Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 13, pp. 40-55, February 2005

ELA
www.math.technion.ac.il/iic/ela

A METHOD FOR COMPUTING QUADRATIC BRUNOVSKY FORMS*

WEN-LONG JIN†

Abstract. In this paper, for continuous, linearly-controllable quadratic control systems with a single input, an explicit, constructive method is proposed for studying their Brunovsky forms, initially studied in [W. Kang and A. J. Krener, Extended quadratic controller normal form and dynamic state feedback linearization of nonlinear systems, *SIAM Journal on Control and Optimization*, 30:1319-1337, 1992]. In this approach, the computation of Brunovsky forms and transformation matrices and the proof of their existence and uniqueness are carried out simultaneously. In addition, it is shown that quadratic transformations in the aforementioned paper can be simplified to prevent multiplicity in Brunovsky forms. This method is extended for studying discrete quadratic systems. Finally, computation algorithms for both continuous and discrete systems are summarized, and examples demonstrated.

Key words. Continuous quadratic systems, Discrete quadratic systems, Linearly-controllable control systems, Quadratic transformations, Quadratically state feedback equivalence, Quadratic Brunovsky forms.

AMS subject classifications. 93B10, 15A04, 93B40.

1. Introduction.

Linear control systems can be continuous in time t,

$$\dot{\xi} = A\xi + b\mu, \tag{1.1}$$

or discrete

$$\xi(t+1) = A\xi(t) + b\mu(t), \tag{1.2}$$

where coefficients $A \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^{n\times 1}$ are generally constant, and state variable, $\xi(\in \mathbb{R}^n)$, and control variable, $\mu(\in \mathbb{R})$, are continuous or discrete respectively. When controllable, both (1.1) and (1.2) admit the Brunovsky form [1], which is derived from controller form, under the following linear change of coordinates and state feedback

$$\xi = Tx, \qquad \mu = u + x^T v, \tag{1.3}$$

where x and u are new state and control variables respectively, and $T \in \mathbb{R}^{n\times n}$ and $v \in \mathbb{R}^{n\times 1}$ are the transformation matrices (vectors) (refer to Chapter 3 of [2]). In the linear Brunovsky form of (1.1) and (1.2), A and b have the following forms:

$$A = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & \ddots & \\ & & & \ddots & 1 \\ & & & & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}. \tag{1.4}$$

*Received by the editors 4 April 2003. Accepted for publication 26 January 2005. Handling Editor: Michael Neumann.

†Institute of Transportation Studies, University of California, 522 Social Science Tower, Irvine, CA 92697-3600, USA (wjin@uci.edu).

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 13, pp. 40-55, February 2005

ELA
www.math.technion.ac.il/iic/ela

In an attempt to extending the Brunovsky forms for non-linear systems, which generally do not even have controller forms, Kang and Krener [3] studied continuous linearly controllable quadratic control systems with a single input, which can be written as

$$\dot{\xi} = A\xi + b\mu + \mathfrak{F}^{[2]}(\xi) + G\xi\mu + O(\xi,\mu)^3, \tag{1.5}$$

where $\mathfrak{F}^{[2]}(\xi) = (\xi^T F_1 \xi, \cdots, \xi^T F_n \xi)^T$ is a vector of n quadratic terms with symmetric $n \times n$ matrices F_i's $(i = 1, \cdots, n)$, $G\xi\mu = (G_1\xi\mu, \cdots, G_n\xi\mu)^T$ is a vector of n bilinear terms with $G \in \mathbb{R}^{n\times n}$, and $O(\xi,\mu)^3$ includes all terms $\xi^a\mu^b$ with $a + b \geq 3$. (We use different notations from [3] for the purpose of clearly presenting our method of computation.) Moreover, the following additional assumptions are made for this system. First, the coefficients in (1.5) and (1.9) A, b, F_i $(i = 1, \cdots, n)$, G and h are assumed to be time-invariant. Second, the two systems are assumed to be linearly controllable; by linearly controllable we mean that the linear parts of the two quadratic systems are controllable and, as a result, the linear parts can be transformed into the Brunovsky form with (1.3).

Kang and Krener [3] first defined quadratically state feedback equivalence up to second-order, or quadratic equivalence for short, under the following quadratic change of coordinates and state feedback,

$$\xi = x + \mathfrak{P}^{[2]}(x) + O(x)^3, \qquad \nu = \mu + x^T Q x + r x\mu + O(x,\mu)^3, \tag{1.6}$$

in which $\mathfrak{P}^{[2]}(x) = (x^T P_1 x, \cdots, x^T P_n x)^T$ is a vector of n quadratic terms, and transformation matrices include symmetric $P_i \in \mathbb{R}^{n\times n}$ $(i = 1, \cdots, n)$, symmetric $Q \in \mathbb{R}^{n\times n}$, and $r \in \mathbb{R}^{1\times n}$. We can see that (1.6) are equivalent to

$$\xi = x + \mathfrak{P}^{[2]}(x) + O(x)^3, \qquad \mu = \nu - x^T Q x - r x\nu + O(x,\nu)^3, \tag{1.7}$$

and hereafter refer to transformations of [3] as (1.7). Then, from all quadratically equivalent systems of a general system (1.5), two types of Brunovsky forms were defined in [3]. In type I forms, the nonlinear terms are reduced into a number of quadratic terms x_i^2; i.e., there are no the cross terms in x_i and x_j $(i \neq j)$ or bilinear terms, $x\nu$. In type II forms, only bilinear terms are kept. In both types of Brunovsky forms, the number of non-zero nonlinear terms is $n(n-1)/2$, compared to $n^2(n+3)/2$ for a general quadratic system.

