Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 13, pp. 352-358, November 2005

ELA
www.math.technion.ac.il/iic/ela

A NOTE ON ESTIMATES FOR THE SPECTRAL RADIUS OF A NONNEGATIVE MATRIX*

SHI-MING YANG† AND TING-ZHU HUANG†

Abstract. Utilizing the concept of Perron complement, a new estimate for the spectral radius of a nonnegative irreducible matrix is presented. A new matrix is derived that preserves the spectral radius while its minimum row sum increases and its maximum row sum decreases. Numerical examples are provided to illustrate the effectiveness of this approach.

Key words. Perron complement, Perron root, Nonnegative matrix, Irreducibility, Spectral radius.

AMS subject classifications. 15A18, 15A42, 15A57, 65F15, 65F99.

1. Introduction. Let A be a nonnegative matrix of order n with eigenvalues $\lambda_1,\ \lambda_2,\ \cdots,\ \lambda_n$. The set of $\{\lambda_1,\ \lambda_2,\ \cdots,\ \lambda_n\}$ is called the spectrum of A. For a nonnegative irreducible matrix A, a fundamental matrix problem is to locate or estimate its Perron root (spectral radius), $\rho(A) = \max|\lambda_i|$ for $1 \le i \le n$. Hence, it is interesting to develop methods giving rise to bounds for $\rho(A)$. It is well known that for such a matrix A, the following inequality holds ([1, 2]):

$$\min_i r_i = r(A) \le \rho(A) \le R(A) = \max_i r_i,$$

where $r_i = \sum_{j=1}^n a_{ij}, 1 \le i \le n$. This result bears a great idea for the estimation of the spectral radius using the elements of A in a simple way. The question arising is how to get sharper bounds by increasing the minimum row sum and decreasing the maximum row sum. This paper concentrates on developing a method of getting sharper bounds for the spectral radius of a nonnegative irreducible matrix, utilizing the concept of Perron complement.

2. Perron complement. In this section, the concept of Perron complement is introduced and some notation is also provided, which will be used in the remainder of this paper.

Meyer [3] introduced the Perron complement and used it to compute the Perron vector of a nonnegative irreducible matrix. Neumann [4] used it to analyze the properties of inverse M-matrices. Fan [5] used it to derive bounds for the Perron root of symmetric irreducible nonnegative matrices and Z-matrices.

Definition 2.1. Let $\langle n\rangle$ denote $\{i \mid 1 \le i \le n\}$, α be a nonempty ordered subset of $\langle n\rangle$ and $\beta = \langle n\rangle\backslash\alpha$. $|\alpha|$ denotes the cardinality of set α.

Definition 2.2. ([3]) Let $A = (a_{ij})$ be a nonnegative irreducible matrix of order n with spectral radius $\rho(A)$. For a certain α, $A[\alpha, \beta]$ denotes the submatrix of A with

*Received by the editors 13 August 2005. Accepted for publication 7 November 2005. Handling Editor: Harm Bart. Work supported by NCET in universities of China.

†School of Applied Mathematics, University of Electronic Science and Technology of China, Chengdu, 610054, P. R. China (shimingyoung@gmail.com, tzhuang@uestc.edu.cn).

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 13, pp. 352-358, November 2005

ELA
www.math.technion.ac.il/iic/ela

elements a_{ij} where $i \in \alpha$ and $j \in \beta$; $A[\alpha]$ denotes $A[\alpha, \alpha]$. The Perron complement of $A[\alpha]$ is defined to be the matrix

$$P(A/A[\alpha]) = A[\beta] + A[\beta, \alpha](\rho(A)I - A[\alpha])^{-1}A[\alpha, \beta].$$

For example, consider a 4×4 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}.$$

Assuming that $\alpha = \{3\}$, $\beta = \langle n \rangle \backslash \alpha = \{1, 2, 4\}$, the Perron complement of $A[\alpha]$ is

$$P(A/A[\alpha]) = \begin{pmatrix} a_{11} & a_{12} & a_{14} \\ a_{21} & a_{22} & a_{24} \\ a_{41} & a_{42} & a_{44} \end{pmatrix} + \begin{pmatrix} a_{13} \\ a_{23} \\ a_{43} \end{pmatrix} (\rho(A)I - a_{33})^{-1} \begin{pmatrix} a_{31} \\ a_{32} \\ a_{34} \end{pmatrix}^T.$$

LEMMA 2.3. ([3]) *If A is a nonnegative irreducible matrix with spectral radius $\rho(A)$, then each Perron complement $P(A/A[\alpha])$ is also a nonnegative irreducible matrix with the same spectral radius $\rho(A)$.*

As we can see, the dimension of $A[\alpha]$'s Perron complement is determined by $|\beta|$, while α is nonempty. Hence, the dimension of the matrix is reduced while its spectral radius is preserved. Furthermore, if α is properly chosen, the row sums of $A[\alpha]$'s Perron complement will change in the desired direction, namely, the maximum row sum decreases and the minimum row sum increases. Besides, it is easy to determine $\rho(A)$, when $\max_i r_i(A) = \min_i r_i(A)$. In the following sections, we assume that $\min_i r_i(A) < \max_i r_i(A)$. Based on these two considerations, we can logically develop our algorithms for the estimation of an upper bound and a lower bound.

