Assume that there is a matrix $B \in \mathfrak{L}$ such that $c \neq 0$ and $h = 0$, so that for any $B \in \mathfrak{L}$ the corresponding entry $m = n = 0$. So we may assume that $m' = n' = 0$. By commutative condition we have $h' = 0$. The commutativity relation of B and C implies that

$$c' = j' = k' = m' = n' = 0.$$

Let

$$X = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

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Then a straightforward computation reveals that $B + tX$ commutes with C for all $t \in \mathrm{F}$. Moreover for $t \neq 0$, $B + tX$ has more than one point in the its spectrum. So, by Lemma 2.4, the triple $(A, B + tX, C)$ belongs to $\overline{G(3, 7)}$ for all $t \in \mathrm{F}$, $t \neq 0$.

(i)-(4) Assume that there is a matrix $B \in \mathfrak{L}$ such that $c = 0$ and $h = 0$. The commutativity relation of B and C implies that

$$c' = h' = m' = n' = d'm = d'n = i'm = i'n = 0.$$

Let

$$X = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & x \end{bmatrix}, \text{ where } x = m \text{ or } x = n.$$

Then a straightforward computation reveals that $B + tX$ commutes with C for all $t \in \mathrm{F}$. If at least one of m and n is nonzero, then for $t \neq 0$, $B + tX$ has more than one point in the its spectrum. So, by Lemma 2.4, the triple $(A, B + tX, C)$ belongs to $\overline{G(3, 7)}$ for all $t \in \mathrm{F}$, $t \neq 0$. Now we may assume that $m = n = 0$. In this case we may assume that $d' \neq 0$, say $d' = 1$. Then we may assume that $d = 0$. Let

$$X = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, Y = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -j' & 0 & -k' & 0 & 0 \end{bmatrix}.$$

Then a straightforward computation reveals that $B + tX$ commutes with $C + tY$ for all $t \in \mathrm{F}$. Since for $t \neq 0$, $B + tX$ has more than one point in the its spectrum, the triple $(A, B + tX, C + tY)$ belongs to $\overline{G(3, 7)}$ for all $t \in \mathrm{F}$, $t \neq 0$.

(ii) Assume that there is a matrix $B \in \mathfrak{L}$ such that $l = 0$. If l' is not zero, then we can change the role of B and C. So we are done by above cases. Thus we may assume that $l' = 0$. The commutative relation of B and C implies that

$$cj' + dm' = c'j + d'm,\ ck' + dn' = c'k + d'n,\ hj' + im' = h'j + i'm,$$
$$hk' + in' = h'k + i'n.$$

(ii)-(1) Suppose that (m, n) and (m', n') are linearly independent. We may assume that $(m, n) = (1, 0)$ and $(m', n') = (0, 1)$. By the rank condition of B, we have $k(ci - dh) = 0$.

(ii)-(1)-(a) Assume that there is a matrix $B \in \mathfrak{L}$ such that $k = 0$. Then we have

$$d' = cj' - c'j,\ d = -ck',\ i' = hj' - h'j,\ i = -hk'.$$

If $c' = 0$, then we could introduce the matrix

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$$Z = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & -j \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

that clearly commutes with both A and B, so that it suffices to prove our case for all triples $(A, B, C + tZ)$ for $t \in \mathrm{F}$, $t \neq 0$. So we may assume that $c^{'} \neq 0$. Let

$$Y = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & g \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then a straightforward computation reveals that $C + tY$ commutes with B for all $t \in \mathrm{F}$. Moreover $rank(C + tY) \geq 4$ for $t \neq 0$. So we can use Theorems 3.1, 4.1, and 5.1. Therefore the triple $(A, B, C + tY)$ belongs to $\overline{G(3, 7)}$ for all $t \in \mathrm{F}$, $t \neq 0$.

(ii)-(1)-(b) Assume that there is a matrix $B \in \mathfrak{L}$ such that $k \neq 0$. Let

$$X = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -h^{'} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -h^{'} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & h & i \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, Y = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & x \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & y & 0 & 0 & 0 & 0 & z \\ 0 & 0 & y & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & h^{'} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & h^{'} \end{bmatrix},$$

where $x = by + b^{'}h^{'}$, $y = -\frac{i^{'}+k^{'}h^{'}}{k}$, $z = gy + g^{'}h^{'}$.

