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CHARACTERIZATION OF CLASSES OF SINGULAR LINEAR DIFFERENTIAL-ALGEBRAIC EQUATIONS*

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Abstract. Linear, possibly over- or underdetermined, differential-algebraic equations are studied that have the same solution behavior as linear differential-algebraic equations with well-defined strangeness index. In particular, three different characterizations are given for differential-algebraic equations, namely by means of solution spaces, canonical forms, and derivative arrays. Two levels of generalization are distinguished, where the more restrictive case contains an additional assumption on the structure of the set of consistent inhomogeneities.

Key words. Singular differential-algebraic equation, Over- and underdetermined system, Solution space, Canonical form, Derivative array.

AMS subject classifications. 34A09.

1. Introduction.

In this paper, we study linear differential-algebraic equations (DAEs)

$$E(t)\dot{x} = A(t)x + f(t), \tag{1.1}$$

where

$$E, A \in C(\mathbb{I}, \mathbb{C}^{m,n}), \quad f \in C(\mathbb{I}, \mathbb{C}^m) \tag{1.2}$$

are assumed to be sufficiently smooth on the interval $\mathbb{I} \subseteq \mathbb{R}$. In particular, we allow (1.1) to be *singular* in the sense that the space of all solutions in $C^1(\mathbb{I}, \mathbb{C}^n)$ of the associated homogeneous problem is infinite-dimensional or that the existence of solutions requires the inhomogeneity to be contained in a proper subspace of $C^\ell(\mathbb{I}, \mathbb{C}^m)$ for a sufficiently large ℓ. Such singular systems arise naturally from control problems (see, e.g., [11, 16]) in form of underdetermined problems or from automatic model generators (see, e.g., [3, 15, 18]) in form of (consistent) overdetermined systems. Throughout the rest of the paper, we use the shorthand notation $k = \min\{m, n\}$. Moreover, we require all occurring functions to be sufficiently smooth, so that all derivatives that arise in the analysis actually exist. Wherever it is possible, we explicitly state the minimal smoothness requirements.

The case of *regular* DAEs, i.e., DAEs for which the solution space of the homogeneous problem is finite-dimensional (such that unique solvability can be achieved by prescribing some initial condition) and for which existence of solutions only requires

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the inhomogeneity to be sufficiently smooth, is well studied. The investigations are typically based on the concept of an index which in principle measures the smoothness of the data we need to discuss existence and uniqueness of solutions. Unfortunately, most index concepts as the differentiation index (see, e.g., [1, 2]) or the tractability index (see, e.g., [4, 13, 14]) exclude singular DAEs by construction. An exception is given by the so-called strangeness index, see [5, 10]. In this concept, no regularity of the DAE is assumed. However, one needs a number of assumptions that certain matrix functions derived from the data have constant rank in order to develop a theory on existence and uniqueness of solutions. These assumptions exclude (already in the regular case) classes of DAEs which behave well in the sense that the solution space has some nice structure.

For regular DAEs, it has been shown in [1] that (1.1) has a well-defined differentiation index and that (1.1) can be transformed to a canonical form. In this canonical form, the DAE splits into two equations. One part, called the algebraic part, is uniquely solvable for sufficiently smooth inhomogeneities without prescribing an initial condition, whereas the other part, called the differential part, constitutes a differential equation for the remaining unknowns. In [7], it has been shown that the concept of the differentiation index is equivalent to the requirement that a hypothesis that only involves matrix functions built from the data E, A and their derivatives, so-called derivative arrays, is satisfied. The key point here is that this hypothesis directly suggests a possible numerical treatment of regular DAEs. In this way, it could also be shown how differentiation and strangeness index are related in the case of regular DAEs.

Concerning singular DAEs, only a few results exist, see, e.g., [5, 6, 8, 11, 9, 12, 17]. As far as linear DAEs are concerned, they all require a number of constant rank assumptions. In particular, it is not clear which singular DAEs are excluded by these assumptions. It is therefore the aim of the present paper to generalize the results of [1, 7] for singular DAEs that behave well with respect to the solution space. We also include a discussion of a subclass for which the space of consistent inhomogeneities can be parameterized in a certain way. After some preliminaries in Section 2, we start from the results for DAEs with well-defined strangeness index to define classes of singular DAEs which have similar properties with respect to solution spaces and consistent inhomogeneities, see Section 3. In Section 4, we derive a (global) canonical form for this class of DAEs. Section 5 yields an equivalent characterization in terms of derivative arrays. In particular, it is shown that all three characterizations (by solution space, by canonical form and by derivative arrays) are equivalent. We close with a summary including a diagram that displays the overall logical structure of the paper and some conclusions in Section 6.

2. Preliminaries. Studying a selected class of problems, all concepts that are introduced should be invariant under a reasonable class of reversible transformations that put a given problem of the class into a problem of the same class. In the case of (1.1), the transformations one should look at are scaling of the equation and scaling of the unknown by pointwise nonsingular matrix functions.

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Definition 2.1. We call two pairs (E, A) and $(\tilde{E}, \tilde{A})$ of matrix functions with $E, A, \tilde{E}, \tilde{A} \in C(\mathbb{I}, \mathbb{C}^{m,n})$ *(globally) equivalent* and write $(E, A) \sim (\tilde{E}, \tilde{A})$ if there exist pointwise nonsingular matrix functions $P \in C(\mathbb{I}, \mathbb{C}^{m,m})$ and $Q \in C^1(\mathbb{I}, \mathbb{C}^{n,n})$ such that

$$\tilde{E} = PEQ, \quad \tilde{A} = PAQ - PE\dot{Q}. \tag{2.1}$$

It is easy to see that this indeed defines an equivalence relation for pairs of matrix functions.

From the regular case, it is known that in some proofs we must work with so-called derivative arrays. Due to an idea of Campbell, see, e.g., [1], one successively differentiates (1.1) with respect to t and gathers all resulting relations up to some differentiation order ℓ into inflated DAEs

$$M_\ell(t)\dot{z}_\ell = N_\ell(t)z_\ell + g_\ell(t) \tag{2.2}$$

with

$$\begin{array}{llll} \text{(a)} & (M_\ell)_{ij} = \binom{i}{j}E^{(i-j)} - \binom{i}{j+1}A^{(i-j-1)}, & & i,j = 0,\ldots,\ell, \\ \text{(b)} & (N_\ell)_{ij} = A^{(i)} \text{ for } j = 0,\ (N_\ell)_{ij} = 0 \text{ else}, & & i,j = 0,\ldots,\ell, \\ \text{(c)} & (g_\ell)_i = f^{(i)}, & & i = 0,\ldots,\ell, \\ \text{(d)} & (z_\ell)_j = x^{(j)}, & & j = 0,\ldots,\ell. \end{array} \tag{2.3}$$

A further advantage of derivative arrays is that one can also deal with them numerically, since only the data functions together with their derivatives are involved. We therefore also aim in characterizations of DAEs on the basis of derivative arrays. The key property of the derivative arrays for our further considerations is the following, see [8, 10].

Theorem 2.2. *Let the pairs (E, A) and $(\tilde{E}, \tilde{A})$ of matrix functions be (globally) equivalent via (2.1) and let (M_ℓ, N_ℓ) and $(\tilde{M}_\ell, \tilde{N}_\ell)$ be the corresponding derivative arrays. Then*

$$\tilde{M}_\ell = \Pi_\ell M_\ell \Theta_\ell, \quad \tilde{N}_\ell = \Pi_\ell N_\ell \Theta_\ell - \Pi_\ell M_\ell \Psi_\ell, \tag{2.4}$$

where

$$\begin{array}{llll} \text{(a)} & (\Pi_\ell)_{ij} = \binom{i}{j}P^{(i-j)}, & & i,j = 0,\ldots,\ell, \\ \text{(b)} & (\Theta_\ell)_{ij} = \binom{i+1}{j+1}Q^{(i-j)}, & & i,j = 0,\ldots,\ell, \\ \text{(c)} & (\Psi_\ell)_{ij} = Q^{(i+1)} \text{ for } j = 0,\ (\Psi_\ell)_{ij} = 0 \text{ else}, & & i,j = 0,\ldots,\ell, \end{array} \tag{2.5}$$

as long as all quantities are defined.

Note that M_ℓ as well as Π_ℓ and Θ_ℓ are block lower triangular, whereas N_ℓ as well as Ψ_ℓ have nontrivial entries only in the first block column. The diagonal entries of Π_ℓ and Θ_ℓ are given by P and Q, respectively. Hence, Π_ℓ and Θ_ℓ are pointwise nonsingular and (2.4) immediately implies that

$$\begin{array}{ll} \text{(a)} & \operatorname{rank} \tilde{M}_\ell = \operatorname{rank} M_\ell, \\ \text{(b)} & \operatorname{rank}[\, \tilde{M}_\ell \ \ \tilde{N}_\ell \,] = \operatorname{rank}[\, M_\ell \ \ N_\ell \,] \end{array} \tag{2.6}$$

These properties will be the basis of further invariance results that we need below.

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3. DAEs with well-defined strangeness index. General DAEs of the form (1.1) are well understood in the theory of the so-called strangeness index where during the construction of a canonical form for (1.1) a number of assumptions that certain arising matrix function have constant rank are involved. To omit details of this theory we do not need in the course of this paper, we introduce the strangeness index as follows.

DEFINITION 3.1. A pair (E, A) of matrix functions with $E, A \in C(\mathbb{I}, \mathbb{C}^{m,n})$ or the corresponding DAE (1.1) is said to have *strangeness index* $\mu \in \mathbb{N}_0$ if

$$(E,A) \sim \left(\begin{bmatrix} I_d & 0 & W \\ 0 & 0 & F \\ 0 & 0 & G \end{bmatrix}, \begin{bmatrix} 0 & L & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_a \end{bmatrix} \right), \tag{3.1}$$

where W, F and G have the block structures

$$\begin{aligned} &\text{(a)} \quad W = \begin{bmatrix} 0 & W_\mu & \cdots & W_1 \end{bmatrix}, \\ &\text{(b)} \quad F = \begin{bmatrix} 0 & F_\mu & & * \\ & \ddots & \ddots & \\ & & \ddots & F_1 \\ & & & 0 \end{bmatrix}, \\ &\text{(c)} \quad G = \begin{bmatrix} 0 & G_\mu & & * \\ & \ddots & \ddots & \\ & & \ddots & G_1 \\ & & & 0 \end{bmatrix}, \end{aligned} \tag{3.2}$$

with the same partitioning with respect to the columns. Furthermore, F_i and G_i together have pointwise full row rank for each $i = 1, \ldots, \mu$.

