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A VARIANT ON THE GRAPH PARAMETERS OF COLIN DE VERDIÈRE: IMPLICATIONS TO THE MINIMUM RANK OF GRAPHS*

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Abstract. For a given undirected graph G, the minimum rank of G is defined to be the smallest possible rank over all real symmetric matrices A whose (i, j)th entry is nonzero whenever $i \neq j$ and $\{i, j\}$ is an edge in G. Building upon recent work involving maximal coranks (or nullities) of certain symmetric matrices associated with a graph, a new parameter ξ is introduced that is based on the corank of a different but related class of symmetric matrices. For this new parameter some properties analogous to the ones possessed by the existing parameters are verified. In addition, an attempt is made to apply these properties associated with ξ to learn more about the minimum rank of graphs – the original motivation.

Key words. Graphs, Minimum rank, Graph minor, Corank, Strong Arnold property, Symmetric matrices.

AMS subject classifications. 15A18, 05C50.

1. Introduction. Recent work and subsequent results have fueled interest in important areas such as spectral graph theory and certain types of inverse eigenvalue problems. Of particular interest here is to bring together some of the pioneering work of Y. Colin de Verdière (specifically his parameter related to planarity of graphs) and the minimum rank of graphs.

All matrices discussed in this paper are real and symmetric. If $A \in M_n$ is a fixed symmetric matrix, the *graph of* A denoted by $G(A)$, has $\{1, ..., n\}$ as vertices, and as edges the unordered pairs $\{i, j\}$ such that $a_{ij} \neq 0$ with $i \neq j$. Graphs G of the form $G = G(A)$ do not have loops or multiple edges, and the diagonal of A is ignored in the determination of $G(A)$. Similarly, for a given graph G, we let

$$\mathcal{S}(G) = \{A \in M_n \mid A = A^T,\, G(A) = G\}.$$

Finally, for any symmetric matrix $A \in M_n$, we let $\mathcal{S}_A = \mathcal{S}(G(A))$.

Suppose that G is a graph on n vertices. Then the *minimum rank of* G is given by

$$\mathrm{mr}(G) = \min_{A \in \mathcal{S}(G)} \operatorname{rank} A.$$

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It is not difficult to verify that $\mathrm{mr}(G) = n - M(G)$, where $M(G)$ is the *maximum multiplicity of* G, and is defined to be

$$M(G) = \max_{A \in \mathcal{S}(G)} \{\mathrm{mult}_A(\lambda) : \lambda \in \sigma(A)\}.$$

Furthermore it is easy to see that for any graph, $\max\{\mathrm{corank}\, A \mid A \in \mathcal{S}(G)\} = M(G)$, where $\mathrm{corank}\, A$ is defined to be the nullity of A. Here $\sigma(A)$ denotes the spectrum of A and $\mathrm{mult}_A(\lambda)$ is the multiplicity of $\lambda \in \sigma(A)$. Also, if $W \subset \{1, 2, \dots, n\}$ and $A \in M_n$, then $A[W]$ means the principal submatrix of A whose rows and columns are indexed by W, and $A(W)$ is the complementary principal submatrix obtained from A by deleting the rows and columns indexed by W. In the special case when $W = \{v\}$ a singleton, we let $A(v) = A(W)$. For a fixed $m \times n$ matrix A, $R(A)$ and $\mathrm{Null}(A)$ denote the range and the null space of A, respectively.

An interesting and still rather unresolved problem is to characterize $\mathrm{mr}(G)$ for a given graph G. Naturally, there have been a myriad of preliminary results, which take on many different forms. For example, if P_n, C_n, K_n, E_n, denote the path on n vertices, the cycle on n vertices, the complete graph on n vertices, and the empty (edgeless) graph on n vertices, respectively, then

$$\mathrm{mr}(P_n) = n - 1, \quad \mathrm{mr}(C_n) = n - 2, \quad \mathrm{mr}(K_n) = 1, \quad \mathrm{mr}(E_n) = 0.$$

Further it is well known that for any connected graph G on n vertices that $\mathrm{mr}(G) = 1$ if and only if G is K_n. Fiedler [8] established that $\mathrm{mr}(G) = n - 1$ if and only if G is P_n. Barrett, van der Holst, and Loewy [4] have characterized all of the graphs on n vertices that satisfy $\mathrm{mr}(G) = 2$.

Other important work pertaining to the class of trees [11], states that $\mathrm{mr}(T) = n - P(T)$, where $P(T)$ is the *path cover number*, namely, the minimum number of vertex disjoint paths occurring as induced subgraphs of T, that cover all the vertices of T. More recently, some modest extensions along these lines have been produced for graphs beyond the class of trees. Namely, for vertex sums and edge sums of so-called non-deficient graphs (which include trees), and for the case of *unicyclic* graphs, i.e., graphs that contain a unique cycle [2, 3].

On a related topic there has been some extremely interesting and exciting work on spectral graph theory that is connected to certain aspects of planarity. For a given graph, a matrix $L = [l_{ij}] \in \mathcal{S}(G)$ is called a *generalized Laplacian matrix of* G if for $i \neq j$, $l_{ij} < 0$ whenever i, j are adjacent in G and $l_{ij} = 0$ otherwise. Colin de Verdière introduced the parameter $\mu(G)$ associated with the nullity of certain generalized Laplacian matrices in $\mathcal{S}(G)$ (see [5, 9, 10] for more specific details). The paper [10] provides a clear exposition and survey of these results, and we will follow much of the notation and treatment given in that paper. The actual definition of $\mu(G)$ will be presented below.

We now turn our attention to the so-called *Strong Arnold Property*, which will be shortened to SAP throughout. We will see that it plays a crucial role in monotonicity, such as the subgraph monotonicity of μ.

We say two matrices are *orthogonal* if, when viewed as n^2-tuples in $\mathbb{R}^{n^2}$, they are orthogonal under the ordinary dot product. Equivalently, B is orthogonal to A

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if and only if $\mathrm{trace}(A^T B) = 0$. The matrix B is *orthogonal to the family of matrices* $\mathcal{F}$ if B is orthogonal to every matrix $C \in \mathcal{F}$. A family we will consider frequently is $\mathcal{F} = \mathcal{S}(G)$ where G is a graph; X orthogonal to $\mathcal{S}(G)$ requires that every diagonal entry of X is 0 and for every edge of G, the corresponding off-diagonal entry of X is 0. Recall that, if $A = [a_{ij}]$ and $B = [b_{ij}]$ are matrices in M_n, the matrix $A \circ B$ defined by $[A \circ B]_{ij} = a_{ij} b_{ij}$ is called the *Hadamard product* of A and B.

DEFINITION 1.1. Let A, X be symmetric $n \times n$ matrices. We say that X *fully annihilates* A if

1. $AX = 0$;
2. $A \circ X = 0$;
3. $I_n \circ X = 0$.

In other words, X fully annihilates A if X is orthogonal to $\mathcal{S}_A$ and $AX = 0$.

DEFINITION 1.2. The matrix A has the Strong Arnold Property (SAP) if the zero matrix is the only symmetric matrix that fully annihilates A.

We begin with a basic, yet useful, observation concerning low corank.

LEMMA 1.3. *If* $\mathrm{corank}\, A \leqslant 1$*, then* A *has SAP.*

Proof. If $\mathrm{corank}\, A = 0$, then A is nonsingular, and the only matrix X that fully annihilates A is the zero matrix. Suppose now $\mathrm{corank}\, A = 1$, and let X fully annihilate A. Therefore, the diagonal of X is 0. Since X is symmetric, this implies X is not a rank 1 matrix. Thus if $X \neq 0$, then $\mathrm{rank}\, X \geqslant 2$, and $AX = 0$ would imply $\mathrm{corank}\, A \geqslant 2$. Thus $X = 0$ and A has SAP. ▯

We are now in a position to formally define the Colin de Verdière parameter, $\mu(G)$. For a given graph G, $\mu(G)$ is defined to be the maximum multiplicity of 0 as an eigenvalue of L, where L satisfies:

1. $L \in \mathcal{S}(G)$, and is a generalized Laplacian matrix;
2. L has exactly one negative eigenvalue (with multiplicity one);
3. L has SAP.