In this paper, we propose a new method for studying the Brunovsky forms, first for (1.5) under transformations (1.7). This method can be carried out in three steps: first, we find the relationships between the coefficients of (1.5), the coefficients of its quadratically equivalent systems, and the corresponding transformation matrices; second, from these relationships, we derive a mapping from the coefficients of (1.5) and its equivalent systems to the transformation matrix P_1, which can be considered as the necessary condition that all equivalent systems of (1.5) should satisfy; third, we show how to compute, from the necessary condition, Brunovsky forms as well as the corresponding transformations. With this method, we find that (1.5) admits the same two types of Brunovsky forms defined in [3]. In contrast, our approach, constructive in

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 13, pp. 40-55, February 2005

ELA
www.math.technion.ac.il/iic/ela

nature, is capable of computing the Brunovsky forms and the transformation matrices simultaneously.

We further show that quadratic transformations, (1.7), can be simplified by setting $r = 0$ to

$$\xi = x + \mathfrak{P}^{[2]}(x) + O(x)^3, \qquad \mu = \nu - x^T Q x + O(x, \nu)^3. \tag{1.8}$$

Still defining the quadratic equivalence in the sense of [3], the new transformations prevent multiple solutions of type I or type II Brunovsky forms. I.e., the Brunovsky form of each type and the corresponding transformation matrices P_i $(i = 1, \cdots, n)$ and Q are uniquely determined by the original system. Moreover, we apply the same method, but with the simplified transformations defined in (1.8), and study the following discrete system

$$\xi(t+1) = A\xi(t) + b\mu(t) + \mathfrak{F}^{[2]}(\xi(t)) + G\xi(t)\mu(t) + h\mu^2(t) + O(\xi, \mu)^3, \tag{1.9}$$

where, similarly, $\mathfrak{F}^{[2]}(\xi(t))$ and $G\xi(t)\mu(t)$ are a vector of quadratic terms and a vector of bilinear terms respectively.[1] We find that for (1.9) there exists only one type of Brunovsky form consisting of $n(n+1)/2$ bilinear terms, which corresponds to type II Brunovsky forms of continuous systems.

The rest of this paper is organized as follows. After reviewing the Brunovsky form of linear systems (Section 2), we study the Brunovsky forms of continuous quadratic systems in Section 3 and of discrete quadratic systems in Section 4. In Section 5, we summarize our method into two computation algorithms: one for continuous systems, and the other for discrete systems. We conclude our paper with Section 6.

2. Review: computation of the Brunovsky form of a controllable linear control system. We first review the computation of the Brunovsky form of a continuous controllable linear control system (1.1).[2] The procedure and the results also apply to the discrete system (1.2). Computation of the Brunovsky form for (1.1) comprises of two steps: first, the linear system is transformed into the controller form with a linear change of coordinates given by the first equation in (1.3); second, the controller form is further reduced into the Brunovsky form with a state feedback given by the second equation in (1.3).

The control matrix for (1.1) is defined as

$$C = [A^{n-1}b \quad \cdots \quad Ab \quad b]. \tag{2.1}$$

Since (1.1) is controllable, rank(C) is n and, therefore, C is invertible. Let d denote the first row of C^{-1}, then we can compute the transformation matrix T as

$$T = \begin{bmatrix} d \\ dA \\ \vdots \\ dA^{n-2} \\ dA^{n-1} \end{bmatrix}, \tag{2.2}$$

[1] Note that the discrete system (1.9) contains a term quadratic in the control variable, $h\mu^2(t)$, where $h \in \mathbb{R}^{n\times 1}$, and (1.9) has $n^2(n+3)/2 + n$ non-zero nonlinear terms.

[2] For further explanation, refer to [1] and Chapter 3 of [2].

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 13, pp. 40-55, February 2005

ELA
www.math.technion.ac.il/iic/ela

and with the linear change of coordinates $\xi = Tx$, (1.1) can be transformed into the following controller form,

$$\dot{x} = \bar{A}x + \bar{b}\mu, \tag{2.3}$$

in which

$$\bar{A} = T^{-1}AT = \begin{bmatrix} 0 & 1 & & & \\ 0 & 0 & 1 & & \\ \vdots & & 0 & \ddots & \\ \vdots & & & \ddots & 1 \\ -v_1 & -v_2 & \cdots & \cdots & -v_n \end{bmatrix}, \qquad \bar{b} = T^{-1}b = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}. \tag{2.4}$$

Then, using a linear state feedback $\mu = u + x^T v$, in which,

$$v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_{n-1} \\ v_n \end{bmatrix}, \tag{2.5}$$

we can transform the controller form (2.3) into the following Brunovsky form,

$$\dot{x} = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & \ddots & \\ & & & \ddots & 1 \\ & & & & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u. \tag{2.6}$$

From the derivation above, we can see that the Brunovsky form exists and is unique, and the transformations can be explicitly computed. In addition, this procedure can be easily integrated in applications related to the Brunovsky form. In the same spirit, we propose a method for directly computing quadratic Brunovsky forms and the corresponding transformations in the following sections.