Observe that due to (3.2) the strangeness index μ (if defined) satisfies $\mu \leq k-1 = \min\{m,n\}-1$. For the derivation of (3.1), the character of the imposed constant rank assumptions and further details in the context of the strangeness index, see [5, 7, 8] or [10]. Since, besides sufficient smoothness of the matrix functions E, A, only constant rank assumptions are involved in the construction of (3.1), the strangeness index has the important property that it is defined on a dense subset of $\mathbb{I}$, see [7]. For this, we assume for simplicity that in the following $\mathbb{I}$ is closed.

THEOREM 3.2. *Let (E, A) be a pair of sufficiently smooth matrix functions. Then there exist pairwise disjoint open intervals $\mathbb{I}_j \subseteq \mathbb{I}$, $j \in \mathbb{N}$, with*

$$\mathbb{I} = \overline{\bigcup_{j\in\mathbb{N}} \mathbb{I}_j} \tag{3.3}$$

such that for every $j \in \mathbb{N}$ the pair (E, A) restricted to $\mathbb{I}_j$ possesses a well-defined strangeness index.

Due to Definition 3.1, a DAE (1.1) with well-defined strangeness index μ can be

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transformed to a DAE of the form

$$\text{(3.4)} \qquad \begin{array}{ll} \text{(a)} & \dot{x}_1 + W(t)\dot{x}_3 = L(t)x_2 + f_1(t), \\ \text{(b)} & F(t)\dot{x}_3 = f_2(t), \\ \text{(c)} & G(t)\dot{x}_3 = x_3 + f_3(t). \end{array}$$

Utilizing the nilpotent structure of G, the third equation has a unique solution x_3 for every $f_3 \in C^{\mu+1}(\mathbb{I}, \mathbb{C}^a)$. This solution can be written in the form

$$\text{(3.5)} \qquad x_3 = \sum_{i=0}^{\mu} D_i f_3^{(i)},$$

with sufficiently smooth coefficients $D_i \in C(\mathbb{I}, \mathbb{C}^{a,a})$. Having determined x_3 and choosing $x_2 \in C^1(\mathbb{I}, \mathbb{C}^u)$ arbitrarily with $u = n - d - a$ then leaves a solvable linear DAE in (3.4a). Thus, for the DAE (3.4) to be solvable, it remains to look at (3.4b) which states a consistency condition for the inhomogeneity. Because of (3.5), this can be written in the form

$$\text{(3.6)} \qquad f_2 = F\dot{x}_3 = \sum_{i=0}^{\mu+1} C_i f_3^{(i)},$$

with sufficiently smooth $C_i \in C(\mathbb{I}, \mathbb{C}^{v,a})$, $v = m - d - a$.

Considering now the homogeneous problem

$$\text{(3.7)} \qquad E(t)\dot{x} = A(t)x$$

associated to (1.1), i.e., setting $f = 0$ gives $x_3 = 0$ and the consistency condition (3.6) is trivially satisfied. Choosing $t_0 \in \mathbb{I}$ fixed and

$$\text{(3.8)} \qquad c \in C^1(\mathbb{I}, \mathbb{C}^u), \quad \alpha \in \mathbb{C}^d$$

arbitrarily, we can parameterize all solutions of (3.4) according to $x_3 = 0$, $x_2 = c$, and x_1 being the solution of the initial value problem

$$\text{(3.9)} \qquad \dot{x}_1 = L(t)c(t), \quad x_1(t_0) = \alpha,$$

hence

$$\text{(3.10)} \qquad x_1(t) = \alpha + \int_{t_0}^{t} L(\tau)c(\tau)\, dt = \alpha + I[c](t).$$

The solution space of the homogeneous problem in the transformed form (3.4) is therefore given by

$$\text{(3.11)} \qquad \tilde{\mathbb{K}} = \{\tilde{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in C^1(\mathbb{I}, \mathbb{C}^n) \mid x_1 = \alpha + I[c],\ x_2 = c,\ x_3 = 0\}.$$

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Denoting the canonical form given in (3.1) by $(\tilde{E}, \tilde{A})$, we have the relation (2.1), where P and Q belong to the equivalence relation (3.1). Back transformation then yields

$$x = Q\tilde{x}. \tag{3.12}$$

With

$$Q = [\,\Phi \;\; \Psi \;\; \Theta\,] \tag{3.13}$$

partitioned conformally with $\tilde{x}$, the solution space of the homogeneous problem associated with the original pair (E, A) is given by

$$\mathbb{K} = \{x \in C^1(\mathbb{I}, \mathbb{C}^n) \mid x_1 = \Phi(\alpha + I[c]) + \Psi c\}. \tag{3.14}$$

Accordingly, one can write the consistency condition (3.6) as

$$f_2 = C[f_3] = \sum_{i=0}^{\mu+1} C_i f_3^{(i)}, \quad Pf = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}, \quad PE\Phi = \begin{bmatrix} I_d \\ 0 \\ 0 \end{bmatrix}. \tag{3.15}$$

Note that both the space $\mathbb{K}$ and the space of all consistent inhomogeneities are parameterized by (3.8) and $f_3 \in C^{\mu+1}(\mathbb{I}, \mathbb{C}^a)$, respectively. Moreover, these properties, if also valid for every restriction to a nontrivial subinterval of $\mathbb{I}$ (i.e., a subinterval of $\mathbb{I}$ with nonempty interior) as in the present case, exclude all possible irregular behavior of the DAE as for example inner point conditions for the inhomogeneity or the existence of local solutions that cannot be extended to solutions on the whole interval.

In this paper we are interested in the characterization of all linear DAEs which show the same properties as DAEs with well-defined strangeness index. In the following, we distinguish two levels of characterizations. On the first more general level A, we only use properties of the solution space. On the second level B, we also include a structure for the space of consistent inhomogeneities. The reason for this will become clear in the next section.

We start with the following two hypotheses which hold for problems with well-defined strangeness index due to the previous discussion in this section.

Hypothesis A.1. *The pair (E, A) of matrix functions and every restriction to a nontrivial subinterval have the following properties:*
1) There exist matrix functions $\Phi \in C^1(\mathbb{I}, \mathbb{C}^{n,d})$ and $\Psi \in C^1(\mathbb{I}, \mathbb{C}^{n,u})$ with $[\,\Phi \;\; \Psi\,]$ having pointwise full column rank such that the associated homogeneous problem (3.7) possesses a solution space of the form

$$\mathbb{K} = \{x \in C^1(\mathbb{I}, \mathbb{C}^n) \mid x = \Phi(\alpha + I[c]) + \Psi c,\ \alpha \in \mathbb{C}^d,\ c \in C^1(\mathbb{I}, \mathbb{C}^u)\} \tag{3.16}$$

with

$$I[c](t) = \int_{t_0}^{t} L(\tau)c(\tau)\, dt, \tag{3.17}$$

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and $t_0 \in \mathbb{I}$ *fixed. Moreover,*

$$\operatorname{rank} E\Phi = d \text{ on } \mathbb{I}. \tag{3.18}$$

2) There are matrix functions $D_i \in C^1(\mathbb{I}, \mathbb{C}^{a,m})$, $i = 0, \ldots, k-1$, *and* $\Theta \in C^1(\mathbb{I}, \mathbb{C}^{n,a})$ *with* $[\,\Phi\ \Psi\ \Theta\,]$ *is pointwise nonsingular such that, if* $f \in C^k(\mathbb{I}, \mathbb{C}^n)$ *is consistent, i.e., if it permits a solution of (1.1), then there exists a particular solution of (1.1) of the form*

$$x = \Phi x_1 + \Theta \sum_{i=0}^{k-1} D_i f^{(i)}, \quad x_1 \in C^1(\mathbb{I}, \mathbb{C}^d). \tag{3.19}$$

HYPOTHESIS B.1. *The pair* (E, A) *of matrix functions and every restriction to a nontrivial subinterval have property 1) of Hypothesis A.1 and the following property:*
3) There exists a pointwise nonsingular $R \in C(\mathbb{I}, \mathbb{C}^{m,m})$ *with*

$$RE\Phi = \begin{bmatrix} I_d \\ 0 \\ 0 \end{bmatrix} \tag{3.20}$$

and $C_i \in C(\mathbb{I}, \mathbb{C}^{v,a})$ *such that for given* $f_3 \in C^k(\mathbb{I}, \mathbb{C}^a)$ *in*

$$Rf = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} \tag{3.21}$$

the DAE (1.1) is solvable if and only if

$$f_2 = C[f_3] = \sum_{i=0}^{k} C_i f_3^{(i)}. \tag{3.22}$$

LEMMA 3.3. *Hypotheses A.1 and B.1 are invariant under (global) equivalence transformations.*

Proof. Let (E, A) satisfy Hypothesis A.1 or Hypothesis B.1, respectively, and let

$$\tilde{E} = PEQ, \quad \tilde{A} = PAQ - PE\dot{Q}, \quad \tilde{f} = Pf, \quad \tilde{x} = Q^{-1}x$$

with P and Q according to Definition 2.1. Defining

$$\tilde{\Phi} = Q^{-1}\Phi, \quad \tilde{\Psi} = Q^{-1}\Psi,$$

we find for the corresponding solution space $\tilde{\mathbb{K}}$ of the homogeneous problem that

$$\begin{aligned} \tilde{\mathbb{K}} &= \{\tilde{x} \in C^1(\mathbb{I}, \mathbb{C}^n) \mid \tilde{x} = Q^{-1}\Phi(\alpha + I[c]) + Q^{-1}\Psi\} = \\ &= \{\tilde{x} \in C^1(\mathbb{I}, \mathbb{C}^n) \mid \tilde{x} = \tilde{\Phi}(\alpha + I[c]) + \tilde{\Psi}\}. \end{aligned}$$

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Moreover, $\tilde\Phi \in C^1(\mathbb{I}, \mathbb{C}^{n,d})$, $\tilde\Psi \in C^1(\mathbb{I}, \mathbb{C}^{n,u})$, and

$$\operatorname{rank} \tilde E\tilde\Phi = \operatorname{rank} PEQQ^{-1}\Phi = \operatorname{rank} E\Phi.$$

Setting

$$[\, \tilde D_0 \;\; \tilde D_1 \;\; \cdots \;\; \tilde D_{k-1} \,] = [\, D_0 \;\; D_1 \;\; \cdots \;\; D_{k-1} \,]\Pi_{k-1}^{-1}$$

for (3.19) with Π_{k-1} from (2.5), we get

$$\begin{aligned} \tilde x &= Q^{-1}x = Q^{-1}(\Phi x_1 + \Theta[\, D_0 \;\; D_1 \;\; \cdots \;\; D_{k-1} \,]g_{k-1}) = \\ &= \tilde\Phi x_1 + \tilde\Theta[\, \tilde D_0 \;\; \tilde D_1 \;\; \cdots \;\; \tilde D_{k-1} \,]\Pi_{k-1}g_{k-1} = \tilde\Phi x_1 + \tilde\Theta \textstyle\sum_{i=0}^{k-1} \tilde D_i \tilde f^{(i)}. \end{aligned}$$

Finally, with

$$\tilde R = RP^{-1}$$

we obtain

$$\tilde R\tilde E\tilde\Phi = RP^{-1}PEQQ^{-1}\Phi = REQ$$

and

$$\tilde R\tilde f = RP^{-1}Pf = Rf.$$

Thus, the claimed invariance is obvious. □

Summarizing the above discussion on pairs (E, A) with well-defined strangeness index, we have shown the following result in terms of invariant properties.