In other words $\mu(G)$ is the maximum *corank* among the matrices satisfying (1)-(3) above. Further observe that $\mu(G) \leqslant M(G) = n - \mathrm{mr}(G)$. Hence there is an obvious relationship between $\mu(G)$ and $\mathrm{mr}(G)$.

Colin de Verdière and others ([5, 9, 10]) have shown that

- $\mu(G) \leqslant 1$ if and only if G is a disjoint union of paths,
- $\mu(G) \leqslant 2$ if and only if G is outerplanar,
- $\mu(G) \leqslant 3$ if and only if G is planar,
- $\mu(G) \leqslant 4$ if and only if G is linklessly embeddable.

A related parameter, also introduced by Colin de Verdière [6] is denoted by $\nu(G)$, and is defined to be the maximum corank among matrices A that satisfy:

1. $A \in \mathcal{S}(G)$;
2. A is positive semidefinite;
3. A has SAP.

Properties analogous to $\mu(G)$ have been established for $\nu(G)$. For example, $\nu(G) \leqslant 2$ if the dual of G is outerplanar, see [6]. Furthermore, $\nu(G)$, like $\mu(G)$ is graph minor monotone – we will come back to this issue later.

One of the motivating issues for this work is an attempt to learn more about the minimum rank of graphs by studying a variant of $\mu(G)$ and $\nu(G)$. Consequently, we

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introduce the following new parameter, which we denote by $\xi(G)$.

DEFINITION 1.4. *For a graph G, $\xi(G)$ is the maximum corank among matrices $A \in \mathcal{S}(G)$ having SAP.*

For a graph parameter ζ defined to be the maximum corank over a family of matrices associated with graph G, we say A is ζ-optimal for G if A is in the family and $\operatorname{corank} A = \zeta(G)$.

REMARK 1.5. Since for any graph G, any μ-optimal or ν-optimal matrix for G is in $\mathcal{S}(G)$ and has SAP, $\mu(G) \leqslant \xi(G)$ and $\nu(G) \leqslant \xi(G)$.

In Example 3.11 we determine a graph G such that $\mu(G) < \xi(G)$ and $\nu(G) < \xi(G)$. For motivational purposes and completeness, we give several examples and observations on the evaluation of ξ.

OBSERVATION 1.6. $\xi(G) = 1$ exactly when G is a disjoint union of paths. Indeed, if $\xi(G) = 1$, we have $\mu(G) \leqslant 1$, so that G is a disjoint union of paths. On the other hand, since $M(P_n) = 1$, and any corank 1 matrix has SAP, $\xi(P_n) = 1$. Then, the converse will follow easily from Theorem 3.2, in which we show that ξ of a disjoint union is the maximum value of ξ on the components.

OBSERVATION 1.7. If $n > 1$, $\xi(K_n) = n - 1$, because J, the all 1's matrix, is in $\mathcal{S}(K_n)$, has corank $n-1$, and has SAP (any matrix in $\mathcal{S}(K_n)$ has SAP since a matrix orthogonal to $\mathcal{S}(K_n)$ is necessarily 0). $\xi(K_1) = 1$ because any corank 1 matrix has SAP. Conversely, it is well known that the only connected graph having $M(G) = |G| - 1$ is K_n, so (again using Theorem 3.2) $\xi(G) = |G| - 1$ implies $G = K_n$ or $G = \overline{K}_2$, the complement of K_2, also denoted by E_2.

EXAMPLE 1.8. $\xi(C_n) = 2$, because $M(C_n) = 2$ (see for example [2, 3]) so $\xi(C_n) \leqslant 2$, but C_n is not a disjoint union of paths so $\xi(C_n) \geqslant 2$.

The next example shows that it is possible to have a matrix A that is ξ-optimal for graph G, and another matrix $B \in \mathcal{S}(G)$ with $\operatorname{corank} A = \operatorname{corank} B$ but B does not have SAP. It also illustrates how computations to establish SAP (or find a matrix X showing failure to have SAP) can be performed.

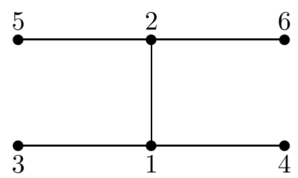

FIG. 1.1. *A graph with $B \in \mathcal{S}(G)$, $\operatorname{corank} B = \xi(G)$, but B does not have SAP*

EXAMPLE 1.9. Let G be the graph in Figure 1.1. Since G is a tree, we can easily compute the maximum corank possible for a matrix in $\mathcal{S}(G)$: $M(G) = P(G) = 2$.

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Let

$$A = \begin{bmatrix} 1 & 2 & 1 & 1 & 0 & 0 \\ 2 & 8 & 0 & 0 & -3 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \end{bmatrix}; \qquad B = \begin{bmatrix} 1 & 2 & 1 & 1 & 0 & 0 \\ 2 & 8 & 0 & 0 & -3 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Clearly $A, B \in \mathcal{S}(G)$, and direct computation shows $\operatorname{corank} A = \operatorname{corank} B = 2$. The matrix

$$X = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{3} & -1 \\ 0 & 0 & 0 & 0 & \frac{1}{3} & 1 \\ 0 & 0 & -\frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \end{bmatrix}$$

fully annihilates B, so that B does not have SAP. To show A does have SAP (and thus is ξ-optimal for G), compute AX for an arbitrary symmetric matrix X orthogonal to $\mathcal{S}(G)$. An examination of the last two columns of AX shows $x_{15} = x_{16} = x_{56} = 0$. The second, third and fourth entries of the first row are $x_{23}+x_{24}, 2x_{23}+x_{34}, 2x_{24}+x_{34}$, forcing $x_{23} = x_{24} = x_{34} = 0$. After substituting these values, we see from the third and fourth columns that $x_{35} = x_{36} = x_{45} = x_{46} = 0$. In other words, $X = 0$.

Example 1.10. If G is regular of degree $|G| - 2$, then $\xi(G) = M(G)$, since any matrix $A \in \mathcal{S}(G)$ has SAP (if X is orthogonal to $\mathcal{S}(G)$, at most one entry in any row is nonzero, which implies easily $X = 0$).

Example 1.11. If T is a tree that is not a path, then $\xi(T) = 2$. This will be proved in Theorem 3.7.

Example 1.12. Let $K_{p,q}$ denote the complete bipartite graph on sets of p and q vertices, $1 \leqslant p \leqslant q$. Observe that $\xi(K_{1,1}) = \xi(P_2) = 1$, and that $\xi(K_{1,2}) = \xi(P_3) = 1$. If $q \geqslant 3$, then $\xi(K_{p,q}) = p + 1$, as will be proved in Corollary 2.8, while $\xi(K_{2,2}) = \xi(C_4) = 2$.

The tools we use to exploit SAP come from manifold theory. As in [10], let $M_1, \ldots, M_k$ be open manifolds embedded in $\mathbb{R}^d$, and let x be a point in their intersection. We say $M_1, \ldots, M_k$ intersect transversally at x if their normal spaces at x are independent. That is, if n_i is orthogonal to M_i for $i = 1, \ldots, k$ and $n_1 + \cdots + n_k = 0$, then $n_i = 0$ for all $i = 1, \ldots, k$. A smooth family of manifolds $M(t)$ in $\mathbb{R}^d$ is defined by a smooth function $f\colon U \times (-1, 1) \to \mathbb{R}^d$, where U is an open set in $\mathbb{R}^s (s \leqslant d - 1)$, and for each $-1 < t < 1$, the function $f(\cdot, t)$ is a diffeomorphism between U and the manifold $M(t)$.

For a given $n \times n$ matrix A, let $\mathcal{R}_A$ be the set of all $n \times n$ matrices B such that $\operatorname{rank} B = \operatorname{rank} A$. The next lemma is from [10].