3. Computation of continuous quadratic Brunovsky forms. Note that, with linear transformations in (1.3), quadratic systems (1.5) and (1.9) will not change their forms except their coefficients. Without loss of generality, therefore, we hereafter assume that A and b in (1.5) (and also in (1.9)) have already been transformed into the forms defined by (1.4). In this section, we study Brunovsky forms of the continuous system (1.5) under transformations (1.7), in a constructive manner. First, we find relationships between coefficients of (1.5), coefficients of its quadratically equivalent systems, and the corresponding transformation matrices. Second, from these relationships, we derive a mapping from the coefficients of (1.5) and those of the equivalent systems to transformation matrix P_1; this mapping is a necessary condition that all the equivalent systems should satisfy. Third, we show how to obtain two types of Brunovsky forms as well as the corresponding transformations.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 13, pp. 40-55, February 2005

ELA
www.math.technion.ac.il/iic/ela

3.1. Quadratically equivalent system of (1.5).

Definition 3.1. Assume $A \in \mathbb{R}^{n\times n}$ is in the linear Brunovsky form given by (1.4), we define a linear operator $\mathbf{L} : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ as

$$\begin{aligned} \mathbf{L}^0 P &= P, \\ \mathbf{L}P &= A^T P + PA, \\ \mathbf{L}^{i+1} P &= \mathbf{L}\cdot\mathbf{L}^i P, \quad i = 0, 1, \cdots . \end{aligned} \tag{3.1}$$

Properties of L. The linear operator $\mathbf{L}$ has the following properties:

1. $\mathbf{L}^i P = 0$ when $i \geq 2n - 1$.
2. The nullity of $\mathbf{L}$ is n; if $P \in ker(\mathbf{L})$; i.e., $\mathbf{L}P = 0$, then P can be written as
$$P_{ij} = \begin{cases} 0, & i + j \leq n, \\ (-1)^i p_{i+j-n}, & \text{otherwise,} \end{cases}$$
in which $p_1, \cdots, p_n$ are independent.
3. $\mathbf{L}$ is not invertible.
4. If P is symmetric, $\mathbf{L}P$ is symmetric.

Theorem 3.2. *The continuous quadratic system (1.5) is equivalent, in the sense of [3], under the quadratic transformations given by (1.7), to a quadratic system, whose ith ($i = 1, \cdots, n$) equation is*

$$\dot{x}_i = x_{i+1} + b_i\nu + x^T \bar{F}_i x + \bar{G}_i x\nu + O(x, \nu)^3, \tag{3.2}$$

where $x_{n+1}(t) \equiv 0$ is a dummy state variable, $\bar{F}_1, \cdots, \bar{F}_n$ are symmetric, $\bar{G}_i$ is the ith row of the matrix $\bar{G}$, and the coefficients $\bar{F}_i$ ($i = 1, \cdots, n$) and $\bar{G}$ are defined by

$$\bar{F}_i = F_i + P_{i+1} - \mathbf{L}P_i - b_i Q, \tag{3.3}$$

$$\bar{G}_i = G_i - 2b^T P_i - b_i r, \tag{3.4}$$

where $P_{n+1} \in \mathbb{R}^{n\times n}$ is a zero dummy transformation matrix, and b_i is the ith element of b defined in (1.4).

Proof. Plugging the transformations defined in (1.7) into (1.5), we obtain

$$\begin{aligned} \dot{x} + \frac{d}{dt}\mathfrak{P}^{[2]}(x) = Ax + b\nu + \mathfrak{F}^{[2]}(x) + Gx\nu \\ + A\mathfrak{P}^{[2]}(x) - bx^T Qx - brx\nu + O(x, \nu)^3, \end{aligned}$$

of which the ith ($i = 1, \cdots, n$) differential equation can be written as

$$\begin{aligned} \dot{x}_i + \frac{d}{dt}(x^T P_i x) = x_{i+1} + b_i\nu + x^T F_i x + G_i x\nu \\ + x^T P_{i+1} x - b_i x^T Qx - b_i rx\nu + O(x, \nu)^3. \end{aligned}$$

After plugging $\dot{x} = Ax + b\nu + O(x, \nu)^2$ into the term $\frac{d}{dt}(x^T P_i x)$ and collecting all terms whose orders are higher than two, we then obtain the equivalent system (3.2) with the coefficients given in (3.3) and (3.4). ∎

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 13, pp. 40-55, February 2005

ELA
www.math.technion.ac.il/iic/ela

Remarks. Equations 3.3 and 3.4 present the relationships between the coefficients of (1.5), those of its quadratically equivalent system (3.2), and the corresponding transformation matrices. We further simplify these relationships in the following subsections.

3.2. A necessary condition for quadratically equivalent systems.

LEMMA 3.3. *The mapping from the coefficients, G and $\bar{G}$, and r to the transformation matrices P_i $(i = 1, \cdots, n)$ is given by*

$$\left[\begin{array}{c} [P_1]_{(n)} \\ \hline [P_2]_{(n)} \\ \hline \vdots \\ \hline [P_n]_{(n)} \end{array}\right] = \frac{1}{2}G - \frac{1}{2}\bar{G} - \frac{1}{2}br, \tag{3.5}$$

where the operator $[\cdot]_{(n)}$ takes the nth row from its object.