Theorem 3.4. *Let (E, A) have a well-defined strangeness index. Then (E, A) satisfies Hypotheses A.1 and B.1.*

4. Global canonical forms. In this section, we study implications for a pair (E, A) of matrix functions that satisfies Hypothesis A.1 or Hypothesis B.1. We start with the common part of both hypotheses, in particular with the special form of the solution space $\mathbb{K}$. From

$$x = \Phi(\alpha + I[c]) + \Psi c \tag{4.1}$$

it follows by differentiation that

$$\dot x = \Phi L c + \dot\Phi(\alpha + I[c]) + \Psi\dot c + \dot\Psi c \tag{4.2}$$

because of

$$I[c](t) = \int_{t_0}^{t} L(\tau)c(\tau)\, dt, \quad \tfrac{d}{dt}(I[c])(t) = L(t)c(t). \tag{4.3}$$

Hence

$$E[\Phi L c + \dot\Phi(\alpha + I[c]) + \Psi\dot c + \dot\Psi c] = A[\Phi(\alpha + I[c]) + \Psi c] \tag{4.4}$$

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for arbitrary $\alpha \in \mathbb{C}^d$, $c \in C^1(\mathbb{I}, \mathbb{C}^u)$. Note that we can combine several choices of α and c in (4.4) into a matrix relation. Thus, for the choice $\alpha = I_d$, $c = 0$, we find that

$$E\dot{\Phi} = A\Phi. \tag{4.5}$$

This reduces (4.4) to

$$E[\Phi L c + \Psi \dot{c} + \dot{\Psi} c] = A\Psi c, \tag{4.6}$$

which still holds for arbitrary $c \in C^1(\mathbb{I}, \mathbb{C}^u)$. For the choice $c = I_u$, we get

$$E[\Phi L + \dot{\Psi}] = A\Psi, \tag{4.7}$$

which reduces (4.6) to

$$E\Psi\dot{c} = 0, \tag{4.8}$$

again for arbitrary $c \in C^1(\mathbb{I}, \mathbb{C}^u)$. Finally, the choice $c = tI_u$ yields

$$E\Psi = 0. \tag{4.9}$$

Since $[\,\Phi\ \ \Psi\,]$ is continuously differentiable and has pointwise full column rank, there exists a matrix function $\Theta \in C^1(\mathbb{I}, \mathbb{C}^a)$ with $a = n - d - u$ such that $[\,\Phi\ \ \Psi\ \ \Theta\,]$ is pointwise nonsingular also under the assumptions of Hypothesis B.1. Hence, on both levels

$$\begin{aligned}(E, A) &\sim ([\,E\Phi\ \ E\Psi\ \ E\Theta\,], [\,A\Phi - E\dot{\Phi}\ \ A\Psi - E\dot{\Psi}\ \ A\Theta - E\dot{\Theta}\,]) = \\ &= ([\,E\Phi\ \ 0\ \ E\Theta\,], [\,0\ \ E\Phi L\ \ A\Theta - E\dot{\Theta}\,]),\end{aligned} \tag{4.10}$$

where we used (4.5), (4.7), and (4.9).

At this point, we first look at Hypothesis A.1. Because of (3.18), there exists a pointwise nonsingular $P \in C(\mathbb{I}, \mathbb{C}^{n,n})$ with

$$PE\Phi = \begin{bmatrix} I_d \\ 0 \end{bmatrix} \tag{4.11}$$

such that

$$(E, A) \sim \left(\begin{bmatrix} I_d & 0 & * \\ 0 & 0 & H \end{bmatrix}, \begin{bmatrix} 0 & L & * \\ 0 & 0 & B \end{bmatrix} \right). \tag{4.12}$$

Consider now the subproblem

$$H(t)\dot{x}_3 = B(t)x_3 + f_2(t), \tag{4.13}$$

where

$$Q^{-1}x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad Pf = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}. \tag{4.14}$$

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Since the first block row in (4.12) is solvable independently of x_3, consistency of $f \in C(\mathbb{I}, \mathbb{C}^m)$ is equivalent with the consistency of $f_2 \in C(\mathbb{I}, \mathbb{C}^{u+a})$ for the subproblem (4.13). Due to the structure of $\mathbb{K}$, the subproblem (4.13) must fix a unique solution for consistent f_2. Moreover, since x_3 does not depend on f_1, the form of the particular solution (3.19) yields that a solution of (4.13) must have the form

$$x_3 = \sum_{i=0}^{k-1} \tilde{D}_i f_2^{(i)}. \tag{4.15}$$

For convenience, we write the derived properties of (E, A) as a new hypothesis for (E, A).

HYPOTHESIS A.2. *The pair (E, A) of matrix functions satisfies*

$$(E, A) \sim \left(\begin{bmatrix} I_d & 0 & * \\ 0 & 0 & H \end{bmatrix}, \begin{bmatrix} 0 & L & * \\ 0 & 0 & B \end{bmatrix} \right), \tag{4.16}$$

where

$$H(t)\dot{x}_3 = B(t)x_3 + f_2(t), \tag{4.17}$$

possesses a unique solution for every consistent sufficiently smooth f. This also holds for every restriction to a nontrivial subinterval of $\mathbb{I}$. In particular, there exist $\tilde{D}_i \in C(\mathbb{I}, \mathbb{C}^{a,u+a})$, $i = 0, \ldots, k-1$, such that the solution of (4.17), if it exists, is of the form

$$x_3 = \sum_{i=0}^{k-1} \tilde{D}_i f_2^{(i)}. \tag{4.18}$$

Note that Hypothesis A.2 is trivially invariant under (global) equivalence transformations. Since the above discussion also holds for every nontrivial subinterval, we have shown the following implication.

THEOREM 4.1. *Hypothesis A.1 implies Hypothesis A.2.*

We return now to (4.10) and concentrate on Hypothesis B.1. From (3.20), we have

$$P^{-1} \begin{bmatrix} I_d \\ 0 \\ 0 \end{bmatrix} = EQ = R^{-1} \begin{bmatrix} I_d \\ 0 \\ 0 \end{bmatrix}. \tag{4.19}$$

Setting

$$Pf = \begin{bmatrix} \tilde{f}_1 \\ \tilde{f}_2 \\ \tilde{f}_3 \end{bmatrix} \tag{4.20}$$

for a given inhomogeneity, it follows with (3.21) that

$$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = Rf = RP^{-1} \begin{bmatrix} \tilde{f}_1 \\ \tilde{f}_2 \\ \tilde{f}_3 \end{bmatrix} = \begin{bmatrix} I_d & P_{12} & P_{13} \\ 0 & P_{22} & P_{23} \\ 0 & P_{32} & P_{33} \end{bmatrix} \begin{bmatrix} \tilde{f}_1 \\ \tilde{f}_2 \\ \tilde{f}_3 \end{bmatrix}. \tag{4.21}$$

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Application of the transformation RP^{-1} to the pair on the right hand side of (4.12) yields

$$(E,A) \sim \left(\begin{bmatrix} I_d & 0 & E_{13} \\ 0 & 0 & E_{23} \\ 0 & 0 & E_{33} \end{bmatrix}, \begin{bmatrix} 0 & L & A_{13} \\ 0 & 0 & A_{23} \\ 0 & 0 & A_{33} \end{bmatrix} \right). \tag{4.22}$$

Note that by construction the transformation of (1.1) according to (4.22) produces (3.21) as inhomogeneity. Hence, the transformed DAE reads

$$\begin{array}{ll} \text{(a)} & \dot{x}_1 + E_{13}(t)\dot{x}_3 = L(t)\dot{x}_2 + A_{13}(t)x_3 + f_1(t), \\ \text{(b)} & E_{23}(t)\dot{x}_3 = A_{23}(t)x_3 + f_2(t), \\ \text{(c)} & E_{33}(t)\dot{x}_3 = A_{33}(t)x_3 + f_3(t). \end{array} \tag{4.23}$$

By Hypothesis B.1, the DAE (1.1) and thus (4.23) is solvable if f_3 is sufficiently smooth and $f_2 = C[f_3]$. In particular, the subsystem (4.23c) is solvable for every sufficiently smooth f_3. Moreover, due to the structure of $\mathbb{K}$, the solution must be unique. It follows that the DAE

$$E_{33}(t)\dot{S} = A_{33}(t)S + I_a \tag{4.24}$$

possesses a unique solution $S \in C^1(\mathbb{I}, \mathbb{C}^{a,a})$. Following the arguments in [1], a small (smooth) perturbation of S yields a pointwise nonsingular matrix function $\tilde{S} \in C^1(\mathbb{I}, \mathbb{C}^{a,a})$ such that

$$J = E_{33}\dot{\tilde{S}} - A_{33}\tilde{S} \tag{4.25}$$

is still pointwise nonsingular. We then get

$$\begin{aligned} (E,A) &\sim \left(\begin{bmatrix} I_d & 0 & E_{13}\tilde{S} \\ 0 & 0 & E_{23}\tilde{S} \\ 0 & 0 & E_{33}\tilde{S} \end{bmatrix}, \begin{bmatrix} 0 & L & A_{13}\tilde{S} - E_{13}\dot{\tilde{S}} \\ 0 & 0 & A_{23}\tilde{S} - E_{23}\dot{\tilde{S}} \\ 0 & 0 & A_{33}\tilde{S} - E_{33}\dot{\tilde{S}} \end{bmatrix} \right) \sim \\ &\sim \left(\begin{bmatrix} I_d & 0 & \tilde{E}_{13} \\ 0 & 0 & \tilde{E}_{23} \\ 0 & 0 & \tilde{E}_{33} \end{bmatrix}, \begin{bmatrix} 0 & L & \tilde{A}_{13} \\ 0 & 0 & \tilde{A}_{23} \\ 0 & 0 & J \end{bmatrix} \right) \sim \\ &\sim \left(\begin{bmatrix} I_d & 0 & W \\ 0 & 0 & F \\ 0 & 0 & G \end{bmatrix}, \begin{bmatrix} 0 & L & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_a \end{bmatrix} \right), \end{aligned} \tag{4.26}$$

where the subsystem

$$G(t)\dot{x}_3 = x_3 + f_3(t) \tag{4.27}$$

possesses a unique solution for every sufficiently smooth f_3. Again the whole construction is valid on every nontrivial subinterval. As before, we formulate the obtained properties as a hypothesis on (E, A).