Lemma 1.13. *The matrix A has SAP if and only if the manifolds $\mathcal{R}_A$ and $\mathcal{S}_A$ intersect transversally at A.*

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The next lemma is a slightly simplified version Lemma 2.1 of [10].

LEMMA 1.14. *Let $M_1(t),\ldots,M_k(t)$ be smooth families of manifolds in $\mathbb{R}^d$ and assume that $M_1(0),\ldots,M_k(0)$ intersect transversally at x. Then there exists $\epsilon > 0$ such that for any t such that $|t| < \epsilon$, the manifolds $M_1(t),\ldots,M_k(t)$ intersect transversally at a point $x(t)$ so that $x(0) = x$ and $x(t)$ depends continuously on t.*

LEMMA 1.15. *[10, Cor. 2.2] Assume that $M_1,\ldots,M_k$ are manifolds in $\mathbb{R}^d$ that intersect transversally at x, and assume that they have a common tangent vector v at x with $\|v\| = 1$. Then for every $\epsilon > 0$ there exists a point $x' \neq x$ such that $M_1,\ldots,M_k$ intersect transversally at x', and*

$$\left\| \frac{(x-x')}{\|x-x'\|} - v \right\| < \epsilon.$$

In Section 2 we establish a graph monotonicity property for ξ, which is also possessed by both μ and ν. In Section 3, from the results in Section 2, we build up many useful tools and facts about ξ and use them to learn more about ξ, and apply these results to $\mathrm{mr}(G)$.

2. Minor Monotonicity and Consequences. Following the previous works of Colin de Verdière as described in [10], we prove that the parameter introduced here, ξ, is also graph minor monotone. We begin with a preliminary result, which also follows from the results in Section 3.

OBSERVATION 2.1. ξ is monotone for deletion of an isolated vertex, i.e., if G' is obtained from G by deleting an isolated vertex of G, then $\xi(G') \leqslant \xi(G)$.

Proof. Let G' be obtained from G by deleting an isolated vertex v of G. Choose a ξ-optimal matrix A' for G'. It is sufficient to construct a matrix $A \in \mathcal{S}(G)$ such that $\operatorname{corank} A = \operatorname{corank} A'$ and A has SAP, for then $\xi(G) \geqslant \operatorname{corank} A = \operatorname{corank} A' = \xi(G')$. Let A be the matrix obtained from A' by adding (in position v) a row and column consisting of 0s except $A_{v,v} = 1$. Then clearly $A \in \mathcal{S}(G)$, $\operatorname{corank} A = \operatorname{corank} A'$, and a simple computation shows A has SAP. ☐

THEOREM 2.2. *ξ is edge deletion monotone, i.e., if G' is obtained from G by deleting an edge of G, then $\xi(G') \leqslant \xi(G)$.*

Proof. Let G' be obtained from G by deleting edge $\{u, w\}$. Proceeding as in Observation 2.1, we choose a ξ-optimal matrix A' for G' and construct the required matrix $A \in \mathcal{S}(G)$. Since A' has SAP, the two manifolds $\mathcal{R}_{A'}$ and $\mathcal{S}_{A'}$ intersect transversally at A'. Let $\mathcal{S}(t)$ be the manifold obtained from $\mathcal{S}_{A'}$ by replacing (in each matrix in $\mathcal{S}_{A'}$) the 0s in positions (u, w) and (w, u) by t. Let $\mathcal{R}(t) = \mathcal{R}_{A'}$. Then by Lemma 1.14, for a sufficiently small positive t, $\mathcal{R}(t)$ and $\mathcal{S}(t)$ intersect transversally at $A = A(t)$. Thus A has SAP. Since $A \in \mathcal{R}(t) = \mathcal{R}_{A'}$, we have $\operatorname{corank} A = \operatorname{corank} A'$, and since $A \in \mathcal{S}(t)$, $A \in \mathcal{S}(G)$. ☐

COROLLARY 2.3. *ξ is subgraph monotone, i.e., if G' is a subgraph of G, then $\xi(G') \leqslant \xi(G)$.*

Recall that for a given edge $e = \{u, v\}$ of G we say *contract e in G* to mean delete e from G, identify its ends u, v in such a way that the resulting vertex is adjacent to exactly the vertices that were originally adjacent to u or v. A *contraction of G* is then defined as any graph obtained from G by contracting an edge.

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THEOREM 2.4. *ξ is contraction monotone, i.e., if G' is obtained from G by contracting an edge, then $\xi(G') \leqslant \xi(G)$.*

Proof. Let $|G| = n$, suppose $\{1,2\} \in E(G)$, and let G' be obtained from G by contracting $\{1,2\}$ (call this new vertex v and place it first in order of the vertices of G'). If vertex 1 is adjacent only to vertex 2, then the result follows by subgraph monotonicity. So we may assume 1 is adjacent to at least one vertex in addition to 2. Renumber if necessary so that vertex 1 is adjacent exactly to the vertices $2, 3, \ldots, r$ ($r \geqslant 3$). By the edge monotonicity of ξ, without loss of generality, we may assume vertex 2 is not adjacent to any of the vertices $3, \ldots, r$. An $n \times n$ symmetric matrix will be written in the following block form

$$B = \left[\begin{array}{cc|c} b_{11} & b_{12} & \mathbf{b}_1^T \;\; \mathbf{b}_1'^T \\ b_{12} & b_{22} & \mathbf{b}_2^T \;\; \mathbf{b}_2'^T \\ \hline \mathbf{b}_1 & \mathbf{b}_2 & \\ \mathbf{b}_1' & \mathbf{b}_2' & \raisebox{1.5ex}{B_0} \end{array}\right],$$

where $\mathbf{b}_1 \in \mathbb{R}^{r-2}$. In addition, let U be the 0-1 matrix with $G(U) = G'$ and $U_{ii} = 1$ for each i. We then have

$$U = \left[\begin{array}{c|c} 1 & \mathbf{1}^T \;\; \mathbf{u}_2'^T \\ \hline \mathbf{1} & \\ \mathbf{u}_2' & \raisebox{1.5ex}{U_0} \end{array}\right],$$

where $\mathbf{1}$ denotes the vector all of whose $r-2$ entries are equal to 1, while $\mathbf{u}_2'$ (U_0) is a suitable 0-1 vector (matrix).

We define three manifolds as follows.

- $\mathcal{M}_1$ is the set of $n \times n$ symmetric matrices B such that $b_{12} = 0$, $\mathbf{b}_1' = 0$ and $B(1) \circ U = B(1)$, that is, $G(B(1))$ can be obtained from G' by (possibly) removing some edges.
- $\mathcal{M}_2$ is the set of $n \times n$ symmetric matrices B such that $\operatorname{corank} B = \xi(G')$.
- $\mathcal{M}_3$ is the set of $n \times n$ symmetric matrices B such that $\operatorname{rank}[\mathbf{b}_1 \;\; \mathbf{b}_2] = 1$.