Proof. Since $b_i = 0$ $(i = 1, \cdots, n-1)$ and $b_n = 1$, we obtain, from (3.4),

$$[P_i]_{(n)} = b^T P_i = \frac{1}{2}G_i - \frac{1}{2}\bar{G}_i - \frac{1}{2}b_i r.$$

Thus we have (3.5). □

LEMMA 3.4. *The mapping from the coefficients F_i and $\bar{F}_i$ $(i = 1, \cdots, n)$ to the transformation matrices P_i $(i = 1, \cdots, n)$ and Q is given by*

$$P_{i+1} = \mathbf{L}^i P_1 - \sum_{j=0}^{i-1} \mathbf{L}^j F_{i-j} + \sum_{j=0}^{i-1} \mathbf{L}^j \bar{F}_{i-j}, \quad i = 1, \cdots, n-1, \tag{3.6}$$

$$Q = \sum_{j=0}^{n-1} \mathbf{L}^j (F_{n-j} - \bar{F}_{n-j}) - L^n P_1. \tag{3.7}$$

Proof. When $i = 1, \cdots, n-1$, from (3.3), we have

$$P_{i+1} = \mathbf{L}P_i - F_i + \bar{F}_i.$$

After iterating this equation with respect to i, we obtain (3.6). When $i = n$, (3.3) can be written as

$$Q = F_n - \bar{F}_n - \mathbf{L}P_n.$$

Since P_n is given in (3.6), we then have (3.7). □

DEFINITION 3.5. With $A \in \mathbb{R}^{n\times n}$ and $\mathbf{L}$ defined in (1.4) and Definition 3.1 respectively, we define a series of linear operators $\mathbf{X}_i : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ $(i = 0, 1, \cdots)$ by

$$\mathbf{X}_0 P = \left[\begin{array}{c} [\mathbf{L}^0 P]_{(n)} \\ \hline [\mathbf{L}^1 P]_{(n)} \\ \hline \vdots \\ \hline [\mathbf{L}^{n-1} P]_{(n)} \end{array}\right], \quad \mathbf{X}_i P = (A^T)^i \mathbf{X}_0 P. \tag{3.8}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 13, pp. 40-55, February 2005

ELA
www.math.technion.ac.il/iic/ela

Properties of $\mathbf{X}_i$. The linear operators $\mathbf{X}_i$ $(i = 0, 1, \cdots)$ have the following properties:

1. $\mathbf{X}_0$ transforms a diagonal matrix into a skew-triangular matrix of the following structure. For a kth $(k = -n+1, \cdots, n-1)$ diagonal matrix P, we denote the kth diagonal elements by $p_l = P_{(|k|-k)/2+l,(|k|+k)/2+l}$ $(l = 1, \cdots, n-|k|)$. Then $\mathbf{X}_0 P$ becomes a diagonal matrix with the following properties: $(\mathbf{X}_0 P)_{n-(|k|-k)/2-l+1,(|k|+k)/2+l} = p_l$, all $(\mathbf{X}_0 P)_{i,j}$ for $i + j = n + k + 1 + 2m$, where $m \geq 1$ and $n + k + 1 + 2m \leq 2n$, are determined by p_l, and the other entries are zeros.[3]
2. From the preceding property, $\mathbf{X}_0$ transforms a lower-triangular matrix into a full matrix and transforms an upper-triangular matrix into a lower skew-triangular matrix Δ, defined by

$$\Delta_{ij} = 0, \qquad \text{when } i + j \leq n + 1. \tag{3.9}$$

3. From Properties 1 and 2, we have that the nullity of $\mathbf{X}_0$ is 0. Therefore, $\mathbf{X}_0$ is invertible.
4. From the definition of $\mathbf{L}$, $\mathbf{X}_i = 0$ when $i \geq n$.
5. From the definition of $\mathbf{X}_i$ $(i = 1, \cdots, n-1)$, $\mathbf{X}_i P$ can be obtained by shifting $\mathbf{X}_0 P$ down by i rows and replacing the first i rows by zeros. From Property 1, therefore, $\mathbf{X}_i$ transforms a diagonal matrix P, whose main diagonal elements are $p_1, \cdots, p_n$ into a lower skew-triangular matrix of the following structure: $(\mathbf{X}_i P)_{n-l+i+1,l} = p_l$ for $l = i+1, \cdots, n$, and all other elements except $(\mathbf{X}_i P)_{i,j}$ for $i + j = n + i + 2m$, where integer $m \geq 1$ and $n + k + 1 + 2m \leq 2n$, are zeros.

THEOREM 3.6. *The mapping from F_i, G, $\bar{F}_i$, $\bar{G}$ $(i = 1, \cdots, n-1)$, and r to P_1 is given by*

$$P_1 = \boldsymbol{X}_0^{-1}\left(\sum_{i=1}^{n-1}\boldsymbol{X}_i F_i + \frac{1}{2}G - \sum_{i=1}^{n-1}\boldsymbol{X}_i\bar{F}_i - \frac{1}{2}\bar{G} - \frac{1}{2}br\right). \tag{3.10}$$

Proof. From (3.6) we can find the nth row of P_{i+1} $(i = 1, \cdots, n-1)$ as

$$[P_{i+1}]_{(n)} = [\mathbf{L}^i P_1]_{(n)} - \sum_{j=0}^{i-1}[\mathbf{L}^j F_{i-j}]_{(n)} + \sum_{j=0}^{i-1}[\mathbf{L}^j \bar{F}_{i-j}]_{(n)}.$$

Hence, we have

$$\begin{bmatrix} [P_1]_{(n)} \\ \hline [P_2]_{(n)} \\ \hline \vdots \\ \hline [P_n]_{(n)} \end{bmatrix} = \mathbf{X}_0 P_1 - \sum_{i=1}^{n-1}\mathbf{X}_i F_i + \sum_{i=1}^{n-1}\mathbf{X}_i\bar{F}_i. \tag{3.11}$$