Then a straightforward computation reveals that $B + tX$ commutes with $C + tY$ for all $t \in \mathrm{F}$. So if at least one of h and $h^{'}$ is nonzero, then the triple $(A, B + tX, C + tY)$ belongs to $\overline{G(3, 7)}$ for all $t \in \mathrm{F}$, $t \neq 0$. Suppose $h = h^{'} = 0$. Let

$$X = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & bc \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & ck \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, Y = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & x \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & c^{'} & 0 & -c & 0 & 0 & y \\ 0 & 0 & c^{'} & 0 & -c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & z \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where $x = bc^{'}$, $y = fc + gc^{'}$, $z = c^{'}k$.

Then a straightforward computation reveals that $B + tX$ commutes with $C + tY$ for all $t \in \mathrm{F}$. If $c \neq 0$, then for $t \neq 0$ $C + tY$ has more than one point in the its spectrum. If $c^{'} \neq 0$ and $c = 0$, then $rank(C + tY) \geq 4$ for $t \neq 0$. So if at least one of c and $c^{'}$ is nonzero, then the triple $(A, B + tX, C + tY)$ belongs to $\overline{G(3, 7)}$ for all $t \in \mathrm{F}$, $t \neq 0$.

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Now we may assume that $c = c^{'} = 0$. Let

$$X = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -b \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -k \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, Y = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -f \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then a straightforward computation reveals that $B + tX$ commutes with $C + tY$ for all $t \in$ F. Moreover for $t \neq 0$, $C + tY$ has more than one point in the its spectrum. So, by Lemma 2.4, the triple $(A, B+tX, C+tY)$ belongs to $\overline{G(3,7)}$ for all $t \in$ F, $t \neq 0$.

(ii)-(2) Suppose that (m, n) and $(m^{'}, n^{'})$ are linearly dependent. We may assume that $(m^{'}, n^{'}) = (0, 0)$.

(ii)-(2)-(a) Assume that there is a matrix $C \in \mathfrak{L}$ such that $d^{'} = i^{'} = 0$. Then there is a projection commuting with A and C.

(ii)-(2)-(b) Assume that there is a matrix $C \in \mathfrak{L}$ such that $d^{'} \neq 0$ and $i^{'} = 0$. We may assume that $d^{'} = 1$ and $d = 0$. Then B and C looks like

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & b & c & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & f & 0 & g & h & i \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & j & 0 & k & 0 & 0 \\ 0 & 0 & m & 0 & n & 0 & 0 \end{bmatrix}, C = \begin{bmatrix} 0 & 0 & 0 & 0 & b^{'} & c^{'} & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & f^{'} & 0 & g^{'} & h^{'} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & j^{'} & 0 & k^{'} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The commutative relation of B and C implies that

$$cj^{'} = c^{'}j + m,\ ck^{'} = c^{'}k + n,\ hj^{'} = h^{'}j,\ hk^{'} = h^{'}k.$$

In this case we may assume that $i \neq 0$. Let

$$X = \begin{bmatrix} 0 & 0 & 0 & x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & z & 0 & w & 0 & -yi \end{bmatrix}, Y = \begin{bmatrix} 0 & 0 & 0 & y & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & y & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where $z = xf^{'} - yf$, $w = xg^{'} - yg$, and $y = j^{'}$ if $x = j$, $y = k^{'}$ if $x = k$, or $y = h^{'}$ if $x = h$. Then a straightforward computation reveals that $B + tX$ commutes with $C + tY$ for all $t \in$ F. Moreover if at least one of $h^{'}$, $j^{'}$, and $k^{'}$ is nonzero, then for $t \neq 0$ $B + tX$ has more than one point in the its spectrum. So, by Lemma 2.4, the triple $(A, B+tX, C+tY)$ belongs to $\overline{G(3,7)}$ for all $t \in$ F, $t \neq 0$. Now we may assume that $h^{'} = j^{'} = k^{'} = 0$. Let

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$$X = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -b^{'} & 0 & 0 \end{bmatrix}, Y = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x & 0 & 0 & 0 & 0 \end{bmatrix},$$

where $x = -\frac{f^{'}}{i}$.