3. Upper and lower bounds. We begin with a theorem.

THEOREM 3.1. *Let A be an irreducible nonnegative matrix. If the kth row attains the minimum row sum in A, let $\alpha = \{k\}$. Then the maximum row sum of $A[\alpha]$'s Perron complement is less than or equal to the maximum row sum of A. That is,*

$$\rho(A) = \rho(P(A/A[\alpha])) \leq R(P(A/A[\alpha])) \leq R(A).$$

Proof. Let

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}.$$

Let $\gamma = \{k \mid r_k = r(A)\}$ denoting the set of all rows which have the minimum row sum. If $|\gamma| > 1$, without losing generality, we randomly choose one element k from γ

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 13, pp. 352-358, November 2005

ELA
www.math.technion.ac.il/iic/ela

and let $\alpha = \{k\}$, $\beta = \langle n\rangle \backslash \alpha$. Then the Perron complement of $A[\alpha]$ is

$$\begin{aligned}
P(A/A[\alpha]) &= (c_{ij}) \qquad i,j \in \beta \\
&= A[\beta] + A[\beta,\alpha](\rho(A) - A[\alpha])^{-1} A[\alpha,\beta] \\
&= \begin{pmatrix}
a_{11} & \cdots & a_{1,k-1} & a_{1,k+1} & \cdots & a_{1,n} \\
 & \vdots & & & \vdots & \\
a_{k-1,1} & \cdots & a_{k-1,k-1} & a_{k-1,k+1} & \cdots & a_{k-1,n} \\
a_{k+1,1} & \cdots & a_{k+1,k-1} & a_{k+1,k+1} & \cdots & a_{k+1,n} \\
 & \vdots & & & \vdots & \\
a_{nn} & \cdots & a_{n,k-1} & a_{n,k+1} & \cdots & a_{n,n}
\end{pmatrix} \\
&\quad + \begin{pmatrix} a_{1,k} \\ \vdots \\ a_{k-1,k} \\ a_{k+1,k} \\ \vdots \\ a_{n,k} \end{pmatrix} \left(\frac{1}{\rho - a_{kk}}\right) \begin{pmatrix} a_{k,1} \\ \vdots \\ a_{k,k-1} \\ a_{k,k+1} \\ \vdots \\ a_{k,n} \end{pmatrix}^T , \\
c_{ij} &= a_{ij} + \frac{a_{ik}a_{kj}}{\rho(A) - a_{kk}}, \qquad i,j \in \beta.
\end{aligned}$$

Each row sum of $P(A/A[\alpha])$ is, for $i \in \beta$,

$$\begin{aligned}
p_i &= \sum_{\substack{j=1 \\ j\neq k}}^{n} c_{ij} \\
&= \sum_{\substack{j=1 \\ j\neq k}}^{n} a_{ij} + \sum_{\substack{j=1 \\ j\neq k}}^{n} \frac{a_{ik}a_{kj}}{\rho(A) - a_{kk}} \\
&= (r_i - a_{ik}) + \frac{r(A)a_{ik} - a_{kk}a_{ik}}{\rho(A) - a_{kk}} \\
&= r_i + \frac{a_{ik}(r(A) - \rho(A))}{\rho(A) - a_{kk}}.
\end{aligned}$$

Since

$$r(A) \le \rho(A), \;\; a_{ik} \ge 0, \;\; \rho(A) > a_{kk},$$

we have $p_i \le r_i$. □

In the proof above, the inequality $\rho(A) > a_{kk}$ always holds because the matrix is nonnegative and irreducible. Hence, we can safely draw the conclusion that all the row sums of $P(A/A[\alpha])$ are less than or equal to their counterparts in A. Equality holds when $a_{ik} = 0$ or $r(A) = \rho(A)$.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 13, pp. 352-358, November 2005

ELA
www.math.technion.ac.il/iic/ela

According to Theorem 3.1, a smaller maximum row sum can be obtained from the Perron complement of $A[\alpha]$, that is

$$\rho(A) = \rho(P(A/A[\alpha]) \le \max_{i\in\beta} p_i \le R(A).$$

Namely

$$\rho(A) \le \max_{i\in\beta}\{r_i + \frac{a_{ik}(r(A)-\rho(A))}{\rho(A)-a_{kk}}\}.$$