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HYPOTHESIS B.2. *The pair (E, A) of matrix functions satisfies*

$$(E,A) \sim \left(\begin{bmatrix} I_d & 0 & W \\ 0 & 0 & F \\ 0 & 0 & G \end{bmatrix}, \begin{bmatrix} 0 & L & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_a \end{bmatrix} \right), \tag{4.28}$$

where

$$G(t)\dot{x}_3 = x_3 + f_3(t) \tag{4.29}$$

possesses a unique solution for every sufficiently smooth f_3. This also holds for every restriction to a nontrivial subinterval of $\mathbb{I}$.

The invariance of Hypothesis B.2 is again trivial and the above discussion can now be formulated as follows.

THEOREM 4.2. *Hypothesis B.1 implies Hypothesis B.2.*

At this point, it becomes clear why the more restrictive Hypothesis B.1 is of interest. Comparing with (3.1), the canonical form given in (4.28) has the same block structure. The main difference to (3.1) is that we do not have the nilpotent structure of the matrix functions F and G in (4.28). The reason for this is that in Hypothesis B.1 we do not require all the constant rank conditions to obtain (3.2).

We close this section with an equivalent formulation of (3.18) in terms of solution properties of the given DAE.

LEMMA 4.3. *An equivalent formulation of Hypothesis A.1 is obtained if the condition (3.18) is replaced by the following property:*

Let $x \in C^1(\hat{\mathbb{I}}, \mathbb{C}^n)$ solve (1.1) with $f \in \operatorname{range} E\Phi$ on a nontrivial subinterval $\hat{\mathbb{I}} \subseteq \mathbb{I}$ and let $\Pi^H x = 0$, where $\Pi \in C^1(\mathbb{I}, \mathbb{C}^{n,u})$ has pointwise full column rank and satisfies $\Pi^H [\, \Phi \;\; \Theta \,] = 0$. Then x can be (uniquely) extended to a function in $C^1(\mathbb{I}, \mathbb{C}^n)$ that solves (1.1).

Proof. In contrast to (3.18), let

$$\operatorname{rank} E(\hat{t})\Phi(\hat{t}) < d$$

for some $\hat{t} \in \mathbb{I}$. Then there exists a $w \in \mathbb{C}^d$, $w \neq 0$, with

$$E(\hat{t})\Phi(\hat{t})w = 0.$$

Choosing $\hat{\mathbb{I}} \subseteq \mathbb{I}$ open such that $\hat{t}$ is a boundary point of $\hat{\mathbb{I}}$ and setting

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} \log(|t-\hat{t}|)w \\ 0 \\ 0 \end{bmatrix}, \quad f(t) = \begin{cases} E(t)\Phi(t)\frac{1}{t-\hat{t}}w & \text{for } t \neq \hat{t}, \\ \frac{d}{dt}(E(t)\Phi(t)w)|_{t=\hat{t}} & \text{for } t = \hat{t}, \end{cases}$$

we have

$$E(t)\Phi(t)\dot{x}_1(t) = E(t)\Phi(t)\frac{1}{t-\hat{t}}w = f(t)$$

for $t \neq \hat{t}$, i.e., $[x_1^T, x_2^T, x_3^T]^T$ with $x_2 = 0$ and $x_3 = 0$ solves the transformed problem given in (4.10). Hence, x given by

$$x(t) = \Phi(t)x_1(t) = \Phi(t)\log(|t-\hat{t}|)w$$

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solves the original DAE on $\hat{\mathbb{I}}$. Moreover,

$$\Pi(t)^H x(t) = \Pi(t)^H \Phi(t) \log(|t - \hat{t}|)w = 0$$

on $\hat{\mathbb{I}}$. But x cannot be extended to a function in $C^1(\mathbb{I}, \mathbb{C}^n)$.

On the other hand, if (3.18) holds, then we can transform the DAE (1.1) according to (4.12). The inhomogeneity is then given by $[f_1^T, f_2^T]^T$, where $f_2 = 0$ for $f \in \operatorname{range} E\Phi$. Let now $x_1 \in C^1(\hat{\mathbb{I}}, \mathbb{C}^d)$, $x_2 = 0$, and $x_3 = 0$ (where the latter two guarantee $\Pi^H x = 0$) solve the transformed DAE. Then the equation corresponding to the second block row is trivially satisfied and the one corresponding to the first block row reduces to $\dot{x}_1 = f_1$, which is solved by x_1 on $\hat{\mathbb{I}}$. It is then obvious that x_1 can be extended to a solution on the entire interval $\mathbb{I}$. □

5. Derivative arrays and reduced DAEs. An obvious advantage of (2.2), at least in the numerical treatment of DAEs, see, e.g., [8], is that only the data functions E, A and f together with their derivatives are involved. One is therefore interested in equivalent characterizations of DAEs in terms of derivative arrays. Moreover, this will also help in proving that all characterizations that belong to the same level are equivalent.

We first assume that Hypothesis A.2 holds. Furthermore, let $(\tilde{M}_\ell, \tilde{N}_\ell)$ be the derivative arrays which belong to the canonical form $(\tilde{E}, \tilde{A})$ given in (4.16). The entry I_d occurring in every diagonal block of $\tilde{M}_\ell$ always contributes to the rank of $\tilde{M}_\ell$. But then the entry L never contributes to the rank of $[\,\tilde{M}_\ell\ \ \tilde{N}_\ell\,]$. Thus the only contribution of $\tilde{N}_\ell$ to the rank of $[\,\tilde{M}_\ell\ \tilde{N}_\ell\,]$ can come from the block column built of H and its derivatives. Since this block column only consists of a columns, we have

$$\operatorname{rank}[\,\tilde{M}_\ell\ \ \tilde{N}_\ell\,] \le \operatorname{rank} \tilde{M}_\ell + a. \tag{5.1}$$

On the other hand, the DAE (4.17) is (uniquely) solvable for all inhomogeneities f_2 of the form

$$f_2 = H\dot{x}_3 - Bx_3 \tag{5.2}$$

for given sufficiently smooth $x_3 \in C^1(\mathbb{I}, \mathbb{C}^a)$. Hence,

$$x_3 = \sum_{i=0}^{k-1} \tilde{D}_i (\tfrac{d}{dt})^i (H\dot{x}_3 - Bx_3)$$

or

$$x_3 = [\,\tilde{D}_0\ \tilde{D}_1\ \cdots\ \tilde{D}_{k-1}\,] \begin{bmatrix} -B & H & & & \\ -\dot{B} & \dot{H} - B & H & & \\ \vdots & \vdots & \ddots & \ddots & \\ -B^{(k-1)} & * & \cdots & * & H \end{bmatrix} \begin{bmatrix} x_3 \\ \dot{x}_3 \\ \vdots \\ x_3^{(k-1)} \end{bmatrix}$$

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for all sufficiently smooth $x_3 \in C^1(\mathbb{I}, \mathbb{C}^a)$. This implies

$$[\, I_a \ \ 0 \ \ \cdots \ \ 0 \,] = [\, \tilde{D}_0 \ \ \tilde{D}_1 \ \ \cdots \ \ \tilde{D}_{k-1} \,] \begin{bmatrix} -B & H & & & \\ -\dot{B} & \dot{H} - B & H & & \\ \vdots & \vdots & \ddots & \ddots & \\ -B^{(k-1)} & * & \cdots & * & H \end{bmatrix}.$$

Thus, defining $\tilde{Z}_3 \in C(\mathbb{I}, \mathbb{C}^{km,a})$ by

$$\tilde{Z}_3^H = [\, 0 \ \ \tilde{D}_0 \,|\, 0 \ \ \tilde{D}_1 \,|\, \cdots \,|\, 0 \ \ \tilde{D}_{k-1} \,], \tag{5.3}$$

we have

$$\tilde{Z}_3^H \tilde{M}_{k-1} = 0, \quad \operatorname{rank} \tilde{Z}_3^H \tilde{N}_{k-1} = a. \tag{5.4}$$

This implies that $\operatorname{rank}[\, \tilde{M}_\ell \ \ \tilde{N}_\ell \,] \geq \operatorname{rank} \tilde{M}_\ell + a$ for $\ell = k-1$. Trivially extending Z_3 by zero blocks shows that this holds for every $\ell \geq k-1$ so that

$$\operatorname{rank}[\, \tilde{M}_\ell \ \ \tilde{N}_\ell \,] = \operatorname{rank} \tilde{M}_\ell + a \text{ for } \ell \geq k-1, \tag{5.5}$$

as long as all quantities are defined. Finally, defining $\tilde{T}_3 \in C(\mathbb{I}, \mathbb{C}^{n,n-a})$ by

$$\tilde{T}_3 = \begin{bmatrix} I_d & 0 \\ 0 & I_u \\ 0 & 0 \end{bmatrix} \tag{5.6}$$

and $\tilde{Z}_1 \in C(\mathbb{I}, \mathbb{C}^{m,d})$ by

$$\tilde{Z}_1 = \begin{bmatrix} I_d \\ 0 \\ 0 \end{bmatrix} \tag{5.7}$$

yields the relations

$$\tilde{Z}_3^H \tilde{N}_{k-1} [\, I_n \ \ 0 \ \ \cdots \ \ 0 \,]^H \tilde{T}_3 = 0, \quad \operatorname{rank} \tilde{Z}_1^H \tilde{E} \tilde{T}_3 = d. \tag{5.8}$$

A pair (E, A) of matrix functions satisfying Hypothesis A.2 therefore satisfies the following hypothesis, at least when (E, A) is given in the canonical form of (4.16).