Then a straightforward computation reveals that $B + tX$ commutes with $C + tY$ for all $t \in \mathrm{F}$. Since for $t \neq 0$ $B + tX$ has more than one point in the its spectrum, the triple $(A, B + tX, C + tY)$ belongs to $\overline{G(3,7)}$ for all $t \in \mathrm{F}$, $t \neq 0$.

(ii)-(2)-(c) Assume that there is a matrix $C \in \mathfrak{L}$ such that $d^{'} = 0$ and $i^{'} \neq 0$. We may assume that $i^{'} = 1$. So now we may assume that $i = 0$. By the rank condition of B we have $h(jn - km) = 0$. If $h = 0$ and $h^{'} \neq 0$. If $d = 0$, then we could introduce the matrix

$$Z = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

that clearly commutes with both A and C, so that it suffices to prove our case for all triples $(A, B + tZ, C)$ for $t \in \mathrm{F}$, $t \neq 0$. So we may assume that $d \neq 0$. Let

$$X = \begin{bmatrix} 0 & 0 & 0 & -c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, Y = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & h^{'} & 1 \\ 0 & 0 & x & 0 & y & 0 & 0 \end{bmatrix},$$

where $x = \frac{cf^{'}}{d}$, $y = \frac{cg^{'}}{d}$.

Then a straightforward computation reveals that $B + tX$ commutes with $C + tY$ for all $t \in \mathrm{F}$. Since for $t \neq 0$ $C + tY$ has more than one point in the its spectrum, the triple $(A, B + tX, C + tY)$ belongs to $\overline{G(3,7)}$ for all $t \in \mathrm{F}$, $t \neq 0$. If $h = 0$ and $h^{'} = 0$. Let

$$X = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, Y = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c^{'} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

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Then a direct computation reveals that $B+tX$ commutes with $C+tY$ for all $t \in \mathrm{F}$. So if at least one of c and $c^{'}$ is nonzero, then the triple $(A, B+tX, C+tY)$ belongs to $\overline{G(3,7)}$ for all $t \in \mathrm{F}$, $t \neq 0$. If $c = c^{'} = 0$. If $k^{'} = 0$, then we could introduce the matrix

$$Z = \begin{bmatrix} 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0 \end{bmatrix}$$

that clearly commutes with both A and B, so that it suffices to prove our case for all triples $(A, B, C+tZ)$ for $t \in \mathrm{F}$, $t \neq 0$. So we may assume that $k^{'} \neq 0$. Let

$$X = \begin{bmatrix} k^{'}&0&0&0&0&0&0\\ 0&k^{'}&0&0&0&0&0\\ 0&0&k^{'}&0&0&0&0\\ 0&-j^{'}&0&0&0&0&0\\ 0&0&-j^{'}&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&x&0&0&0&0 \end{bmatrix}, Y = \begin{bmatrix} 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&y&0&z&0&0 \end{bmatrix},$$

$$\text{where } x = g^{'}j^{'} - f^{'}k^{'},\ y = -\frac{b^{'}j^{'}}{d},\ z = -\frac{b^{'}k^{'}}{d}.$$

Then a direct computation reveals that $B+tX$ commutes with $C+tY$ for all $t \in \mathrm{F}$. Since for $t \neq 0$ $B+tX$ has more than one point in the its spectrum, the triple $(A, B+tX, C+tY)$ belongs to $\overline{G(3,7)}$ for all $t \in \mathrm{F}$, $t \neq 0$. If $h \neq 0$. The commutativity relation of B and C implies that

$$cj^{'} = c^{'}j,\ ck^{'} = c^{'}k,\ hj^{'} = h^{'}j + m,\ hk^{'} = h^{'}k + n.$$

From the rank condition we have $jn - km = 0$. So we have $jk^{'} = j^{'}k$. Let

$$X = \begin{bmatrix} 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&w&0&0&0&0 \end{bmatrix}, Y = \begin{bmatrix} -k&0&0&x&0&0&0\\ 0&-k&0&0&x&0&0\\ 0&0&-k&0&0&0&0\\ 0&j&0&0&0&0&0\\ 0&0&j&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&y&0&z&n&-k \end{bmatrix},$$

$$\text{where } x = \frac{ck+dn}{h},\ y = \frac{xf-bj}{d},\ z = \frac{xg-bk}{d},\ w = gj - fk.$$