[3]Note that, for $n = 2$, there is no solution to m. In this case, all entries are zeros except $(\mathbf{X}_0 P)_{ij}$ for $i + j = n + k + 1$.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 13, pp. 40-55, February 2005

ELA
www.math.technion.ac.il/iic/ela

From (3.5) and (3.11), we obtain

$$\mathbf{X}_0 P_1 = \sum_{i=1}^{n-1} \mathbf{X}_i F_i + \frac{1}{2} G - \sum_{i=1}^{n-1} \mathbf{X}_i \bar{F}_i - \frac{1}{2}\bar{G} - \frac{1}{2} br.$$

Multiplying both sides by the inverse of $\mathbf{X}_0$, we then have (3.10). □

Remarks. Note that P_1 is assumed to be symmetric. Therefore quadratically equivalent coefficients $\bar{F}_i$ $(i = 1, \cdots, n-1)$ and $\bar{G}$ have to ensure the symmetry of the right hand side of (3.10). Thus (3.10) constitutes a necessary condition for all equivalent systems of (1.5), including the Brunovsky forms.

3.3. Computation of the Brunovsky forms and the transformation matrices. In this section, given r as well as the coefficients of (1.5), F_i $(i = 1, \cdots, n)$ and G, we show how to choose $\bar{F}_i$ $(i = 1, \cdots, n)$ and $\bar{G}$ in Brunovsky forms, which, first, satisfy the necessary condition (3.10) and, second, have the smallest number of non-zero terms.

Since $\bar{F}_n$ does not appear in the necessary condition, (3.10), we simply let $\bar{F}_n = 0$ in Brunovsky forms. To determine other coefficients, we decompose the terms related to original system (1.5) and r on the right hand side of (3.10) as

$$\mathbf{X}_0^{-1}\left(\sum_{i=1}^{n-1} \mathbf{X}_i F_i + \frac{1}{2} G - \frac{1}{2} br\right) = L + D + U, \tag{3.12}$$

where L, D, U are strictly lower triangular, diagonal, and strictly upper triangular matrices respectively. In order to add $\mathbf{X}_0^{-1}(\sum_{i=1}^{n-1} \mathbf{X}_i \bar{F}_i + \frac{1}{2}\bar{G})$ to the right hand side of (3.10) to obtain a symmetric matrix, we have, according to the property of the decomposition, the following three cases.

1. If $L - U^T = 0$, $L + D + U$ is already symmetric, and we simply set $\bar{F}_i$ $(i = 1, \cdots, n-1)$ and $\bar{G}$ to be 0. In this case, the Brunovsky form of (1.5) is a linear system; i.e., (1.5) can be linearized.
2. When $L - U^T \neq 0$, we can set $\mathbf{X}_0^{-1}(\sum_{i=1}^{n-1} \mathbf{X}_i \bar{F}_i + \frac{1}{2}\bar{G})$ to a lower-triangular matrix, $L - U^T$. In this case, $P_1 = U^T + D + U$. According to the properties of $\mathbf{X}_0$, $\sum_{i=1}^{n-1} \mathbf{X}_i \bar{F}_i + \frac{1}{2}\bar{G}$ is a full matrix.
3. When $L - U^T \neq 0$, we can also set $\mathbf{X}_0^{-1}(\sum_{i=1}^{n-1} \mathbf{X}_i \bar{F}_i + \frac{1}{2}\bar{G})$ to be an upper-triangular matrix, $U - L^T$, $\sum_{i=1}^{n-1} \mathbf{X}_i \bar{F}_i + \frac{1}{2}\bar{G}$ is a triangular matrix defined by (3.9), which has $n(n-1)/2$ non-zero terms as $U - L^T$. In this case, we have

   $$\sum_{i=1}^{n-1} \mathbf{X}_i \bar{F}_i + \frac{1}{2}\bar{G} = \mathbf{X}_0 (U - L^T) \equiv \Delta_1, \tag{3.13}$$

   and the first transformation matrix is given by

   $$P_1 = L + D + L^T. \tag{3.14}$$

4. In addition to the aforementioned cases, $\mathbf{X}_0^{-1}(\sum_{i=1}^{n-1} \mathbf{X}_i \bar{F}_i + \frac{1}{2}\bar{G})$ can be also either $L - U^T$ or $U - L^T$ plus an arbitrary symmetric matrix.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 13, pp. 40-55, February 2005

ELA
www.math.technion.ac.il/iic/ela

Comparing Cases 2, 3, and 4, we can see that the solutions in Cases 2 and 4 have more non-linear terms than in Case 3. Therefore, in the Brunovsky forms, we set $\mathbf{X}_0^{-1}(\sum_{i=1}^{n-1}\mathbf{X}_i\bar{F}_i + \frac{1}{2}\bar{G})$ to be $U - L^T$, and from (3.13), we can obtain two types of Brunovsky forms as follows.