As shown in [10],

- if $B \in \mathcal{M}_1$, the normal space of $\mathcal{M}_1$ at B is the set of symmetric matrices X such that

$$X = \left[\begin{array}{cc|c} 0 & x_{12} & \mathbf{0}^T \;\; \mathbf{x}_1'^T \\ x_{12} & 0 & \mathbf{x}_2^T \;\; \mathbf{x}_2'^T \\ \hline \mathbf{0} & \mathbf{x}_2 & \\ \mathbf{x}_1' & \mathbf{x}_2' & \raisebox{1.5ex}{X_0} \end{array}\right], \qquad X(1) \circ U = 0; \tag{2.1}$$

- if $B \in \mathcal{M}_2$, the normal space of $\mathcal{M}_2$ at B is the set of symmetric matrices Y such that

$$BY = 0; \tag{2.2}$$

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- if $B \in \mathcal{M}_3$, and $\mathbf{b}_1 = \gamma \mathbf{b}_2$, the normal space of $\mathcal{M}_3$ at B is the set of symmetric matrices Z such that

$$Z = \left[\begin{array}{cc|cc} 0 & 0 & \mathbf{z}_1^T & \mathbf{0}^T \\ 0 & 0 & -\gamma \mathbf{z}_1^T & \mathbf{0}^T \\ \hline \mathbf{z}_1 & -\gamma \mathbf{z}_1 & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ \mathbf{0} & \mathbf{0} & & \end{array}\right], \qquad \mathbf{z}_1^T \mathbf{b}_2 = 0. \tag{2.3}$$

Define

$$P = \left[\begin{array}{cc|cc} 1 & 0 & \mathbf{0}^T & \mathbf{0}^T \\ 0 & p_{22} & \mathbf{p}_2^T & \mathbf{p}_2'^T \\ \hline \mathbf{0} & \mathbf{p}_2 & \multicolumn{2}{c}{\multirow{2}{*}{P_0}} \\ \mathbf{0} & \mathbf{p}_2' & & \end{array}\right],$$

where $P(1)$ is a ξ-optimal matrix for G'. Note that $\operatorname{rank} P = 1 + \operatorname{rank} P(1)$; so $\operatorname{corank} P = \xi(G')$. In addition, P is in each of the $\mathcal{M}_i$'s (note that $\mathbf{p}_2 \neq 0$, so that $P \in \mathcal{M}_3$). As shown in [10], the three manifolds intersect transversally at P, and the matrix

$$T = \left[\begin{array}{cc|cc} 0 & 0 & \mathbf{p}_2^T & \mathbf{0}^T \\ 0 & 0 & \mathbf{0}^T & \mathbf{0}^T \\ \hline \mathbf{p}_2 & \mathbf{0} & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ \mathbf{0} & \mathbf{0} & & \end{array}\right]$$

is a common tangent to all three manifolds at P. Thus, by Lemma 1.15 (or [10, Cor. 2.2]), there is a matrix Q in the intersection of the $\mathcal{M}_i$'s such that the $\mathcal{M}_i$'s intersect transversally at Q, and $Q - P$ is "almost parallel" to T. By a judicious choice of ϵ, we can ensure that Q has nonzero entries everywhere P or T has nonzero entries. In other words we can write

$$Q = \left[\begin{array}{cc|cc} q_{11} & 0 & \mathbf{q}_1^T & \mathbf{0}^T \\ 0 & q_{22} & \mathbf{q}_2^T & \mathbf{q}_2'^T \\ \hline \mathbf{q}_1 & \mathbf{q}_2 & \multicolumn{2}{c}{\multirow{2}{*}{Q_0}} \\ \mathbf{0} & \mathbf{q}_2' & & \end{array}\right],$$

where $G(Q(1)) = G'$. In particular, $\mathbf{q}_2$ has no zero components. Moreover, since $Q \in \mathcal{M}_3$, there exists $\gamma \neq 0$ such that $\mathbf{q}_1 = \gamma \mathbf{q}_2$, so that $\mathbf{q}_1$ has no zero components as well.

Let $S = I_n - \frac{1}{\gamma} E_{21}$, and $A = SQS^T$. Easy computations show that $G(A) = G$, and $\operatorname{corank} A = \operatorname{corank} Q = \xi(G')$. So it is enough to show that A has SAP. Suppose, by way of contradiction, that there exists a nonzero matrix W that fully annihilates

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A. Then $A \circ W = 0$ and $I_n \circ W = 0$, so that

$$W = \left[\begin{array}{cc|cc} 0 & 0 & \mathbf{0}^T & \mathbf{w}_1'^T \\ 0 & 0 & \mathbf{w}_2^T & \mathbf{w}_2'^T \\ \hline \mathbf{0} & \mathbf{w}_2 & \multicolumn{2}{c}{\multirow{2}{*}{W_0}} \\ \mathbf{w}_1' & \mathbf{w}_2' & & \end{array}\right].$$

In addition $AW = 0$, that is, $SQS^TW = 0$, and since S is invertible, we can write $QS^TWS = 0$. Define

$$Y = S^TWS = \left[\begin{array}{cc|cc} 0 & 0 & -\frac{1}{\gamma}\mathbf{w}_2^T & \mathbf{w}_1'^T - \frac{1}{\gamma}\mathbf{w}_2'^T \\ 0 & 0 & \mathbf{w}_2^T & \mathbf{w}_2'^T \\ \hline -\frac{1}{\gamma}\mathbf{w}_2 & \mathbf{w}_2 & \multicolumn{2}{c}{\multirow{2}{*}{W_0}} \\ \mathbf{w}_1' - \frac{1}{\gamma}\mathbf{w}_2' & \mathbf{w}_2' & & \end{array}\right].$$

If $\mathbf{w}_2 \neq 0$, define further

$$Z = \left[\begin{array}{cc|cc} 0 & 0 & -\frac{1}{\gamma}\mathbf{w}_2^T & \mathbf{0}^T \\ 0 & 0 & \mathbf{w}_2^T & \mathbf{0}^T \\ \hline -\frac{1}{\gamma}\mathbf{w}_2 & \mathbf{w}_2 & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ \mathbf{0} & \mathbf{0} & & \end{array}\right],$$

and $X = Y - Z$. By using (2.1), (2.2), and (2.3), we see that X, Y, and Z are normal at Q to $\mathcal{M}_1$, $\mathcal{M}_2$, and $\mathcal{M}_3$, respectively, so that the $\mathcal{M}_i$'s do not intersect transversally at Q, which is a contradiction. On the other hand, if $\mathbf{w}_2 = 0$, Y would be normal at Q to both $\mathcal{M}_1$ and $\mathcal{M}_2$, which is again a contradiction. □

For a given graph G, we call H a *minor of* G if H is obtained from G by a sequence of deletions of edges, deletions of isolated vertices, and contractions of edges. We are now in a position to state the minor monotonicity result of ξ which both μ and ν also satisfy.

Corollary 2.5. *ξ is minor monotone, i.e., if G' is a minor of G, then $\xi(G') \leqslant \xi(G)$.*

As noted in [10], this implies that the Robertson-Seymour graph minor theory applies to ξ, so that the graphs G that have the property $\xi(G) \leqslant k$ can be characterized by a finite set of forbidden minors. The Robertson-Seymour graph minor theorem is an extremely powerful tool; consult the last chapter [7] for further discussion.

Using the results thus far, we continue to derive more properties of the parameter ξ, while at the same time adding to the list of examples in which ξ can be calculated. The first result below is a direct consequence of Corollary 2.3 and Observation 1.7.

Corollary 2.6. *If K_p is a subgraph of G then $\xi(G) \geqslant p - 1$.*

Corollary 2.7. *Suppose $V(G)$ has disjoint subsets $W_i, i = 1, \ldots, q$ such that for all $i = 1, \ldots, q$ the subgraph of G induced by W_i is a path, and for all $i \neq j$, there is no edge in G between a vertex in W_i and a vertex in W_j. Then $\xi(G) \leqslant$*

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$|G| - (|W_1| + \ldots + |W_q|) + 1$. *In particular, if G has a set of q independent vertices (i.e., each W_i is a singleton), then $\xi(G) \leqslant |G| - q + 1$.*

Proof. Label the vertices in W_i that are the endpoints of the path induced by W_i as u_i and v_i (if $|W_i| = 1$ then $u_i = v_i$; otherwise they are distinct). Create a new graph G' by adding the edges $\{v_i, u_{i+1}\}$ for $i = 1, \ldots, q-1$. Since there is no edge in G between a vertex in W_i and a vertex in W_j, the subgraph H induced in G' by $\bigcup_{i=1}^{q} W_i$ is a path on $|W_1| + \ldots + |W_q|$ vertices, so $\mathrm{mr}H = |W_1| + \ldots + |W_q| - 1$. Since H is induced, $\mathrm{mr}(H) \leqslant \mathrm{mr}(G')$, so $M(G') = |G'| - \mathrm{mr}(G') \leqslant |G| - (|W_1| + \ldots + |W_q|) + 1$. By edge monotonicity (Theorem 2.2), $\xi(G) \leqslant \xi(G') \leqslant M(G') \leqslant |G| - (|W_1| + \ldots + |W_q|) + 1$. ☐

Since $\mu(G), \nu(G) \leqslant \xi(G)$, any graph G satisfying the hypotheses of Corollary 2.7 also has $\mu(G) \leqslant |G| - (|W_1| + \ldots + |W_q|) + 1$ and $\nu(G) \leqslant |G| - (|W_1| + \ldots + |W_q|) + 1$.