Then a straightforward computation reveals that $B+tX$ commutes with $C+tY$ for all $t \in \mathrm{F}$. If $k \neq 0$, then for $t \neq 0$ $C+tY$ has more than one point in the its spectrum. So the triple $(A, B+tX, C+tY)$ belongs to $\overline{G(3,7)}$ for all $t \in \mathrm{F}$, $t \neq 0$. Now we may assume that $k = 0$. From the rank condition of B we have $jn = 0$. So we will consider 3 cases. If $j = 0$ and $n \neq 0$. From the commutative condition we have that $k^{'} \neq 0$, $c = 0$, $hj^{'} = m$, and $hk^{'} = n$. Let

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$$Y = \begin{bmatrix} 0 & 0 & 0 & d & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & d & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & f & 0 & g & h & 0 \end{bmatrix}.$$

Then a straightforward computation reveals that $C + tY$ commutes with B for all $t \in \mathrm{F}$. Since $d \neq 0$ and $h \neq 0$, $rank(C+tY) \geq 4$ for $t \neq 0$. So the triple $(A, B, C+tY)$ belongs to $\overline{G(3,7)}$ for all $t \in \mathrm{F}$, $t \neq 0$. If $j \neq 0$ and $n = 0$. From the commutative condition we have that $k^{'} = 0$, $cj^{'} = c^{'}j$, and $hj^{'} = h^{'}j + m$. Let

$$X = \begin{bmatrix} 0 & 0 & 0 & \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, Y = \begin{bmatrix} 0 & 0 & 0 & y & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & y & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & z & 0 & w & v & x \end{bmatrix},$$

where $x = hj$, $y = -dhj^{'}$, $z = -fhj^{'} + f^{'}hj$, $w = -ghj^{'} + g^{'}hj$, $v = -mh$, $\alpha = -dhj$. Then a straightforward computation reveals that $B + tX$ commutes with $C + tY$ for all $t \in \mathrm{F}$. Then for $t \neq 0$, $C + tY$ has more than one point in the its spectrum. So, by Lemma 2.4, the triple $(A, B + tX, C + tY)$ belongs to $\overline{G(3,7)}$ for all $t \in \mathrm{F}$, $t \neq 0$. If $j = 0$ and $n = 0$. From the commutative condition we have that $k^{'} = 0$, $cj^{'} = 0$, and $hj^{'} = m$. Let

$$Y = \begin{bmatrix} 0 & 0 & 0 & d & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & d & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & f & 0 & g & h & 0 \end{bmatrix}.$$

Then a straightforward computation reveals that $C + tY$ commutes with B for all $t \in \mathrm{F}$. Since $d \neq 0$ and $h \neq 0$, $rank(C+tY) \geq 4$ for $t \neq 0$. So the triple $(A, B, C+tY)$ belongs to $\overline{G(3,7)}$ for all $t \in \mathrm{F}$, $t \neq 0$.

(ii)-(2)-(d) Assume that there is a matrix $C \in \mathfrak{L}$ such that $d^{'} \neq 0$ and $i^{'} \neq 0$. We may assume that $d^{'} = 1$. So now we may assume that $d = 0$. By the rank condition of B we have $ci(jn - km) = 0$. If $i = 0$, then we could introduce the matrix

$$Z = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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that clearly commutes with both A and C, so that it suffices to prove our case for all triples $(A, B+tZ, C)$ for $t \in \mathrm{F}$, $t \neq 0$. So we may assume that $i \neq 0$. If $jn - mk = 0$ and $h^{'} - i^{'}c^{'} \neq 0$. Let

$$X = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & j & x \\ 0 & 0 & 0 & 0 & 0 & m & y \end{bmatrix}, Y = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & j^{'} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & j^{'} \end{bmatrix},$$

$$\text{where } x = \frac{ij^{'}}{h^{'} - i^{'}c^{'}},\ y = -\frac{ic^{'}j^{'}}{h^{'} - i^{'}c^{'}}.$$