First, by setting $\bar{G} = 0$, we have from (3.13) that $\sum_{i=1}^{n-1}\mathbf{X}_i\bar{F}_i = \Delta_1$, which is a lower skew-triangular matrix. To obtain the smallest number of, i.e., $n(n-1)/2$, non-zero items in the Brunovsky forms, we can select $\bar{F}_i$ to be a diagonal matrix with main diagonal elements as $(0,\cdots,0,\bar{f}_{i,i+1},\cdots,\bar{f}_{i,n})$, which can be uniquely computed as follows. From properties of $\mathbf{X}_i$ $(i = 1,\cdots,n-1)$, we first have

$$\bar{f}_{1,j} = (\mathbf{X}_1\bar{F}_1)_{n-j+2,j} = (\Delta_1)_{n-j+2,j}, j = 2,\cdots,n, \tag{3.15}$$

and

$$\Delta_2 = \Delta_1 - \mathbf{X}_1\bar{F}_1. \tag{3.16}$$

Then, we can compute $\bar{f}_{2,j}$ for $j = 3,\cdots,n$ from Δ_2 and $\Delta_3 = \Delta_2 - \mathbf{X}_2\bar{F}_2$ in the same fashion. By repeating this process, we can obtain all non-zero elements in $\bar{F}_i$ for $i = 1,\cdots,n-1$. Since all these matrices are uniquely determined by Δ_1, the original system is uniquely equivalent to the following system,

$$\dot{x}_i = x_{i+1} + b_i\nu + \sum_{j=i+1}^{n}\bar{f}_{ij}x_j^2 + O(x,\nu)^3, \qquad i = 1,\cdots,n, \tag{3.17}$$

which is type I, complete-quadratic Brunovsky form in [3].

Second, by setting $\bar{F}_i = 0$ $(i = 1,\cdots,n-1)$, we have $\bar{G} = 2\Delta_1$ and the corresponding equivalent system,

$$\dot{x}_i = x_{i+1} + b_i\nu + \sum_{j=n-(i+1)}^{n}\bar{G}_{ij}x_j\nu + O(x,\nu)^3, \qquad i = 1,\cdots,n, \tag{3.18}$$

which is type II, bilinear Brunovsky form in [3]. We can see that $\bar{G}$ is also uniquely determined by the original system.

Once we have the Brunovsky forms of a continuous quadratic system, we can solve the corresponding transformation matrices P_i $(i = 1,\cdots,n)$ and Q from (3.14), (3.6), and (3.7). These transformation matrices are also unique corresponding to each Brunovsky form.

3.4. Discussions. In the preceding subsection, we finished computing the two types of Brunovsky forms and the corresponding transformation matrices of a linearly controllable quadratic system, (1.5). Besides, we showed that the Brunovsky form of each type and the corresponding transformation matrices are unique. However, the uniqueness is dependent on r. That is, for different values of r, a system can have multiple solutions of type I or type II Brunovsky forms. A straightforward strategy to prevent the multiplicity is to set $r = 0$, and we have a simplified version of transformations, defined by (1.8). That is, with (1.8), a quadratic system has unique

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 13, pp. 40-55, February 2005

ELA
www.math.technion.ac.il/iic/ela

type I and type II Brunovsky forms. Further, one can follow the arguments of Kang and Krener [3] but with $r = 0$ and prove that (1.8) still defines quadratic equivalence in the same sense.[4]

4. Computation of discrete quadratic Brunovsky forms. In this section, we apply the method proposed in the preceding section and solve Brunovsky forms for a discrete quadratic control system (1.9), but with simplified transformations defined in (1.8). We carry out our study in the same three-step, constructive procedure as for the continuous system. Compared to continuous system, the discrete system has one more term, and the relationships between coefficients and transformations and the resulted Brunovsky forms are fundamentally different from those of continuous systems.

4.1. Quadratically equivalent systems of (1.9).

DEFINITION 4.1. Assume $A \in \mathbb{R}^{n\times n}$ is in the linear Brunovsky form defined in (1.4), we define a linear operator $\mathbf{L} : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ by

$$\begin{array}{lcl} \mathbf{L}^0 P & = & P, \\ \mathbf{L}P & = & A^T PA, \\ \mathbf{L}^{i+1} P & = & \mathbf{L}\cdot\mathbf{L}^i P, \quad i = 0, 1, \cdots . \end{array}$$

Properties of L. The linear operator $\mathbf{L}$ has the following properties:

1. $\mathbf{L}^i P = 0$ when $i \geq n$.
2. The nullity of $\mathbf{L}$ is $2n - 1$; if $P \in ker(\mathbf{L})$, then $P_{ij} = 0$ when $(n - i)(n - j) > 0$, and elements in nth row and nth column can be arbitrarily selected.
3. $\mathbf{L}$ is not invertible.
4. If P is symmetric, $\mathbf{L}P$ is symmetric; but the inverse proposition may not be true.

Remarks. Note that the linear operator $\mathbf{L}$ for the discrete system is different from that for continuous systems. This fundamental difference leads to the difference in the relationships between coefficients and transformations and the Brunovsky forms.

THEOREM 4.2. *The Discrete system (1.9) is quadratically equivalent, in the sense of [3], under the quadratic transformations in (1.8), to a system, whose ith ($i = 1, \cdots, n$) equation is*

$$x_i(t+1) = x_{i+1}(t) + b_i\nu(t) + x^T(t)\bar{F}_i x(t) + \bar{G}_i x(t)\nu(t) + O(x,\nu)^3, \tag{4.1}$$

where $x_{n+1}(t) \equiv 0$ is a dummy state variable, $\bar{F}_1, \cdots, \bar{F}_n$ are symmetric, $\bar{G}_i$ is the ith row of the matrix $\bar{G}$, and the coefficients $\bar{F}_i$ and h_i ($i = 1, \cdots, n$) and $\bar{G}$ are defined by

$$\bar{F}_i = F_i + P_{i+1} - \mathbf{L}P_i - b_i Q, \tag{4.2}$$

$$\bar{G}_i = G_i - 2b^T P_i A, \tag{4.3}$$

$$h_i = (P_i)_{nn}, \tag{4.4}$$

[4]Due to the difference in notations, $r = 0$ is equivalent to saying $\beta^{[1]} = 0$ in [3].