The next corollary follows from the previous one and the facts that $\mu(K_{p,q}) = p+1$ for $q \geqslant p \geqslant 1$ and $q \geqslant 3$ [10], and $\mu(G) \leqslant \xi(G)$ for any graph G.

Corollary 2.8. *If $q \geqslant p \geqslant 1$ and $q \geqslant 3$ then $\xi(K_{p,q}) = p + 1$.*

The next corollary is an immediate consequence of edge monotonicity. Note that the only distinction between matrices considered when maximizing corank for M and for ξ is SAP.

Corollary 2.9. *If it is possible to add an edge to G, obtaining graph G', and have $M(G') < M(G)$ then $\xi(G) < M(G)$, i.e., any matrix A in $\mathcal{S}(G)$ with* $\mathrm{rank}\, A = \mathrm{mr}(G)$ *does not have SAP.*

We have seen that SAP is sufficient for edge monotonicity. In fact, SAP also appears to be necessary for edge monotonicity; we have results for several families of graphs G that when $\xi(G) < M(G)$, it is possible to add an edge and reduce M. For example, the proof of Corollary 2.7 shows how to add edges between the q independent vertices of $K_{p,q}$ to obtain a graph G with $M(G) < M(K_{p,q})$, provided $q \geqslant 4$. Note that it follows from Corollary 2.8 that $\xi(K_{p,q}) < M(K_{p,q})$ is true exactly when $q \geqslant 4$. We do not know of any examples with $\xi(G) < M(G)$ where it is not possible to add an edge and reduce M (see also Proposition 3.8).

3. Constructions. In this section we examine the behavior of ξ under various constructions, such as disjoint union, vertex sum, joins, etc. In contrast to the previous section, where the results closely paralleled those for μ, and where the methods of proof were often the same as those in [10], here the results for ξ sometimes differ from those for μ, and in most cases even when the result is the same, the method of proof is different. We will need numerous technical lemmas.

Lemma 3.1. *Let B be the direct sum of matrices B_i, $i = 1, \ldots, k$. Then B has SAP if and only if at most one B_i is singular, and such B_i has SAP.*

Proof. Let B have SAP, and suppose two of the B_i, say B_1 and B_2, are singular. Then there exist nonzero vectors $\mathbf{x}_1$, $\mathbf{x}_2$ such that $B_1\mathbf{x}_1 = B_2\mathbf{x}_2 = 0$. Let $\hat{\mathbf{x}}_1 = [\mathbf{x}_1^T \;\; 0 \;\; 0]^T$ and $\hat{\mathbf{x}}_2 = [0 \;\; \mathbf{x}_2^T \;\; 0]^T$. Then $X = \hat{\mathbf{x}}_1\hat{\mathbf{x}}_2^T + \hat{\mathbf{x}}_2\hat{\mathbf{x}}_1^T$ fully annihilates B, so that B does not have SAP. Therefore, at most one of the B_i, say B_1, is singular. Suppose now that B_1 does not have SAP. Then, there exists a nonzero matrix X_1 that fully annihilates B_1, so that the matrix $X = \begin{bmatrix} X_1 & 0 \\ 0 & 0 \end{bmatrix}$ fully annihilates B, that is, B does not have SAP.

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Conversely, if all B_i are nonsingular, then B is nonsingular, and has SAP by Lemma 1.3. If exactly one of the B_i, say B_1, is singular and has SAP, then we can write $B = \begin{bmatrix} B_1 & 0 \\ 0 & B' \end{bmatrix}$, where B' is nonsingular. Let $X = \begin{bmatrix} X_{11} & X_{21}^T \\ X_{21} & X_{22} \end{bmatrix}$ fully annihilate B. Then $BX = 0$ implies $B'X_{21} = B'X_{22} = 0$, that is, $X_{21} = X_{22} = 0$. Thus we have $B_1X_{11} = 0$, and, finally, $X_{11} = 0$, since B_1 has SAP. In other words, X is necessarily the zero matrix, so that B has SAP. ◻

The next theorem follows immediately from Lemma 3.1 and the monotonicity of ξ on submatrices. The analogous result is true for μ.

Theorem 3.2. *If G is not connected, and the components of G are the graphs $G_1, \ldots, G_k$, then $\xi(G) = \max_{i=1}^k \xi(G_i)$.*

Lemma 3.3. *Let*

$$B = \begin{bmatrix} \beta & \mathbf{c}_1^T & \mathbf{c}_2^T & \mathbf{c}_3^T \\ \mathbf{c}_1 & B_1 & 0 & 0 \\ \mathbf{c}_2 & 0 & B_2 & 0 \\ \mathbf{c}_3 & 0 & 0 & B_3 \end{bmatrix},$$

and suppose $\operatorname{Null} \begin{bmatrix} \mathbf{c}_i^T \\ B_i \end{bmatrix} \neq 0$ *for $i = 1, 2$. Then B does not have SAP.*

Proof. For $i = 1, 2$, select $\mathbf{x}_i \in \operatorname{Null} \begin{bmatrix} \mathbf{c}_i^T \\ B_i \end{bmatrix}$, $\mathbf{x}_i \neq 0$. Then, the matrix

$$X = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{x}_1\mathbf{x}_2^T & 0 \\ 0 & \mathbf{x}_2\mathbf{x}_1^T & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

fully annihilates B, so that B does not have SAP. ◻

Observation 3.4. In particular, note that the condition $\operatorname{Null} \begin{bmatrix} \mathbf{c}_i^T \\ B_i \end{bmatrix} \neq 0$ is satisfied exactly when either $\operatorname{corank} B_i \geqslant 2$, or when $\operatorname{corank} B_i = 1$ and $\mathbf{c}_i \in R(B_i)$.

Lemma 3.5. *Let $\alpha, \gamma \in \mathbb{R}$, and consider the matrices*

$$A = \begin{bmatrix} \alpha & \mathbf{b}_1^T & \mathbf{b}_2^T \\ \mathbf{b}_1 & A_1 & 0 \\ \mathbf{b}_2 & 0 & A_2 \end{bmatrix}; \qquad \widetilde{A}_1 = \begin{bmatrix} \gamma & \mathbf{b}_1^T \\ \mathbf{b}_1 & A_1 \end{bmatrix}.$$

If A has SAP, then

i. A_1 has SAP if $\mathbf{b}_1 \in R(A_1)$;

ii. $\widetilde{A}_1$ has SAP for each γ such that $\operatorname{rank} \widetilde{A}_1 > \operatorname{rank} A_1$;

iii. If $\operatorname{rank} A = \operatorname{rank} A_1 + \operatorname{rank} A_2$, then $\widetilde{A}_1$ has SAP for each γ.

Proof. Given X_1 orthogonal to $\mathcal{S}_{A_1}$, define

$$\widehat{X}_1 = \begin{bmatrix} 0 & 0^T & 0^T \\ 0 & X_1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

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If A_1 does not have SAP, then there exists a nonzero X_1 that fully annihilates A_1. If $\mathbf{b}_1 \in R(A_1)$, then $\mathbf{b}_1^T X_1 = 0$, so that $\widehat{X}_1$ fully annihilates A, that is, A does not have SAP. This establishes (i).