Then a straightforward computation reveals that $B + tX$ commutes with $C + tY$ for all $t \in \mathrm{F}$. If $j^{'} \neq 0$, then for $t \neq 0$ $C + tY$ has more than one point in the its spectrum. So the triple $(A, B+tX, C+tY)$ belongs to $\overline{G(3,7)}$ for all $t \in \mathrm{F}$, $t \neq 0$. If $j^{'} = 0$ and $j \neq 0$, then for $t \neq 0$ $B + tX$ has more than one point in the its spectrum. So the triple $(A, B+tX, C+tY)$ belongs to $\overline{G(3,7)}$ for all $t \in \mathrm{F}$, $t \neq 0$. Now we may assume that $j^{'} = j = 0$. Let

$$X = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & k & x \\ 0 & 0 & 0 & 0 & 0 & n & y \end{bmatrix}, Y = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & k^{'} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & k^{'} \end{bmatrix},$$

$$\text{where } x = \frac{ik^{'}}{h^{'} - i^{'}c^{'}},\ y = -\frac{ic^{'}k^{'}}{h^{'} - i^{'}c^{'}}.$$

Then a straightforward computation reveals that $B + tX$ commutes with $C + tY$ for all $t \in \mathrm{F}$. If $k^{'} \neq 0$, then for $t \neq 0$ $C + tY$ has more than one point in the its spectrum. So the triple $(A, B+tX, C+tY)$ belongs to $\overline{G(3,7)}$ for all $t \in \mathrm{F}$, $t \neq 0$. If $k^{'} = 0$ and $k \neq 0$, then for $t \neq 0$ $B + tX$ has more than one point in the its spectrum. So the triple $(A, B+tX, C+tY)$ belongs to $\overline{G(3,7)}$ for all $t \in \mathrm{F}$, $t \neq 0$. Now we may assume that $k^{'} = k = 0$. If $c^{'} = 0$, then we could introduce the matrix

$$Z = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

that clearly commutes with both A and B, so that it suffices to prove our case for all triples $(A, B, C+tZ)$ for $t \in \mathrm{F}$, $t \neq 0$. So we may assume that $c^{'} \neq 0$. Let

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$$X = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -c^{'} \end{bmatrix}.$$

Then a straightforward computation reveals that $B + tX$ commutes with C for all $t \in \mathrm{F}$. Since for $t \neq 0$ $B + tX$ has more than one point in the its spectrum, the triple $(A, B + tX, C)$ belongs to $\overline{G(3, 7)}$ for all $t \in \mathrm{F}$, $t \neq 0$. If $jn - mk = 0$ and $h^{'} - i^{'}c^{'} = 0$. Let

$$X = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -c^{'} \end{bmatrix}.$$

Then a direct computation reveals that $B + tX$ commutes with C for all $t \in \mathrm{F}$. If $c^{'} \neq 0$, then for $t \neq 0$, $B + tX$ has more than one point in the its spectrum. So the triple $(A, B + tX, C)$ belongs to $\overline{G(3, 7)}$ for all $t \in \mathrm{F}$, $t \neq 0$. Now we may assume that $c^{'} = 0$. Then, by assumption, we have $h^{'} = 0$. The commutative relation of B and C implies that

$$cj^{'} = m,\ ck^{'} = n,\ hj^{'} = i^{'}m,\ hk^{'} = i^{'}n.$$

If $c = 0$, then we could introduce the matrix

$$Z = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & i^{'} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & j^{'} & 0 & k^{'} & 0 & 0 \end{bmatrix}$$

that clearly commutes with both A and C, so that it suffices to prove our case for all triples $(A, B + tZ, C)$ for $t \in \mathrm{F}$, $t \neq 0$. So we may assume that $c \neq 0$. Let

$$X = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & n & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & n \end{bmatrix}, Y = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & k^{'} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then a straightforward computation reveals that $B + tX$ commutes with $C + tY$ for all $t \in \mathrm{F}$. If $n \neq 0$, then for $t \neq 0$, $B + tX$ has more than one point in the its spectrum. So, by Lemma 2.4, the triple $(A, B + tX, C + tY)$ belongs to $\overline{G(3, 7)}$ for all $t \in \mathrm{F}$, $t \neq 0$. Now we may assume that $n = 0$. Since $n = ck^{'}$ and $c \neq 0$, we have $k^{'} = 0$. Let