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 13, pp. 40-55, February 2005

ELA
www.math.technion.ac.il/iic/ela

where P_{n+1} is a zero dummy matrix, b_i is the ith element of b, and $(P_i)_{nn}$ is the (n,n) entry of the matrix P_i.

The proof is similar to that of Theorem 3.2 and, therefore, omitted.

Remarks. Note that (4.3) is different from (3.4), and we have an additional equation (4.4).

4.2. A necessary condition for the equivalent systems.

LEMMA 4.3. *The mapping from the coefficients G and $\bar{G}$ to the transformation matrices P_i $(i = 1, \cdots, n)$ is given by*

$$\begin{bmatrix} [P_1]_{(n)} \\ \hline [P_2]_{(n)} \\ \hline \vdots \\ \hline [P_n]_{(n)} \end{bmatrix} \cdot A = \frac{1}{2}G - \frac{1}{2}\bar{G}. \tag{4.5}$$

The proof is similar to that of Lemma 3.3 and, therefore, omitted.

LEMMA 4.4. *The mapping from the coefficients F_i, $\bar{F}_i$ $(i = 1, \cdots, n)$ and h to the transformation matrices P_i $(i = 1, \cdots, n)$ and Q is given by*

$$P_{i+1} = \mathbf{L}^i P_1 - \sum_{j=0}^{i-1} \mathbf{L}^j F_{i-j} + \sum_{j=0}^{i-1} \mathbf{L}^j \bar{F}_{i-j}, \quad i = 1, \cdots, n-1, \tag{4.6}$$

$$Q = \sum_{j=0}^{n-1} \mathbf{L}^j (F_{n-j} - \bar{F}_{n-j}), \tag{4.7}$$

and

$$\begin{array}{lcll} (P_1)_{n-i\,n-i} & = & \sum_{j=0}^{i-1} (\mathbf{L}^j F_{i-j})_{nn} + h_{i+1}, & \qquad i = 1, \cdots, n-1, \\ (P_1)_{nn} & = & h_1. & \end{array} \tag{4.8}$$

Proof. The derivation of (4.6) and (4.7) from (4.2) is similar to that in the proof of Lemma 3.4 and, therefore, omitted. Here we only show how (4.8) can be derived as follows. From (4.6), we have $(i = 1, \cdots, n-1)$

$$(P_{i+1})_{nn} = (\mathbf{L}^i P_1)_{nn} - \sum_{j=0}^{i-1} (\mathbf{L}^j F_{i-j})_{nn} = (P_1)_{n-i\,n-i} - \sum_{j=0}^{i-1} (\mathbf{L}^j F_{i-j})_{nn}.$$

Comparing this with (4.4), we then obtain (4.8), from which we can see that the diagonal entries of P_1 are uniquely determined by the coefficients F_i $(i = 1, \cdots, n)$ and h. ◻

DEFINITION 4.5. With $A \in \mathbb{R}^{n\times n}$ and $\mathbf{L}$ given by (1.4) and Definition 4.1 respectively, we define a series of linear operators $\mathbf{X}_i : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ $(i = 0, 1, \cdots)$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 13, pp. 40-55, February 2005

by

$$\begin{aligned} \mathbf{X}_0 P &= \left[\begin{array}{c} [\mathbf{L}^0 P]_{(n)} \\ \hline [\mathbf{L}^1 P]_{(n)} \\ \hline \vdots \\ \hline [\mathbf{L}^{n-1} P]_{(n)} \end{array} \right], \\ \mathbf{X}_i P &= (A^T)^i \mathbf{X}_0 P. \end{aligned} \tag{4.9}$$

Properties of $\mathbf{X}_i$. The linear operators $\mathbf{X}_i$ $(i = 0, 1, \cdots)$ have the following properties:

1. $\mathbf{X}_i = 0$ when $i \geq n$.
2. The rank of $\mathbf{X}_0$ is $n(n+1)/2$; denote the (i,j)th entry of P by P_{ij} $(i, j = 1, \cdots, n)$, then the (i,j)th entry of $\mathbf{X}_0 P$ can be written as

$$(\mathbf{X}_0 P)_{ij} = \begin{cases} P_{n-i+1, j-i+1}, & j \geq i; \\ 0, & j < i. \end{cases} \tag{4.10}$$

3. From Property 2, $\mathbf{X}_0$ is not invertible. Note that its continuous counterpart in Definition 3.5 is invertible.

THEOREM 4.6. *The mapping from the coefficients F_i, G, $\bar{F}_i$, and $\bar{G}$ $(i = 1, \cdots, n)$ to P_1 is given by*

$$\boldsymbol{X}_0 P_1 A = \sum_{i=1}^{n-1} \boldsymbol{X}_i F_i A + \frac{1}{2} G - \sum_{i=1}^{n-1} \boldsymbol{X}_i \bar{F}_i A - \frac{1}{2} \bar{G} \tag{4.11}$$

The proof is similar to that of Theorem 3.6 and omitted.

Remarks. According to (4.10), we can find the (i,j)th entry of $\mathbf{X}_0 P A$,

$$(\mathbf{X}_0 P_1 A)_{ij} = \begin{cases} (P_1)_{n-i+1, j-i}, & j > i; \\ 0 & j \leq i. \end{cases} \tag{4.12}$$

That is, $\mathbf{X}_0 P A$, the left hand side of (4.11), is an upper-triangular matrix, which contains all the non-diagonal entries of P_1. Therefore, $\bar{F}_i$ $(i = 1, \cdots, n)$ and $\bar{G}$ of a quadratically equivalent system of (1.9) must satisfy that the right hand side of (4.11) be also upper triangular. Thus, (4.11) is a necessary condition for all equivalent systems of (1.9), including the Brunovsky form.