Suppose $\widetilde{A}_1$ does not have SAP. Then there exists nonzero $\widetilde{X}_1 = \begin{bmatrix} 0 & \mathbf{y}^T \\ \mathbf{y} & X_1 \end{bmatrix}$ that fully annihilates $\widetilde{A}_1$. In particular, $\widetilde{A}_1\widetilde{X}_1 = 0$ yields

- $\gamma\mathbf{y}^T + \mathbf{b}_1^T X_1 = 0$ (from the (1,2)-block)
- $\mathbf{b}_1\mathbf{y}^T + A_1X_1 = 0$ (from the (2,2)-block)

We will show that $\mathbf{y} = 0$, causing these two equations to reduce to $\mathbf{b}_1^T X_1 = 0$ and $A_1X_1 = 0$, so that $\widehat{X}_1$ above contradicts the hypothesis that A has SAP. If $\mathbf{y} \neq 0$, then from the (2,2)-block equation multiplied by $\mathbf{y}$, $\mathbf{b}_1 = -\frac{1}{\mathbf{y}^T\mathbf{y}}A_1X_1\mathbf{y} \in R(A_1)$. Thus if $\mathbf{b}_1 \notin R(A_1)$, then $\mathbf{y} = 0$. So now assume $\mathbf{b}_1 \in R(A_1)$, that is, there exists a vector $\mathbf{u}$ such that $\mathbf{b}_1 = A_1\mathbf{u}$. Thus, the (2,2)-block equation becomes $A_1\mathbf{u}\mathbf{y}^T + A_1X_1 = 0$ and so $(\mathbf{u}^T A_1\mathbf{u})\mathbf{y}^T + \mathbf{u}^T A_1X_1 = 0$. From the (1,2)-block equation, $\gamma\mathbf{y}^T + \mathbf{u}^T A_1X_1 = 0$. Therefore $(\mathbf{u}^T A_1\mathbf{u})\mathbf{y}^T = \gamma\mathbf{y}^T$. Since $\operatorname{rank}\widetilde{A}_1 > \operatorname{rank}A_1$ yields $\gamma \neq \mathbf{u}^T A_1\mathbf{u}$, we conclude necessarily $\mathbf{y} = 0$, and if $\widetilde{A}_1$ does not have SAP, neither does A. This establishes (ii).

To prove (iii), we first note that $\operatorname{rank}A = \operatorname{rank}A_1 + \operatorname{rank}A_2$ implies $\mathbf{b}_1 = A_1\mathbf{u}_1$, $\mathbf{b}_2 = A_2\mathbf{u}_2$, and $\alpha = \mathbf{u}_1^T A_1\mathbf{u}_1 + \mathbf{u}_2^T A_2\mathbf{u}_2$ for suitable vectors $\mathbf{u}_1, \mathbf{u}_2$. In addition, by part (ii), we only need to prove that $\widetilde{A}_1$ has SAP whenever $\operatorname{rank}\widetilde{A}_1 = \operatorname{rank}A_1$, that is, when $\widetilde{A}_1 = \begin{bmatrix} \mathbf{u}_1^T A_1\mathbf{u}_1 & \mathbf{u}_1^T A_1 \\ A_1\mathbf{u}_1 & A_1 \end{bmatrix}$. Suppose $\widetilde{X}_1 = \begin{bmatrix} 0 & \mathbf{y}_1^T \\ \mathbf{y}_1 & X_1 \end{bmatrix}$ fully annihilates $\widetilde{A}_1$. Let

$$X = \begin{bmatrix} 0 & \mathbf{y}_1^T & 0 \\ \mathbf{y}_1 & X_1 & -\mathbf{y}_1\mathbf{u}_2^T \\ 0 & -\mathbf{u}_2\mathbf{y}_1^T & 0 \end{bmatrix}.$$

Clearly X is orthogonal to $\mathcal{S}_A$. Since $\widetilde{A}_1\widetilde{X}_1 = 0$, $A_1\mathbf{y}_1 = 0$ and $A_1\mathbf{u}_1\mathbf{y}_1^T + A_1X_1 = 0$, and then a computation shows $AX = 0$. Since A has SAP, $X = 0$. Thus $\widetilde{X}_1 = 0$ and so $\widetilde{A}_1$ has SAP. ☐

LEMMA 3.6. *Let A be a matrix in the form*

$$A = \begin{bmatrix} \alpha & \mathbf{b}_1^T & \mathbf{b}_2^T & \dots & \mathbf{b}_k^T \\ \mathbf{b}_1 & A_1 & 0 & \dots & 0 \\ \mathbf{b}_2 & 0 & A_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{b}_k^T & 0 & 0 & \dots & A_k \end{bmatrix},$$

where, for $i = 1, \dots, k-1$, $\operatorname{corank}A_i \geqslant \operatorname{corank}A_{i+1}$. If A has SAP, then

1. $\operatorname{corank}A_2 \leqslant 1$, *and, if* $\operatorname{corank}A_2 = 1$, *then* $\mathbf{b}_1 \notin R(A_1)$ *or* $\mathbf{b}_2 \notin R(A_2)$*;*
2. $\operatorname{corank}A_3 \leqslant 1$, *and, if* $\operatorname{corank}A_3 = 1$, *then* $\operatorname{corank}A_1 = \operatorname{corank}A_2 = 1$, *and* $\mathbf{b}_i \notin R(A_i)$ *for* $i = 1, 2, 3$*;*
3. $\operatorname{corank}A_i = 0$ *for* $i \geqslant 4$.

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Proof. By Observation 3.4, if $\operatorname{corank} A_2 \geqslant 2$, as well if $\operatorname{corank} A_2 = 1$ and $\mathbf{b}_1 \in R(A_1)$, $\mathbf{b}_2 \in R(A_2)$, we have $\operatorname{Null} \begin{bmatrix} \mathbf{b}_i^T \\ A_i \end{bmatrix} \neq 0$, $i = 1, 2$. Lemma 3.3 implies that A does not have SAP. This establishes (1).

Concerning (2), $\operatorname{corank} A_3 \leqslant 1$ follows by (1). If $\operatorname{corank} A_3 = 1$ and $\operatorname{corank} A_1 > 1$, then Observation 3.4 and Lemma 3.3, using $B_1 = A_1$ and $B_2 = A_2 \oplus A_3$, shows that A does not have SAP. Finally, if, for some $i = 1, 2, 3$, $\mathbf{b}_i \in R(A_i)$, then again Observation 3.4 and Lemma 3.3, using $B_1 = A_i$, $B_2 = \bigoplus_{j=1, j\neq i}^{3} A_j$, shows that A does not have SAP.

Finally, if $\operatorname{corank} A_4 \geqslant 1$, it suffices to define $B_1 = A_1 \oplus A_2$, $B_2 = A_3 \oplus A_4$ and proceed as in the previous cases, to conclude again that A does not have SAP. □

Lemma 3.6 has immediate application to computing ξ for trees.

THEOREM 3.7. *If T is a tree that is not a path, then $\xi(T) = 2$.*

Proof. Let T be a tree that is not a path, so $\xi(T) \geqslant 2$. Let A be a ξ-optimal matrix for T. Since $\operatorname{corank} A \geqslant 2$, by the Parter-Wiener Theorem [12], there is a vertex v such that $\operatorname{corank} A(v) = \operatorname{corank} A + 1$, and 0 is an eigenvalue of at least 3 principal submatrices A_i corresponding to components of $T - v$. Then by Lemma 3.6 (renumbering the vertices if necessary so $v = 1$ and the coranks are ordered as in the Lemma 3.6), the maximum possible number of singular A_i is 3, and the corank of each of these principal submatrices is 1, i.e., $\operatorname{corank} A(v) = 3$. Thus $\xi(T) = \operatorname{corank} A = 2$.□

Picking up from the remarks following Corollary 2.9, we now establish for trees that if there is a gap between ξ and M, then an edge can be added to the tree to reduce M.