Note that $\rho(A)$ is unknown, and there is no guarantee that the row, having the maximum row sum in A, is still the one whose row sum is the greatest in $P(A/A[\alpha])$. Hence, the row i of maximum row sum cannot be determined explicitly. However, if we view p_i as a function of $\rho(A)$,

$$\begin{aligned} p_i(\rho(A)) &= r_i - \frac{a_{ik}(\rho(A)-r(A))}{\rho(A)-a_{kk}} \\ &= r_i - a_{ik}[1 + \frac{a_{kk}-r(A)}{\rho(A)-a_{kk}}], \end{aligned}$$

we see that p_i increases as $\rho(A)$ increases. Therefore, solving $|\beta|$ inequalities

$$\rho(A) \le r_i + \frac{a_{ik}(r(A)-\rho(A))}{\rho(A)-a_{kk}} \qquad (i\in\beta),$$

or simply,

$$\rho(A)^2 - (a_{kk} + r_i - a_{ik})\rho(A) + r_i a_{kk} - a_{ik} r(A) \le 0 \qquad (i\in\beta),$$

and letting $b_i = a_{kk} - r_i - a_{ik}$, $c_i = r_i a_{kk} - a_{ik} r(A)$, the upper bound of all the solution intervals d is as desired:

$$\rho(A) = d \le \max_{i\in\beta}\{d_i \mid d_i = \frac{b_i + \sqrt{b_i^2 - 4c_i}}{2}\}.$$

Here, d guarantees that the estimation of $\rho(A)$ is obtained via the maximum row sum of $P(A/A[\alpha])$. Consequently, we propose the following algorithm for the improved upper bound of the spectral radius, taking into consideration the case that more rows than one attain the minimum row sum of A.

Algorithm 1

1. Calculate all the row sums $r_i(A)$ and set $r(A) = \min_i r_i(A)$.
   Let $\gamma = \{l \mid r_l = r(A)\}$, $l \in \langle n\rangle$; set $d = \infty$.
2. Get one element from γ and assign its value to k. Then delete this element from γ. Update $\alpha = \{k\}, \beta = \langle n\rangle\backslash\alpha$.
3. Let $b_i = a_{kk} + r_i - a_{ik}$, $c_i = r_i a_{kk} - a_{ik} r(A)$. Set

$$d = \min\{\max_{i\in\beta} \frac{b_i + \sqrt{b_i^2 - 4c_i}}{2}, d\}.$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 13, pp. 352-358, November 2005

ELA
www.math.technion.ac.il/iic/ela

4. If γ is nonempty, go to step 2; otherwise, the desired new upper bound is d.

Analogously, a lower bound can be obtained. A similar theorem and algorithm are given in what follows without proof. Our discussion on the upper bound is also applicable to the lower bound.

THEOREM 3.2. *Let A be an irreducible nonnegative matrix. If the Kth row attains the maximum row sum, let $\alpha = \{K\}$. Then the minimum row sum of $A[\alpha]$'s Perron complement is greater than or equal to the minimum row sum of A. That is,*

$$r(A) \leq r(P(A/A[\alpha])) \leq \rho(P(A/A[\alpha])) = \rho(A).$$

According to Theorem 3.2, a greater minimum row sum can be obtained from the Perron complement of $A[\alpha]$, that is

$$r(A) \leq \min_{i\in\beta} p_i \leq \rho(P(A/A[\alpha])) = \rho(A).$$

Again, we propose an algorithm for the improved lower bound of spectral radius, supposing that more rows than one attain the maximum row sum of A.

Algorithm 2

1. Calculate all the row sums $r_i(A)$ and set $R(A) = \max_i r_i(A)$.
   Let $\gamma = \{l \mid r_l = R(A)\}$, $l \in \langle n\rangle$; set $d = 0$.
2. Get one element from γ and assign its value to K. Then delete it from γ. Update $\alpha = \{K\}$, $\beta = \langle n\rangle\backslash\alpha$.
3. Let $b_i = a_{KK} + r_i - a_{iK}$, $c_i = r_i a_{KK} - a_{iK}R(A)$. Set
$$d = \max\{\min_{i\in\beta} \frac{b_i + \sqrt{b_i^2 - 4c_i}}{2}, d\}.$$
4. If γ is nonempty, go to step 2; otherwise, the desired new lower bound is d.

REMARK 3.3. For a nonnegative irreducible matrix A, no improvement will be gotten by applying Theorem 3.1, if $\max_i r_i(A) = \min_i r_i(A)$. Besides, if $a_{ik} = 0$ in the row having maximum row sum of $P(A/A[\alpha])$, the upper bound will not decrease. Unfortunately, we cannot determine i in advance and have no idea whether $a_{ik} = 0$ or not. Similarly, this is also true for Theorem 3.1.