Hypothesis A.3. *The pair (E, A) of matrix functions with its derivative arrays (M_ℓ, N_ℓ) has the following properties:*
1) There exists a matrix function $Z_3 \in C(\mathbb{I}, \mathbb{C}^{km,a})$ with pointwise full column rank and

$$Z_3^H M_{k-1} = 0, \quad \operatorname{rank} Z_3^H N_{k-1} = a \tag{5.9}$$

implying that there exists a matrix function $T_3 \in C(\mathbb{I}, \mathbb{C}^{n,n-a})$ with pointwise full column rank and

$$Z_3^H N_{k-1} [\, I_n \ \ 0 \ \ \cdots \ \ 0 \,]^H T_3 = 0. \tag{5.10}$$

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2) For every $t \in \mathbb{I}$ and every matrix Z_4 whose columns form a basis of corange $M_k(t)$, *we have*

$$Z_4^H N_k(t) = a. \tag{5.11}$$

3) There exists a matrix function $Z_1 \in C(\mathbb{I}, \mathbb{C}^{n,d})$ with pointwise full column rank and

$$\operatorname{rank} Z_1^H E T_3 = d. \tag{5.12}$$

LEMMA 5.1. *Hypothesis A.3 is invariant under (global) equivalence transformations.*

Proof. Let (E, A) with its derivative arrays (M_ℓ, N_ℓ) satisfy Hypothesis A.3 and let

$$\tilde{E} = PEQ, \quad \tilde{A} = PAQ - PE\dot{Q}, \quad \tilde{f} = Pf, \quad \tilde{x} = Q^{-1}x$$

with P and Q according to Definition 2.1. Furthermore, let $(\tilde{M}_\ell, \tilde{N}_\ell)$ be the derivative arrays of $(\tilde{E}, \tilde{A})$. Then (2.4) holds. Defining

$$\tilde{Z}_3^H = Z_3^H \Pi_{k-1}^{-1}, \quad \tilde{T}_3 = Q^{-1}T_3, \quad \tilde{Z}_4^H = Z_4^H \Pi_k^{-1}(t), \quad \tilde{Z}_1^H = Z_1^H P^{-1}$$

yields

$$\tilde{Z}_3^H \tilde{M}_{k-1} = Z_3^H \Pi_{k-1}^{-1} \Pi_{k-1} M_{k-1} \Theta_{k-1} = Z_3^H M_{k-1} \Theta_{k-1} = 0$$

and

$$\operatorname{rank} \tilde{Z}_3^H \tilde{N}_{k-1} = \operatorname{rank} Z_3^H \Pi_{k-1}^{-1} (\Pi_{k-1} N_{k-1} \Theta_{k-1} - \Pi_{k-1} M_{k-1} \Psi_{k-1}) = a.$$

Property 2) follows accordingly. Furthermore,

$$\begin{aligned} \tilde{Z}_3^H \tilde{N}_{k-1} [\, I_n \ 0 \ \cdots \ 0 \,]^H \tilde{T}_3 &= Z_3^H N_{k-1} \Theta_{k-1} [\, I_n \ 0 \ \cdots \ 0 \,]^H Q^{-1} T_3 = \\ &= Z_3^H N_{k-1} [\, Q \ * \ \cdots \ * \,]^H Q^{-1} T_3 = Z_3^H N_{k-1} [\, I_n \ 0 \ \cdots \ 0 \,]^H T_3, \end{aligned}$$

since only the first block column of N_{k-1} is nontrivial. Finally,

$$\operatorname{rank} \tilde{Z}_1^H \tilde{E} \tilde{T}_3 = \operatorname{rank} Z_1^H P^{-1} PEQQ^{-1} T_3 = \operatorname{rank} Z_1^H E T_3 = d.$$

□

Again, the previous discussion together with the invariance of the developed hypothesis shows that the following implication holds.

THEOREM 5.2. *Hypothesis A.2 implies Hypothesis A.3.*

We now assume that Hypothesis B.2 holds and show that it implies Hypothesis A.2. The principle part of the corresponding proof can already be found in [7]. Nevertheless we present a detailed proof, since we need the same techniques later in the course of our discussion. It is sufficient to concentrate on the part belonging to (4.29). We therefore consider

$$(E, A) = (G, I_a), \tag{5.13}$$

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with G as in (3.2c), and the corresponding derivative arrays (M_ℓ, N_ℓ) and assume that the associated DAE is uniquely solvable for every sufficiently smooth inhomogeneity.

Suppose that there exists $\hat{t} \in \mathbb{I}$ with $\operatorname{corank}[\, M_\ell(\hat{t}) \;\; N_\ell(\hat{t}) \,] > 0$, where the corank is defined to be the rank deficiency with respect to the rows. Then there exists a $v_\ell \in \mathbb{C}^{(l+1)a}$, $v_\ell \neq 0$, with $v_\ell^H [\, M_\ell(\hat{t}) \;\; N_\ell(\hat{t}) \,] = 0$ and an arbitrarily smooth function $f = f_3$ with $v_\ell^H g_\ell(\hat{t}) \neq 0$ for the corresponding g_ℓ defined by (2.3c). But this is in contradiction to the solvability of (4.29) which implies (2.2) and thus $v_\ell^H g_\ell(\hat{t}) = 0$. Hence, we have

$$\operatorname{corank}[\, M_\ell \;\; N_\ell \,] = 0 \text{ on } \mathbb{I}. \tag{5.14}$$

Since N_ℓ has only a nontrivial columns this implies

$$\operatorname{corank} \, M_\ell \leq a \text{ on } \mathbb{I}. \tag{5.15}$$

On the other hand, there exist disjoint open intervals $\mathbb{I}_j \subseteq \mathbb{I}$ with (3.3) such that the strangeness index μ is well-defined for (5.13) restricted to a selected $\mathbb{I}_j$. Because of the unique solvability of the associated DAE (5.13) on $\mathbb{I}_j$ due to Hypothesis B.2, its canonical form from (3.1) can only consist of the part (G, I_a). Recall that the other parts would allow for a free choice of initial values $x_1(t_0)$ or of functions x_2 according to (3.4) and the following discussion. Hence, we may assume on $\mathbb{I}_j$ that G has the nilpotent structure (3.2c). The corresponding derivative arrays M_ℓ are given by

$$M = \begin{bmatrix} G & & & \\ \dot{G} - I & G & & \\ \ddot{G} & 2\dot{G} - I & G & \\ \vdots & \ddots & \ddots & \ddots \end{bmatrix}, \tag{5.16}$$

where we formally consider M to be an infinite matrix function as suggested in [7]. The expressions that will be developed in the following will turn out to be finite when taking into account that due to the nilpotent structure of G all $(\mu+1)$-fold products, where each factor is G or one of its derivatives, vanish.

We are interested in the corange (i.e., in the orthogonal complement of the range) of M. Thus, we look for a nontrivial Z of maximal rank with

$$Z^H M = 0. \tag{5.17}$$

With $Z^H = [\, Z_0^H \;\; Z_1^H \;\; Z_2^H \;\; \cdots \,]$, this can be written as

$$[\, Z_0^H \;\; Z_1^H \;\; Z_2^H \;\; \cdots \,] \left\{ \begin{bmatrix} G & & & \\ \dot{G} & G & & \\ \ddot{G} & 2\dot{G} & G & \\ \vdots & \ddots & \ddots & \ddots \end{bmatrix} - \begin{bmatrix} 0 & & & \\ I & 0 & & \\ 0 & I & 0 & \\ \vdots & \ddots & \ddots & \ddots \end{bmatrix} \right\} = 0. \tag{5.18}$$

Setting $Z_0^H = I$ and solving for the other blocks of Z gives

$$[\, Z_1^H \;\; Z_2^H \;\; \cdots \,] = [\, G \;\; 0 \;\; \cdots \,](I - X)^{-1}, \tag{5.19}$$

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where

$$X = \begin{bmatrix} \dot{G} & G & & \\ \ddot{G} & 2\dot{G} & G & \\ \vdots & \ddots & \ddots & \ddots \end{bmatrix} \tag{5.20}$$

is nilpotent, hence

$$(I - X)^{-1} = \sum_{i \geq 0} X^i. \tag{5.21}$$

A simple induction argument then yields that Z_j^H is a sum of at least j-fold products, where each factor is G or one of its derivatives. Thus, $Z_j^H = 0$ for $j > \mu$ and all expressions are indeed finite. Moreover, we have shown that

$$\operatorname{corank} M_\ell \geq a \text{ on } \mathbb{I}_j \text{ for } \ell \geq \mu. \tag{5.22}$$

Because of (2.6a), this also holds when G does not necessarily have the nilpotent structure of (3.2c). Observing that corank $M_{\ell+1} \geq$ corank M_ℓ for every ℓ due to (2.3), we get

$$\operatorname{corank} M_\ell \geq a \text{ on } \bigcup_{j \in \mathbb{N}} \mathbb{I}_j \text{ for } \ell \geq \hat{\mu}, \tag{5.23}$$

where $\hat{\mu} \leq \min\{m, n\} - 1 = k - 1$ is the maximum of all strangeness indices for the subintervals $\mathbb{I}_j$. Since the rank function is lower semicontinuous, this implies

$$\operatorname{corank} M_\ell \geq a \text{ on } \mathbb{I} \text{ for } \ell \geq \hat{\mu}. \tag{5.24}$$

Together with (5.15), this gives

$$\operatorname{corank} M_\ell = a \text{ on } \mathbb{I} \text{ for } \ell \geq \hat{\mu}. \tag{5.25}$$

In particular, M_{k-1} has constant rank on $\mathbb{I}$. Hence, there exists a matrix function $Z_3 \in C(\mathbb{I}, \mathbb{C}^{ka,a})$ with

$$Z_3^H M_{k-1} = 0, \quad \operatorname{rank} N_{k-1} [\, I_a \;\; 0 \;\; \cdots \;\; 0 \,]^H = a, \tag{5.26}$$

the latter because of (5.14) and the special form of N_{k-1}. In particular, Z_3^H has the form

$$Z_3^H = [\, I_a \;\; \tilde{D}_1 \;\; \cdots \;\; \tilde{D}_{k-1} \,] \tag{5.27}$$

with appropriately defined matrix functions $\tilde{D}_i$, $i = 1, \ldots, k-1$. The inflated DAE (2.2) for (5.13) with $\ell = k - 1$ implies

$$0 = x_3 + Z_3^H g_{k-1} \tag{5.28}$$

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due to the special form of N_ℓ. Hence, the solution of (4.29) is given by

$$x_3 = -Z_3^H g_{k-1}. \tag{5.29}$$

Observing the definition of g_{k-1}, we can write x_3 as

$$x_3 = \sum_{i=0}^{k-1} \tilde{D}_i f_3^{(i)}. \tag{5.30}$$

But this is exactly the form of solution representation as required in (4.18) such that we have shown the following result.