PROPOSITION 3.8. *If T is a tree and $\xi(T) < M(T)$, then we can add an edge to T to obtain graph G such that $M(G) < M(T)$*

Proof. Let T be tree such that $\xi(T) < M(T) = h$. Then T is not a path, so $\xi(T) = 2 < M(T) = P(T)$. Choose a minimal path cover $\mathcal{P}$ for T, and let $P_1, P_2, P_3 \in \mathcal{P}$. We claim that we can choose $i, j \in \{1, 2, 3\}$ such that no vertices of P_i are adjacent to vertices of P_j. Suppose, by way of contradiction, that there exist (not necessarily distinct) vertices $u_i, v_i \in P_i$, $i = 1, 2, 3$, such that v_1u_2, v_2u_3, and v_3u_1 are edges in T. Then T would contain the cycle $u_1v_1u_2v_2u_3v_3u_1$, that is, a contradiction. Therefore we can assume that the vertices of P_1 are not adjacent to the vertices of P_2. Join by an edge e an endpoint of P_1 to an endpoint of P_2 to obtain a new path P. Thus, $P, P_3, \ldots, P_h$ provide a path covering of $G = T + e$. Since G is unicyclic, [3], $M(G) \leqslant P(G) \leqslant h - 1 < P(T) = M(T)$. □

We are now in a position to state and prove an important result for calculating ξ for vertex sums of graphs. Let $G_1, \ldots, G_k$ be disjoint graphs. For each i, we select a vertex $v_i \in V(G_i)$ and join all G_i's by identifying all v_i's as a unique vertex v. The resulting graph is called the *vertex-sum* at v of the graphs $G_1, \ldots, G_k$.

THEOREM 3.9. *Let G be vertex-sum at v of graphs $G_1, \ldots, G_k$. Then*

$$\max_{i=1}^{k} \xi(G_i) \leqslant \xi(G) \leqslant \max_{i=1}^{k} \xi(G_i) + 1.$$

Proof. By subgraph monotonicity, $\xi(G) \geqslant \max_{i=1}^{k} \xi(G_i)$. Again by subgraph monotonicity we may assume each of the $G_i - v$ is connected. Let A be ξ-optimal for

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G. Renumber the vertices of G and the order of the G_i so that $v = 1$ and A can be written as

$$A = \begin{bmatrix} \alpha & \mathbf{b}_1^T & \mathbf{b}_2^T & \dots & \mathbf{b}_k^T \\ \mathbf{b}_1 & A_1 & 0 & \dots & 0 \\ \mathbf{b}_2 & 0 & A_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{b}_k^T & 0 & 0 & \dots & A_k \end{bmatrix} \tag{3.1}$$

where, for each i, $G(A_i) = G_i - v$, and $\operatorname{corank} A_i \geqslant \operatorname{corank} A_{i+1}$. This renumbering process does not affect SAP.

By [13], $\operatorname{rank} A = \sum_{i=1}^k \operatorname{rank} A_i + \delta$ with $\delta \in \{0, 1, 2\}$, and $\delta = 2$ if and only if there is an i such that $\mathbf{b}_i \notin R(A_i)$. Therefore, $\operatorname{corank} A = \sum_{i=1}^k \operatorname{corank} A_i + 1 - \delta$.

Case 1: $\operatorname{corank} A_3 = 1$. By Lemma 3.6, $\operatorname{corank} A_1 = \operatorname{corank} A_2 = 1$, and $\mathbf{b}_i \notin R(A_i)$ for $i = 1, 2, 3$. So in this case $\delta = 2$, and $\xi(G) = \operatorname{corank} A = \sum_{i=1}^3 \operatorname{corank} A_i + 1 - 2 = 2 \leqslant \xi(G_1) + 1$.

Case 2: $\operatorname{corank} A_3 = 0$, and $\mathbf{b}_1 \notin R(A_1)$ or $\mathbf{b}_2 \notin R(A_2)$. Again, $\delta = 2$. By Lemma 3.6, $\operatorname{corank} A_2 \leqslant 1$. Define $\widetilde{A}_1 = \begin{bmatrix} \gamma & \mathbf{b}_1^T \\ \mathbf{b}_1 & A_1 \end{bmatrix}$, where γ is any real number such that $\operatorname{rank} \widetilde{A}_1 > \operatorname{rank} A_1$. By Lemma 3.5, $\widetilde{A}_1$ has SAP. So

$$\begin{aligned} \xi(G) &= \operatorname{corank} A && (3.2)\\ &= \operatorname{corank} A_1 + \operatorname{corank} A_2 - 1 && (3.3)\\ &\leqslant \operatorname{corank} A_1 && (3.4)\\ &\leqslant \operatorname{corank} \widetilde{A}_1 + 1 && (3.5)\\ &\leqslant \xi(G_1) + 1. && (3.6) \end{aligned}$$

Case 3: $\operatorname{corank} A_3 = 0$, and $\mathbf{b}_1 \in R(A_1)$, $\mathbf{b}_2 \in R(A_2)$. From Lemma 3.6, $\operatorname{corank} A_i = 0$ for $i \geqslant 2$, and so $\mathbf{b}_i \in R(A_i)$ for all i. In particular, $\delta \leqslant 1$, and $\mathbf{b}_1 = A_1 \mathbf{u}_1$ for some vector $\mathbf{u}_1$. Let $\widetilde{A}_1 = \begin{bmatrix} \gamma & \mathbf{b}_1^T \\ \mathbf{b}_1 & A_1 \end{bmatrix}$, $\gamma \in \mathbb{R}$. If $\delta = 1$, we choose $\gamma \neq \mathbf{u}_1^T A_1 \mathbf{u}_1$, so that $\operatorname{rank} \widetilde{A}_1 = \operatorname{rank} A_1 + 1$, and $\widetilde{A}_1$ has SAP by Lemma 3.5ii. On the other hand, if $\delta = 0$, we choose $\gamma = \mathbf{u}_1^T A_1 \mathbf{u}_1$, so that $\operatorname{rank} \widetilde{A}_1 = \operatorname{rank} A_1$, and $\widetilde{A}_1$ has SAP by Lemma 3.5iii. Note that, in any case, $\operatorname{corank} \widetilde{A}_1 = \operatorname{corank} A_1 + 1 - \delta$. Therefore

$$\begin{aligned} \xi(G) &= \operatorname{corank} A \\ &= \operatorname{corank} A_1 + 1 - \delta \\ &= \operatorname{corank} \widetilde{A}_1 \\ &\leqslant \xi(G_1). \;\square \end{aligned}$$

Observation 3.10. *If $\xi(G) = \max_{i=1}^k \xi(G_i) + 1$ and A is ξ-optimal A, then, with regard to (3.1), if we let $\mathbf{b}^T = (\mathbf{b}_1^T, \dots, \mathbf{b}_k^T)$, $A' = \bigoplus_{i=1}^k A_i$, we have $\mathbf{b} \notin R(A')$. Furthermore, either*

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1. G is a generalized star, namely, all G_i are paths, or,
2.a. $\operatorname{corank} A_1 > 2$*;*
b. $\operatorname{corank} A_2 = 1$*;*
c. $\operatorname{corank} A_i = 0$ *for* $i = 3, ..., k$*;*
d. $\mathbf{b}_1 \notin R(A_1)$*;*
e. $\xi(G_1 - v) < M(G_1 - v)$.

Proof. From the proof of Theorem 3.9, $\xi(G) = \max_{i=1}^{k} \xi(G_i) + 1$ occurs only in Case 1 and in Case 2, and in both cases $\delta = 2$, that is, $\mathbf{b} \notin R(A')$. In addition, in Case 1 we obtained $\xi(G) = 2$, so that $\xi(G_i) = 1$ for each i. Therefore all G_i are paths, and G is a generalized star. In Case 2, we showed $\operatorname{corank} A_i = 0$ for $i \geqslant 3$. Moreover, in (3.2-3.6) we have all equalities. In particular, equality in (3.4) yields $\operatorname{corank} A_2 = 1$, while equality in (3.5) yields $\operatorname{rank} \widetilde{A}_1 = \operatorname{rank} A_1 + 2$, so that $\mathbf{b}_1 \notin R(A_1)$. Finally, from (3.6), we get $M(G_1 - v) \geqslant \operatorname{corank} A_1 > \operatorname{corank} \widetilde{A}_1 = \xi(G_1) \geqslant \xi(G_1 - v)$ □.

Clearly a generalized star that is not a path realizes $\xi(G) = \max_{i=1}^{k} \xi(G_i) + 1$. The next example also has $\xi(G) = \max_{i=1}^{k} \xi(G_i) + 1$, in addition to $\xi(G) > \mu(G)$ and $\xi(G) > \nu(G)$.