In both Algorithms 1 and 2, it is assumed that more rows than one have maximum or minimum row sum, namely, $|\gamma| > 1$. Hence, each algorithm loops between step 2 and step 4 to obtain sharper bounds. However, if we randomly choose one element from γ without looping, we can also get improved bounds (may not be the sharpest possible).

4. Examples. Thus far, the approach for sharper bounds has been established in a theoretical sense. The considerations are also applicable to column sums. Here, a simple numerical example is provided to illustrate the bounds and the method proposed in the previous section. Then we examine ten 4×4 matrices to see how our estimates improve the ones by Frobenius.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 13, pp. 352-358, November 2005

ELA
www.math.technion.ac.il/iic/ela

Example 1 Consider the 3×3 nonnegative irreducible matrix

$$\begin{pmatrix} 4 & 5 & 3 \\ 6 & 6 & 2 \\ 5 & 3 & 7 \end{pmatrix},$$

where $r(A) = 12$, $\gamma = \{1\}$, $a_{11} = 4$, $\alpha = \{1\}$. The Perron complement of $A[\alpha]$ is

$$P(A/A[\alpha]) = \begin{pmatrix} 6 & 2 \\ 3 & 7 \end{pmatrix} + \begin{pmatrix} 30 & 18 \\ 25 & 15 \end{pmatrix} \left(\frac{1}{\rho(A) - 4} \right).$$

Solving the following two inequalities,

$$\rho(A) \leq 6 + \frac{30}{\rho(A) - 4} + 2 + \frac{18}{\rho(A) - 4}$$

$$\rho(A) \leq 3 + \frac{25}{\rho(A) - 4} + 7 + \frac{15}{\rho(A) - 4},$$

we get

$$\rho(A) \leq 13.2111, \text{ and } \rho(A) \leq 14.0000.$$

Hence, the new upper bound is the larger one, 14.0000. In a similar way, the new lower bound can be found to be 13.0000. Using column sums instead of row sums, we get that the lower and upper bounds are 12.7650 and 13.7202, respectively.

Example 2. Here we examine the improvement of the bounds using ten 4×4 matrices generated randomly by Matlab (all the elements range between 1 and 10). In Figure 4.1, bars are arranged in the following sequence: $r(A)$, lower bound, upper bound, $R(A)$.

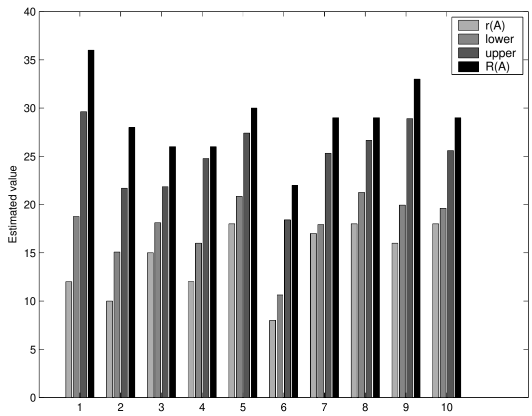

FIG. 4.1. *Comparison of bounds*

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 13, pp. 352-358, November 2005

ELA
www.math.technion.ac.il/iic/ela

Example 3. We considered a 600×600 nonnegative irreducible matrix A, randomly generated by Matlab (with $0-1$ average distribution). Its minimum row sum was 276.4040 and maximum row sum is 323.1155. Using the new algorithms, we obtained the improved bounds

$$276.5627 < \rho(A) < 323.0851.$$

We also considered a Vandermonde matrix of order 6 with minimum row sum 6 and maximum row sum 9331, while the new bounds were 99.6126 and 1579.6. More experiments showed that the distribution of the elements, especially the difference between the maximum (minimum) row sum and the other sums, affect how great an improvement our bounds offer. Further research can be done on this issue.

REFERENCES

[1] A. Berman and R.J. Plemmons. *Nonnegative Matrices in the Mathematical Sciences*, SIAM, Philadelphia, 1994.
[2] R.B. Bapat and T.E.S. Raghavan. *Nonnegative Matrices and Applications*, Cambridge University Press, Cambridge, MA, 1997.
[3] C.D. Meyer. Uncoupling the Perron eigenvector problem. *Linear Algebra Appl.*, 114/115:69–94, 1989.
[4] M. Neumann. Inverse of Perron complements of inverse M-matrices. *Linear Algebra Appl.*, 313:163–171, 2000.
[5] Y.Z. Fan. Schur complement and its applications to symmetric nonnegative and Z-matrices. *Linear Algebra Appl*, 353:289–307, 2002.