Theorem 5.3. *Hypothesis B.2 implies Hypothesis A.2.*

At this point, it seems to be convenient to first study implications of Hypothesis A.3 before we proceed with further implications of Hypothesis B.2. Given a $Z_2 \in C(\mathbb{I}, \mathbb{C}^{(k+1)m,v})$ with $v = m - d - a$ satisfying

$$Z_2^H M_k = 0, \quad Z_2^H N_k = 0, \tag{5.31}$$

Hypothesis A.3 fixes a so-called reduced DAE

$$\begin{bmatrix} \hat{E}_1(t) \\ 0 \\ 0 \end{bmatrix} \dot{x} = \begin{bmatrix} \hat{A}_1(t) \\ 0 \\ \hat{A}_3(t) \end{bmatrix} x + \begin{bmatrix} \hat{f}_1(t) \\ \hat{f}_2(t) \\ \hat{f}_3(t) \end{bmatrix}, \tag{5.32}$$

where

$$\begin{array}{lll} \hat{E}_1 = Z_1^H E, & \hat{A}_1 = Z_1^H A, & \hat{A}_3 = Z_3^H N_{k-1}[\, I_n \;\; 0 \;\; \cdots \;\; 0 \,]^H, \\ \hat{f}_1 = Z_1^H f, & \hat{f}_2 = Z_2^H g_k, & \hat{f}_3 = Z_3^H g_{k-1}. \end{array} \tag{5.33}$$

Obviously, every (sufficiently smooth) solution x of (1.1) must also solve (5.32) implying (pointwise)

$$g_k \in \operatorname{range}[\, M_k \;\; N_k \,] \tag{5.34}$$

and thus we must have $\hat{f}_2 = 0$. If such a Z_2 does not occur, then we can set $\hat{f}_2 = 0$ anyway. On the other hand, one can show that (5.34) implies solvability of (1.1).

Theorem 5.4. *Let (E, A) satisfy Hypothesis A.3. Furthermore, let f satisfy (5.34). Then x solves (1.1) if and only if it solves (5.32).*

Proof. As already mentioned, if x solves (1.1) it is immediately clear by construction that it also solves (5.32). Let x now be a solution of (5.32). According to Theorem 3.3, we restrict the problem to an interval $\mathbb{I}_j$ and transform there to the canonical form $(\tilde{E}, \tilde{A})$ given in (3.1). Due to Hypothesis A.3 the block-sizes of both canonical forms must coincide. Let the corresponding derivative arrays be denoted by (M_ℓ, N_ℓ) and $(\tilde{M}_\ell, \tilde{N}_\ell)$, respectively. In the notation of Hypothesis A.3 and

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Lemma 5.1, we have

$$\begin{array}{l}\tilde{Z}_1^H\tilde{E} = Z_1^H P^{-1}PEQ = Z_1EQ,\\ \tilde{Z}_1^H\tilde{A} = Z_1^H P^{-1}(PAQ - PE\dot{Q}) = Z_1AQ - Z_1E\dot{Q},\\ \tilde{Z}_3^H\tilde{N}_{k-1}[\, I_n \;\; 0 \;\; \cdots \;\; 0\,]^H =\\ \quad = Z_3^H\Pi_{k-1}^{-1}(\Pi_{k-1}N_{k-1}\Theta_{k-1} - \Pi_{k-1}M_{k-1}\Psi_{k-1})[\, I_n \;\; 0 \;\; \cdots \;\; 0\,]^H =\\ \quad = Z_3^H N_{k-1}\Theta_{k-1}[\, I_n \;\; 0 \;\; \cdots \;\; 0\,]^H = Z_3^H N_{k-1}[\, Q \;\; * \;\; \cdots \;\; *\,]^H =\\ \quad = Z_3^H N_{k-1}[\, I_n \;\; 0 \;\; \cdots \;\; 0\,]^H Q.\end{array}$$

This shows that the reduced problem transforms covariantly with Q. Thus, it is sufficient to consider the problem in the canonical form of (3.1). Hence, we are allowed to assume that

$$E = \begin{bmatrix} I_d & 0 & W \\ 0 & 0 & F \\ 0 & 0 & G \end{bmatrix}, \quad A = \begin{bmatrix} 0 & L & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_a \end{bmatrix}, \quad f = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix},$$

where F and G have the nilpotent structure of (3.2). Using again formally infinite matrix functions, we get from (5.18) that we can choose

$$Z_3^H = [\, 0 \;\; 0 \;\; I_a \mid 0 \;\; 0 \;\; Z_{31}^H \mid 0 \;\; 0 \;\; Z_{32}^H \mid \cdots\,]$$

with

$$[\, Z_{31}^H \;\; Z_{32}^H \;\; \cdots\,] = [\, G \;\; 0 \;\; \cdots\,](I - X)^{-1}$$

satisfying (5.9). In the same way, we can choose

$$Z_2^H = [\, 0 \;\; I_v \;\; 0 \mid 0 \;\; 0 \;\; Z_{21}^H \mid 0 \;\; 0 \;\; Z_{22}^H \mid \cdots\,]$$

with

$$[\, Z_{21}^H \;\; Z_{22}^H \;\; \cdots\,] = [\, F \;\; 0 \;\; \cdots\,](I - X)^{-1}$$

satisfying (5.31) because of

$$\begin{array}{l} Z_2^H M_k = [\, F \;\; 0 \;\; \cdots\,] + [\, Z_{21}^H \;\; Z_{22}^H \;\; \cdots\,](X - I) = 0,\\ Z_2^H N_k = I_v 0 + [\, Z_{21}^H \;\; Z_{22}^H \;\; \cdots\,]0 = 0.\end{array}$$

Finally, we can choose

$$Z_1^H = [\, I_d \;\; 0 \;\; 0\,].$$

The corresponding reduced problem thus reads

$$\begin{bmatrix} I_d & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & L & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_a \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \hat{f}_1 \\ \hat{f}_2 \\ \hat{f}_3 \end{bmatrix}$$

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with

$$\begin{array}{l} \hat{f}_1 = f_1, \\ \hat{f}_2 = f_2 + FV^H(I-X)^{-1}g, \\ \hat{f}_3 = f_3 + GV^H(I-X)^{-1}g, \end{array} \qquad V = \begin{bmatrix} I_a \\ 0 \\ \vdots \end{bmatrix}, \quad g = \begin{bmatrix} \dot{f}_3 \\ \ddot{f}_3 \\ \vdots \end{bmatrix}.$$

In particular, we have $\hat{f}_2 = 0$ due to the assumption on g_k. The reduced problem at once yields $x_3 = -\hat{f}_3$. With the block up-shift matrix

$$S = \begin{bmatrix} 0 & I & & \\ & 0 & I & \\ & & \ddots & \ddots \end{bmatrix},$$

we have the identities $\dot{g} = S^H g$,

$$(I-X)^{-1}S^H = S^H(I-X)^{-1} - (I-X)^{-1}\dot{X}(I-X)^{-1},$$

see [7], and

$$\begin{aligned} V^H(I-X)^{-1} &= V^H \sum_{i\geq 0} X^i = V^H + V^H X \sum_{i\geq 0} X^i = \\ &= V^H + (\dot{G}V^H + GV^H S^H)(I-X)^{-1}. \end{aligned}$$

We then find that

$$\begin{aligned} \hat{f}_2 - F\dot{\hat{f}}_3 &= f_2 + FV^H(I-X)^{-1}g - F\dot{f}_3 - F\dot{G}V^H(I-X)^{-1}g - \\ &\quad -FGV^H(I-X)^{-1}\dot{X}(I-X)^{-1}g - FGV^H(I-X)^{-1}\dot{g} = \\ &= f_2 + FV^H g + F\dot{G}V^H(I-X)^{-1}g + FGV^H S^H(I-X)^{-1}g - \\ &\quad -F\dot{f}_3 - F\dot{G}V^H(I-X)^{-1}g - FGV^H S^H(I-X)^{-1}g + \\ &\quad +FGV^H(I-X)^{-1}S^H g - FGV^H(I-X)^{-1}S^H g = \\ &= f_2 + FV^H g - F\dot{f}_3 = f_2. \end{aligned}$$

Replacing F with G yields in the same way that $\hat{f}_3 - G\dot{\hat{f}}_3 = f_3$. Hence,

$$\begin{aligned} F\dot{x}_3 &= -F\dot{\hat{f}}_3 = f_2 - \hat{f}_2 = f_2, \\ G\dot{x}_3 &= -G\dot{\hat{f}}_3 = f_3 - \hat{f}_3 = x_3 + f_3. \end{aligned}$$

This shows that the transformed x solves the transformed DAE (1.1) on $\mathbb{I}_j$. Thus, x solves (1.1) on $\mathbb{I}_j$ for every $j \in \mathbb{N}$ and therefore on a dense subset of $\mathbb{I}$. Since all functions are continuous, the given x solves (1.1) on the entire interval $\mathbb{I}$. $\square$

THEOREM 5.5. *Hypothesis A.3 implies Hypothesis A.1.*

Proof. Extending T_3 from (5.10) to a smooth pointwise nonsingular matrix function $[\,T_3\ T_4\,]$ and splitting T_3 into $[\,T_1\ T_2\,]$ such that $Z_1^H E T_1$ is pointwise nonsingular and $Z_1^H E T_2 = 0$, the pair of matrix functions belonging to the reduced problem (5.32)