Example 3.11. Let G be the graph shown in Figure 3.1. G is the vertex sum of

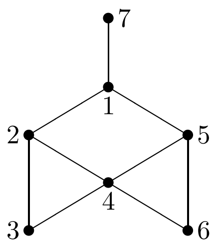

Fig. 3.1. *A vertex sum G with $\xi(G)$ greater than the maximum of ξ on vertex summands*

$G_1 = \langle 1, 2, 3, 4, 5, 6 \rangle$ and $G_2 = \langle 1, 7 \rangle$. Here $\langle W \rangle$ indicates the subgraph of G induced by the vertices $W \subset \{1, 2, \ldots, n\}$. Then $\xi(G) = 3$, because the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

has corank 3 and SAP. Moreover, $\xi(G_1) = 2$ because G_1 has an induced path of length 4, and $\xi(G_2) = 1$. Thus $\xi(G) = \max \xi(G_i) + 1$. Now $\mu(G) = 2$ because G is outerplanar. In addition we establish that $\nu(G) = 2$. Suppose $B \in \mathcal{S}(G)$ has SAP and $\operatorname{corank} B = 3$. Let $\mathbf{b} = (b_{12}, b_{13}, \ldots, b_{17})^T$. Since B is ξ-optimal and $\xi(G) = \max \xi(G_i) + 1$, $\mathbf{b} \notin R(B[2, 3, 4, 5, 6, 7])$, so $\operatorname{rank} B[2, 3, 4, 5, 6, 7] = \operatorname{rank} B - 2 = 2$. Since $\operatorname{mr}(\langle 2, 3, 4, 5, 6 \rangle) = 2$, this forces $b_{77} = 0$. But then $B[1, 7]$ is not positive semi-definite, so neither is B.

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It is also interesting to contrast the behaviour of ξ and μ on clique sums. If G is a clique sum on K_s of $G_i, i = 1, \ldots, k$ with $s < \max_{i=1}^{k}\mu(G_i)$ (as is the case for a vertex sum provided at least one G_i is not a path), then $\mu(G) = \max_{i=1}^{k}\mu(G_i)$ (see [10]).

In [10, Theorem 2.7] it is shown that $\mu(G) \leqslant \mu(G-v)+1$, where v is an arbitrary vertex in G, and if v is connected to every other vertex and $G-v$ is connected of order greater than 1, then equality holds. Do the same results hold for ξ? We have some partial results.

LEMMA 3.12. *If G is connected, $|G| > 1$, and G' is obtained by joining v to each vertex of G, then $\xi(G') \geqslant \xi(G) + 1$.*

Proof. Let the new vertex be n. Choose a matrix A that is ξ-optimal for G, so $A \in \mathcal{S}(G)$, $\operatorname{corank} A = \xi(G)$ and A has SAP. Since G is connected and $|G| > 1$, there is no column of A consisting entirely of 0's. So there is a $\mathbf{b} \in R(A)$ such that every entry b_i is nonzero. Since $\mathbf{b} \in R(A)$, there exists $\mathbf{u} \in \mathbb{R}^{n-1}$ such that $\mathbf{b} = A\mathbf{u}$. Let $A' = \begin{bmatrix} A & A\mathbf{u} \\ \mathbf{u}^T A & \mathbf{u}^T A\mathbf{u} \end{bmatrix}$. Then $\operatorname{rank} A' = \operatorname{rank} A$ so $\operatorname{corank} A' = \operatorname{corank} A + 1$. Since $b_i \neq 0$ for all i, $A' \in \mathcal{S}(G')$. Let X' fully annihilate A'. Then $X' = \begin{bmatrix} X & 0 \\ 0^T & 0 \end{bmatrix}$ (since n is joined to every other vertex). In addition, $A'X' = 0$ implies $AX = 0$, and since A has SAP, we conclude $X = 0$, that is, $X' = 0$. Thus A' has SAP, so $\xi(G') \geqslant \operatorname{corank} A' = \operatorname{corank} A + 1 = \xi(G) + 1$. ☐

LEMMA 3.13. *If there exists a ξ-optimal A for G with $\mathbf{b} \in R(A(v))$, then $\xi(G) \leqslant \xi(G-v)+1$.*

Proof. Without loss of generality, $v = 1$. Let A be ξ-optimal for G with $\mathbf{b} \in R(A_0)$ where $A = \begin{bmatrix} \alpha & \mathbf{b}^T \\ \mathbf{b} & A_0 \end{bmatrix}$. Then by Lemma 3.5, A_0 has SAP. rank $A_0 \leqslant \operatorname{rank} A$, so $\operatorname{corank} A_0 \geqslant \operatorname{corank} A - 1 = \xi(G) - 1$. Since A_0 has SAP, $\xi(G-v) \geqslant \operatorname{corank} A_0$. ☐

A graph G is called *vertex transitive* if, for any two distinct vertices u, v of G, there is an automorphism (that is, a bijection from V to V that preserves adjacency) of G mapping u to v.

COROLLARY 3.14. *If G is vertex transitive, then $\xi(G) \leq \xi(G-v)+1$.*

Proof. Without loss of generality, $v = 1$. Let G be vertex transitive and let A be ξ-optimal for G. Since A is singular, there is a row, say k, that is a linear combination of other rows of A. Since G is vertex transitive, there is a graph automorphism ψ such that $\psi(1) = k$. Let P_ψ be the permutation matrix for ψ, i.e., row i of P_ψ is row $\psi(i)$ of the identity matrix. Then since ψ is an automorphism of G, $P_\psi A P_\psi^T \in \mathcal{S}(G)$ and row 1 of $P_\psi A P_\psi^T$ is a linear combination of the other rows. Thus by Lemma 3.13, $\xi(G) \leqslant \xi(G-v)+1$. ☐

If G and H are two graphs, then the *join of G and H*, denoted by $G \vee H$, is the graph obtained from the disjoint union of G and H by adding an edge from each vertex of G to each vertex of H.

LEMMA 3.15. *If $G = G_1 \vee G_2$, $A \in \mathcal{S}(G)$ is partitioned as $\begin{bmatrix} A_1 & B \\ B^T & A_2 \end{bmatrix}$ with $A_i \in \mathcal{S}(G_i)$, $i = 1, 2$, and A_i has SAP for $i = 1, 2$, then A has SAP.*

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Proof. Suppose X fully annihilates A. Then X has the form $X = \begin{bmatrix} X_1 & 0 \\ 0^T & X_2 \end{bmatrix}$, and $AX = \begin{bmatrix} A_1X_1 & BX_2 \\ B^TX_1 & A_2X_2 \end{bmatrix} = 0$, so that $A_iX_i = 0$ for $i = 1, 2$. Since A_i has SAP, $X_i = 0$ and A has SAP. □

THEOREM 3.16. *If G is connected, $|G| > 1$ and $\xi(G) = M(G)$, then*

$$\xi(G \vee K_r) = \xi(G) + r = M(G \vee K_r).$$

Proof. The graph $G \vee K_r$ can be obtained from G by adjoining one vertex at a time to all vertices of previous graph. By Lemma 3.12, $\xi(G \vee K_r) \geqslant \xi(G) + r$. Since $\xi(G) = M(G)$, $\xi(G) = |G| - \mathrm{mr}(G)$. Thus $\xi(G) + r \leqslant \xi(G \vee K_r) \leqslant M(G \vee K_r) \leqslant |G \vee K_r| - \mathrm{mr}(G \vee K_r) \leqslant |G| + r - \mathrm{mr}(G) = \xi(G) + r$. □

COROLLARY 3.17. *There is a graph on n vertices with minimum rank k having $\frac{n(n-1)}{2} - \frac{k(k-1)}{2}$ edges. This can be obtained as $G = P_{k+1} \vee K_{n-k-1}$ or from K_n by deleting $\frac{k(k-1)}{2}$ (specific) edges. This graph satisfies $\xi(G) = M(G)$.*

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