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$$X = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & m & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & m \end{bmatrix}, Y = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & j^{'} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then a straightforward computation reveals that $B + tX$ commutes with $C + tY$ for all $t \in \mathrm{F}$. If $m \neq 0$, then for $t \neq 0$, $B + tX$ has more than one point in the its spectrum. So the triple $(A, B + tX, C + tY)$ belongs to $\overline{G(3, 7)}$ for all $t \in \mathrm{F}$, $t \neq 0$. Now we may assume that $m = 0$. Then $j^{'} = 0$. Let

$$X = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & i \end{bmatrix}, Y = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & x \\ 0 & 0 & 0 & 0 & 0 & 0 & y \end{bmatrix},$$

where $x = \frac{i}{c}$, $y = \frac{i^{'}c - h}{c}$.

Then a straightforward computation reveals that $B + tX$ commutes with $C + tY$ for all $t \in \mathrm{F}$. Since for $t \neq 0$ $B + tX$ has more than one point in the its spectrum, the triple $(A, B + tX, C + tY)$ belongs to $\overline{G(3, 7)}$ for all $t \in \mathrm{F}$, $t \neq 0$. If $jn - mk \neq 0$ and $h^{'} - i^{'}c^{'} \neq 0$. Since $ic(jn - mk) = 0$ and $i \neq 0$, we have $c = 0$. Let

$$X = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x & z \\ 0 & 0 & 0 & 0 & 0 & w & y \end{bmatrix}, Y = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & j & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & j \end{bmatrix},$$

where $x = \frac{hj}{h^{'} - i^{'}c^{'}}$, $y = -c^{'}z$, $z = \frac{ij}{h^{'} - i^{'}c^{'}}$, $w = -c^{'}x$.

Then a straightforward computation reveals that $B + tX$ commutes with $C + tY$ for all $t \in \mathrm{F}$. If $j \neq 0$, then for $t \neq 0$, $C + tY$ has more than one point in the its spectrum. So the triple $(A, B + tX, C + tY)$ belongs to $\overline{G(3, 7)}$ for all $t \in \mathrm{F}$, $t \neq 0$. If $j = 0$, then by the commutative condition of B and C we have $m = 0$. So $jn - mk = 0$. It can not occur. If $jn - mk \neq 0$ and $h^{'} - i^{'}c^{'} = 0$. Let

$$X = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -c^{'} \end{bmatrix}.$$

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Then a straightforward computation reveals that $B + tX$ commutes with C for all $t \in$ F. If $c^{'} \neq 0$, then for $t \neq 0$, $B + tX$ has more than one point in the its spectrum. So the triple $(A, B + tX, C)$ belongs to $\overline{G(3, 7)}$ for all $t \in$ F, $t \neq 0$. If $c^{'} = 0$, then by the commutative condition of B and C we have $m = n = 0$. So $jn - mk = 0$. It can not occur. □

7. The main result. Now we arrive at the main result. We need to consider only the case of a 3 dimensional linear space $\mathfrak{L}$ of nilpotent commuting matrices of 7×7. The case of a single nonzero Jordan block is [8], the case of square zero is [8], and the case of at most two Jordan blocks is [7]. So this leaves only the case of Jordan blocks of orders $4 + 2 + 1$, $3 + 3 + 1$, $3 + 2 + 2$, or $3 + 2 + 1 + 1$ which are handled in the previous 4 sections. So we can have the following main result.

Theorem 7.1. *Any triple of nilpotent commuting triples of* 7×7 *over an algebraically closed field of characteristic zero belongs to* $\overline{G(3, 7)}$.

Corollary 7.2. *If* A, B, *and* C *are three commuting* $n \times n$ *matrices with* $n < 8$, *then the algebra they generate has at most dimension* n.

Proof. Let $\mathcal{A} = \mathcal{A}(A, B, C)$ be the algebra generated by A, B, and C. Then $dim\mathcal{A} \leq r$ is just a closed condition. Thus $\mathcal{V} = \{(A, B, C) \in C(3, n) | dim\mathcal{A} \leq n\}$ is a closed subvariety. On the other hand it contains $G(3, n)$ which is dense in $C(3, n)$. Thus it is the whole variety. □

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