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can be transformed to the canonical form of (3.1) according to

$$
\begin{aligned}
&\left(\begin{bmatrix} \hat{E}_1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \hat{A}_1 \\ 0 \\ \hat{A}_3 \end{bmatrix}\right) \sim \left(\begin{bmatrix} Z_1^H E T_1 & 0 & Z_1^H E T_4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} * & * & * \\ 0 & 0 & 0 \\ 0 & 0 & \hat{A}_3 T_4 \end{bmatrix}\right) \sim \\
&\sim \left(\begin{bmatrix} I_d & 0 & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} J & * & * \\ 0 & 0 & 0 \\ 0 & 0 & I_a \end{bmatrix}\right) \sim \\
&\sim \left(\begin{bmatrix} Y & 0 & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} JY & * & * \\ 0 & 0 & 0 \\ 0 & 0 & I_a \end{bmatrix} - \begin{bmatrix} \dot{Y} & 0 & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) \sim \\
&\sim \left(\begin{bmatrix} Y & 0 & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & * & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_a \end{bmatrix}\right) \sim \\
&\sim \left(\begin{bmatrix} I_a & 0 & W \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & L & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_a \end{bmatrix}\right),
\end{aligned}
$$

where Y is chosen as a solution of the differential equation $\dot{Y} = JY$ with some nonsingular initial value. See [5] for more details. Let now P and Q denote the matrix functions associated with this transformation to canonical form. Comparing with (3.1) shows that the reduced problem has a vanishing strangeness index. Thus, Theorem 3.4 yields that the reduced problem satisfies Hypothesis A.1 with Φ and Ψ from $Q = [\,\Phi\ \Psi\ \Theta\,]$. Due to Theorem 5.4, the solution space of the homogeneous DAE associated with the original pair (E, A) then has the required form (3.16). Furthermore, we have

$$
P \begin{bmatrix} \hat{E}_1 \\ 0 \\ 0 \end{bmatrix} \Phi = \begin{bmatrix} I_d \\ 0 \\ 0 \end{bmatrix}
$$

implying that $\operatorname{rank} \hat{E}_1 \Phi = \operatorname{rank} Z_1^H E \Phi = d$ and therefore $\operatorname{rank} E\Phi = d$. Assume now that the original DAE and thus the reduced DAE is solvable. Then, with $x = \Phi x_1 + \Psi x_2 + \Theta x_3$ the reduced DAE yields $x_3 = -(\hat{A}_3 T_4)^{-1} \hat{f}_3$ with $\hat{f}_3 = Z_3^H g_{k-1}$. Choosing $x_2 = 0$ then gives a solution of the reduced DAE and thus of the original DAE which has exactly the required form (3.19). Finally, if (E, A) satisfies Hypothesis A.3, then every restriction of (E, A) to a nontrivial subinterval also satisfies Hypothesis A.3. □

Let now $(\tilde{M}_\ell, \tilde{N}_\ell)$ be the derivative arrays belonging to the canonical form $(\tilde{E}, \tilde{A})$ from (4.28). Since the part (G, I_a) satisfies Hypothesis A.2, we already know from Hypothesis A.3 that there exists a $\tilde{Z}_3 \in C(\mathbb{I}, \mathbb{C}^{km,a})$ of the form

$$
\tilde{Z}_3^H = [\,0\ 0\ \tilde{D}_0 \,|\, 0\ 0\ \tilde{D}_1 \,|\, \cdots \,|\, 0\ 0\ \tilde{D}_{k-1}\,] \tag{5.35}
$$

with

$$
\tilde{Z}_3^H \tilde{M}_{k-1} = 0, \quad \operatorname{rank} \tilde{Z}_3^H \tilde{N}_{k-1} = a. \tag{5.36}
$$

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Furthermore, the DAE belonging to the canonical form from (4.28) is solvable if and only if $F\dot{x}_3 = f_2$ in the notation as in (3.4). Replacing x_3 with the help of (5.30) gives

$$f_2 = \sum_{i=0}^{k} \tilde{C}_i f_3^{(i)} \tag{5.37}$$

with $\tilde{C}_i \in C(\mathbb{I}, \mathbb{C}^{v,a})$. We then define $\tilde{Z}_2 \in C(\mathbb{I}, \mathbb{C}^{(k*1)m,v})$ by

$$\tilde{Z}_2^H = [\, 0 \;\; I_v \;\; -\tilde{C}_0 \,|\, 0 \;\; 0 \;\; -\tilde{C}_1 \,|\, \cdots \,|\, 0 \;\; 0 \;\; -\tilde{C}_k \,]. \tag{5.38}$$

For every sufficiently smooth $\tilde{x} \in C^1(\mathbb{I}, \mathbb{C}^n)$, the DAE belonging to the canonical form from (4.28) with $\tilde{f} = \tilde{E}\dot{\tilde{x}} - \tilde{A}\tilde{x}$ is obviously solvable. Hence, we have $\tilde{Z}_2^H \tilde{g}_k = 0$ for $\tilde{g}_k$ being the inhomogeneity of the corresponding inflated DAE. It follows that

$$\tilde{Z}_2^H \tilde{M}_k \dot{\tilde{z}}_k = \tilde{Z}_2^H \tilde{N}_k \tilde{z}_k \tag{5.39}$$

must hold for all sufficiently smooth $\tilde{x} \in C^1(\mathbb{I}, \mathbb{C}^n)$ implying

$$\tilde{Z}_2^H \tilde{M}_k = 0, \quad \tilde{Z}_2^H \tilde{N}_k = 0. \tag{5.40}$$

These properties of $(\tilde{E}, \tilde{A})$ lead to the following formulation of a characterizing hypothesis.

Hypothesis B.3. *The pair (E, A) of matrix functions satisfies properties 1)–3) of Hypothesis A.3 and the following property:*
4) There is a pointwise nonsingular $R \in C(\mathbb{I}, \mathbb{C}^{m,m})$ with (3.20), where Φ is as in the proof of Theorem 5.5, such that for the transformed problem (RE, RA) with derivative arrays $(\tilde{M}_\ell, \tilde{N}_\ell)$ there exists a $\tilde{Z}_2 \in C(\mathbb{I}, \mathbb{C}^{(k+1)m,v})$, $v = m - d - a$, of the form

$$\tilde{Z}_2^H = [\, 0 \;\; I_v \;\; -C_0 \,|\, 0 \;\; 0 \;\; -C_1 \,|\, \cdots \,|\, 0 \;\; 0 \;\; -C_k \,] \tag{5.41}$$

satisfying

$$\tilde{Z}_2^H \tilde{M}_k = 0, \quad \tilde{Z}_2^H \tilde{N}_k = 0. \tag{5.42}$$

Moreover, there exists a $\tilde{Z}_3 \in C(\mathbb{I}, \mathbb{C}^{km,a})$ of the form

$$\tilde{Z}_3^H = [\, 0 \;\; 0 \;\; D_0 \,|\, 0 \;\; 0 \;\; D_1 \,|\, \cdots \,|\, 0 \;\; 0 \;\; D_{k-1} \,] \tag{5.43}$$

with the properties of 1) for the transformed problem.

Since by construction property 4) of Hypothesis B.3 is invariant under (global) equivalence transformations, the invariance of Hypothesis B.3 follows from the invariance of Hypothesis A.3. Observing that for (E, A) satisfying Hypothesis B.2, we can choose R to be the transformation P behind the equivalence in (4.28), the discussion that led to Hypothesis B.3 proves the following result.

Theorem 5.6. *Hypothesis B.2 implies Hypothesis B.3.*

In the overall setting of characterizing classes of singular DAEs there is now only one result missing.

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THEOREM 5.7. *Hypothesis B.3 implies Hypothesis B.1.*

Proof. Since we have already shown that Hypothesis B.3 implies Hypothesis A.1, we only must show property 3) of Hypothesis B.1. Moreover, we can use the results of the beginning of Section 4 up to (4.22) and (4.23) with $Q = [\,\Phi\ \ \Psi\ \ \Theta\,]$ from the proof of Theorem 5.5 and R as given by Hypothesis B.3. In particular, we have

$$(REQ, RAQ) = \left(\begin{bmatrix} I_d & 0 & E_{13} \\ 0 & 0 & E_{23} \\ 0 & 0 & E_{33} \end{bmatrix}, \begin{bmatrix} 0 & L & A_{13} \\ 0 & 0 & A_{23} \\ 0 & 0 & A_{33} \end{bmatrix} \right). \tag{5.44}$$

Since $\tilde{Z}_2$ and $\tilde{Z}_3$ are not affected by the part Q of equivalence transformations (see proof of Lemma 5.1), we can assume that $(\tilde{M}_\ell, \tilde{N}_\ell)$ are the derivative arrays belonging to (REQ, RAQ). Because of property 1) with the special form of $\tilde{Z}_3$, the part (E_{33}, A_{33}) satisfies Hypothesis A.3 with $n = a$ and thus $d = v = 0$. The corresponding reduced problem only consists of the part (5.32c) which is uniquely solvable as long as $\hat{f}_3$ is defined, i.e., as long as f_3 is sufficiently smooth. The proof of Theorem 5.4 then yields that the obtained solution of the reduced DAE also solves (4.23c) and that it is unique.

Consider now (1.1) with sufficiently smooth f and corresponding f_2 in (4.23). If f_2 does not satisfy (3.22) with the C_i chosen from $\tilde{Z}_2$, the inflated DAE

$$\tilde{M}_k \dot{\tilde{z}}_k = \tilde{N}_k \tilde{z}_k + \tilde{g}_k$$

belonging to the transformed problem produces $\tilde{Z}_2^H \tilde{g}_k \neq 0$ and (1.1) cannot have a solution. If on the other hand f_2 satisfies (3.22), we take x_3 to be the unique solution of (5.32c). Then

$$\begin{bmatrix} E_{33} & & & \\ \dot{E}_{33} - A_{33} & E_{33} & & \\ \vdots & \ddots & \ddots & \\ E_{33}^{(k)} - kA_{33}^{(k-1)} & \cdots & k\dot{E}_{33} - A_{33} & E_{33} \end{bmatrix} \begin{bmatrix} \dot{x}_3 \\ * \\ \vdots \\ * \end{bmatrix} = \begin{bmatrix} A_{33} \\ \dot{A}_{33} \\ \vdots \\ A_{33}^{(k)} \end{bmatrix} x_3 + \begin{bmatrix} f_3 \\ \dot{f}_3 \\ \vdots \\ f_3^{(k)} \end{bmatrix}$$

and multiplication with $[\,C_0\ C_1\ \cdots\ C_k\,]$ yields

$$E_{23}\dot{x}_3 = A_{23}x_3 + f_2$$

because of (5.42). Thus, x_3 also solves (4.23b) implying that (4.23) and therefore (1.1) is solvable. □

REMARK 5.8. For the numerical treatment of linear DAEs, it is clear that we cannot deal with consistency conditions as in property 4) of Hypothesis B.3. On the other hand, Hypothesis A.3 is sufficient to define the reduced DAE (5.32). This reduced DAE is numerically accessible except for $\hat{f}_2$ and non-smooth scalings from the left. The latter do not affect the numerical solution, since they simply cancel out in the standard discretization schemes like BDF methods. Furthermore, we can simply set $\hat{f}_2 = 0$ or leave out the corresponding equation to make the reduced DAE solvable. We can then fix the free part of the unknown function by some appropriate

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TABLE 1
Summary of implications

Hyp. A.1	$\overset{\text{Th. 4.1}}{\Longrightarrow}$	Hyp. A.2	$\overset{\text{Th. 5.2}}{\Longrightarrow}$	Hyp. A.3	$\overset{\text{Th. 5.5}}{\Longrightarrow}$	Hyp. A.1
		$\Uparrow$ Th. 5.3				
Hyp. B.1	$\overset{\text{Th. 4.2}}{\Longrightarrow}$	Hyp. B.2	$\overset{\text{Th. 5.6}}{\Longrightarrow}$	Hyp. B.3	$\overset{\text{Th. 5.7}}{\Longrightarrow}$	Hyp. B.1
$\Uparrow$ Th. 3.4						
Hyp. C.0						

additional condition, see, e.g., [6]. Moreover, the consistency of the inhomogeneity can be checked numerically if one determines an approximation to the residual

$$r = E\dot{x} - Ax - f \tag{5.45}$$

by using a discretized version of it. See also [6] for a similar statement.

REMARK 5.9. Up to now, we have not yet addressed property 2) of Hypothesis A.3. This is due to the fact that it is actually implied by the other properties. Nevertheless, we have included it to make the following procedure possible. Let (E, A) satisfy Hypothesis A.3. Then there is a minimal value $\hat{\mu}$, such that Hypothesis A.3 is fulfilled with $\hat{\mu}$ replacing $k-1$. Property 2) of Hypothesis A.3 then guarantees that the quantities a and d are uniquely fixed and that the theory concerning the reduced DAE still works for the smaller derivative arrays. If μ_j is the strangeness index of (E, A) restricted to $\mathbb{I}_j$ from (3.3) for $j \in \mathbb{N}$, it is possible to show that

$$\hat{\mu} = \max_{j \in \mathbb{N}} \{\mu_j\}. \tag{5.46}$$

In particular, one can consider $\hat{\mu}$ as a generalization of the strangeness index for such a pair (E, A). Cp. [7] in the case of regular DAEs.

6. Summary and Conclusions. We started with properties of pairs of matrix functions and the associated DAEs when they possess a well-defined strangeness index. We then examined pairs of matrix functions which exhibit the same properties. In particular, the investigations ran on two levels, where in the more restrictive case additional structure of the space of consistent inhomogeneities was considered. The results of this paper are that on both levels we have obtained three equivalent characterizations of the corresponding class of pairs. In particular, they were by means of spaces, of canonical forms and of derivative arrays. To give an overview over all theorems that contributed to these characterizations, we first introduce the following hypothesis for completeness.

HYPOTHESIS C.0. *The pair (E, A) of matrix functions has a well-defined strangeness index.*

The course of our presentation can then be drawn from Table 1 which shows all involved theorems with their implications. This diagram can then be simplified to show the three levels of classes of singular pairs of matrix functions and DAEs

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TABLE 2
Levels and equivalences

Hyp. A.1	$\Longleftrightarrow$	Hyp. A.2	$\Longleftrightarrow$	Hyp. A.3
		$\Uparrow$		
Hyp. B.1	$\Longleftrightarrow$	Hyp. B.2	$\Longleftrightarrow$	Hyp. B.3
		$\Uparrow$		
		Hyp. C.0		

(when one includes the most special level of a well-defined strangeness index) and their equivalent characterizations, see Table 2.

Of course, the most important level is the most general top level. For numerical purposes it is therefore worth mentioning that the properties of DAEs belonging to this level allow for a numerical treatment via the associated reduced problem. Overall, we have obtained classifications for several different classes of possibly over- or underdetermined DAEs.

We finish up with a small example in order to illustrate the various characterizations we have dealt with in this paper. It should, however, be noted that such an example cannot cover all aspects we have touched here.

EXAMPLE 6.1. Let $E, A \in C(\mathbb{R}, \mathbb{C}^{2,2})$ and $f \in C(\mathbb{R}, \mathbb{C}^2)$ be given by

$$E(t) = \begin{bmatrix} -t & t^2 \\ -1 & t \end{bmatrix}, \quad A(t) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad f(t) = \begin{bmatrix} h_1(t) \\ h_2(t) \end{bmatrix}.$$

Defining $P \in C(\mathbb{R}, \mathbb{C}^{2,2})$ and $Q \in C^1(\mathbb{R}, \mathbb{C}^{2,2})$ by

$$P(t) = \begin{bmatrix} 0 & -1 \\ 1 & -t \end{bmatrix}, \quad Q(t) = \begin{bmatrix} -t & 1 \\ -1 & 0 \end{bmatrix},$$

a short computation yields

$$P(t)E(t)Q(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad P(t)A(t)Q(t) - P(t)E(t)\dot{Q}(t) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Hence, the pair (E, A) of matrix functions has a well-defined strangeness index $\mu = 1$. In particular, we have $F, G \in C(\mathbb{R}, \mathbb{C}^{1,1})$ with $F(t) = 1$ and $G(t) = 0$ in (3.1). It is then obvious that (E, A) satisfies Hypothesis B.2.

Writing down the associated DAE (1.1) as

$$-t\dot{x}_1 + t^2\dot{x}_2 = -x_1 + h_1(t), \quad -\dot{x}_1 + t\dot{x}_2 = -x_2 + h_2(t),$$

we can multiply the second equation with t and subtract the so obtained relation from the first equation. This yields

$$x_1 = tx_2 + h_1(t) - th_2(t).$$

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We can then differentiate x_1 and eliminate $\dot{x}_1$ in the second equation. In this way, we obtain the consistency condition

$$\dot{h}_1(t) - t\dot{h}_2(t) = 0,$$

which is certainly not of the form (3.22). To obtain all solutions of the corresponding homogeneous problem, we can choose $x_2 = -c$ with arbitrary $c \in C^1(\mathbb{R}, \mathbb{C})$ to get $x_1 = -tc$. Splitting Q according to (3.13), the part Φ is empty due to $d = 0$ and the parts Ψ, Θ are given by

$$\Psi(t) = \begin{bmatrix} -t \\ -1 \end{bmatrix}, \quad \Theta(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Hence, the solution space $\mathbb{K}$ has the required form (3.16). Furthermore, if the inhomogeneity is consistent, then the function x defined by

$$x(t) = \begin{bmatrix} h_1(t) - th_2(t) \\ 0 \end{bmatrix} = \Theta(h_1(t) - th_2(t))$$

is a particular solution of the DAE of the form (3.19). In order to show that the given pair (E, A) satisfies Hypothesis B.1, we must show that there exists a suitable matrix function $R \in C(\mathbb{R}, \mathbb{C}^{2,2})$ such that consistency of the inhomogeneity is characterized by a relation of the form (3.22). The property (3.20) holds here for every R because of $d = 0$. Choosing $R = P$ as suggested in the general setting and transforming the original DAE by multiplying with R from the left gives

$$\dot{x}_1 - t\dot{x}_2 = x_2 + f_1(t), \quad 0 = -x_1 + tx_2 + f_2(t),$$

with

$$R(t)f(t) = \begin{bmatrix} 0 & -1 \\ 1 & -t \end{bmatrix} \begin{bmatrix} h_1(t) \\ h_2(t) \end{bmatrix} = \begin{bmatrix} -h_2(t) \\ h_1(t) - th_2(t) \end{bmatrix} = \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}.$$

Note that the numbering of the components is here different from that in (3.21). Solving again for x_1, differentiating, and eliminating $\tilde{x}_1$ gives the consistency condition

$$f_1(t) = \dot{f}_2(t),$$

which obviously is of the form (3.22).

Turning to derivative arrays, we must consider M_1, N_1 due to $k = 2$. These are given by

$$M_1 = \left[\begin{array}{cc|cc} -t & t^2 & 0 & 0 \\ -1 & t & 0 & 0 \\ \hline 0 & 2t & -t & t^2 \\ 0 & 2 & -1 & t \end{array}\right], \quad N_1 = \left[\begin{array}{cc|cc} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Following Hypothesis A.3, we can choose the matrix functions Z_3, T_3 according to

$$Z_3(t)^H = [\, 1 \;\; -t \mid 0 \;\; 0 \,], \quad T_3(t) = \begin{bmatrix} t \\ 1 \end{bmatrix},$$

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while all possible matrices Z_4 can be obtained from the choice

$$Z_4^H = \left[\begin{array}{cc|cc} 1 & -t & 0 & 0 \\ 0 & 0 & 1 & -t \end{array}\right]$$

by multiplying with some nonsingular matrix from the left. Recalling $d = 0$, the part 3) of Hypothesis A.3 is trivially satisfied. In order to show part 4) of Hypothesis B.3, we again choose $R = P$. The corresponding derivative arrays are given by

$$\tilde{M}_2 = \left[\begin{array}{cc|cc|cc} 1 & -t & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & -2 & 1 & -t & 0 & 0 \\ 1 & -t & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -3 & 1 & -t \\ 0 & -2 & 1 & -t & 0 & 0 \end{array}\right], \quad \tilde{N}_2 = \left[\begin{array}{cc|cc|cc} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & t & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Possible choices for the matrix functions $\tilde{Z}_2, \tilde{Z}_3$ are given by

$$\tilde{Z}_2(t)^H = [\,1\;\;0\,|\,0\;\;-1\,|\,0\;\;0\,], \quad \tilde{Z}_3(t)^H = [\,0\;\;1\,|\,0\;\;0\,].$$

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