4.3. Computation of the Brunovsky form and the transformation matrices.

THEOREM 4.7. *In the Brunovsky form of (1.9), the coefficients are*

$$\begin{aligned} \bar{F}_i &= 0, \quad \forall i = 1, \cdots, n, \\ \bar{G} &= 2(L + D), \end{aligned} \tag{4.13}$$

where L and D are the strictly lower triangular part and the diagonal part of $\sum_{i=1}^{n-1} \boldsymbol{X}_i F_i A + \frac{1}{2} G$.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 13, pp. 40-55, February 2005

ELA
www.math.technion.ac.il/iic/ela

Proof. The first two terms on the right hand side of (4.11) can be written as

$$\sum_{i=1}^{n-1} \mathbf{X}_i F_i A + \frac{1}{2} G = L + D + U, \tag{4.14}$$

where L, D, U are strictly lower triangular, diagonal, and strictly upper triangular matrices respectively. To ensure the right hand side of (4.11) to be an upper triangular matrix, we set $\sum_{i=1}^{n-1} \mathbf{X}_i \bar{F}_i A + \frac{1}{2}\bar{G}$ to be

$$\sum_{i=1}^{n-1} \mathbf{X}_i \bar{F}_i A + \frac{1}{2} \bar{G} = L + D, \tag{4.15}$$

in which there are $n(n+1)/2$ non-zero elements at most, and we hence have

$$\mathbf{X}_0 P_1 A = U. \tag{4.16}$$

Due to the properties of $\mathbf{X}_i$, no simple $\bar{F}_i$'s with $n(n+1)/2$ non-zero elements can satisfy (4.15). We simply set the coefficients as in (4.13), and obtain the only type of Brunovsky form of (1.9):

$$x_i(t+1) = x_{i+1}(t) + b_i \nu(t) + \sum_{j=1}^{i} \bar{G}_{ij} x_j(t) \nu + O(x, \nu)^3, \quad i = 1, \cdots, n. \tag{4.17}$$ □

Remarks. The Brunovsky form of discrete systems corresponds to type II form of continuous systems and has $n(n+1)/2$ bilinear terms. After finding the Brunovsky form of a discrete quadratic system, we can solve the corresponding transformation matrices P_i ($i = 1, \cdots, n$) and Q as follows. From (4.12) and (4.16), we can find the non-diagonal elements of P_1, and the diagonal elements from (4.8). Then P_i ($i = 2, \cdots, n$) and Q can be calculated from (4.6) and (4.7).

5. Algorithms and examples. In this section, we summarize the algorithms for computing Brunovsky forms and the corresponding transformations of both continuous and discrete linearly controllable quadratic systems with a single input. To prevent multiplicity in Brunovsky forms, we use the simplified version of transformations, (1.8), for both systems. Thus, $r \equiv 0$ for all formulas in Section 3. Following each algorithm, an example is given.

ALGORITHM 5.1. *Computation of continuous quadratic Brunovsky forms.*

Step 1 *Compute* $\boldsymbol{X}_0^{-1} \left(\sum_{i=1}^{n-1} \boldsymbol{X}_i F_i + \frac{1}{2} G \right)$ *and decompose it into* $L + D + U$.

Step 2 *Compute* $P_1 = L + D + L^T$ *and* $\Delta_1 = \boldsymbol{X}_0 (U - L^T)$.

Step 3 *For type I Brunovsky forms, set* $\bar{G} = 0$ *and solve the equation*

$$\sum_{i=1}^{n-1} \boldsymbol{X}_i \bar{F}_i = \Delta_1$$

to compute $\bar{F}_i$ ($i = 1, \cdots, n-1$) *as in (3.15) and (3.16). Go to Step 5.*

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 13, pp. 40-55, February 2005

Step 4 *For type II Brunovsky forms, set $\bar{F}_i = 0$ $(i = 1, \cdots, n)$ and $\bar{G} = 2\Delta_1$. Go to step 5.*
Step 5 *Compute P_i $(i = 2, \cdots, n)$ and Q from (3.6) and (3.7).*

EXAMPLE 5.2. *Find type I Brunovsky form of a continuous quadratic control system, which is already in type II Brunovsky form,*

$$\begin{aligned} \dot{\xi}_1 &= \xi_2 + O(\xi,\mu)^3, \\ \dot{\xi}_2 &= \mu + \xi_2\mu + O(\xi,\mu)^3. \end{aligned} \tag{5.1}$$

Solutions. In this system, $n = 2$, and coefficients are

$$F_1 = F_2 = 0,$$
$$G = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Following Algorithm 5.1, we find type I Brunovsky form,

$$\begin{aligned} \dot{x}_1 &= x_2 + \tfrac{1}{2}x_2^2 + O(x,\nu)^3, \\ \dot{x}_2 &= \nu + O(x,\nu)^3, \end{aligned} \tag{5.2}$$

and the corresponding transformations,

$$\begin{aligned} \xi_1 &= x_1, \\ \xi_2 &= x_2 + \tfrac{1}{2}x_2^2, \\ \mu &= \nu. \end{aligned} \tag{5.3}$$

Reversely, if given (5.2), we find the transformations for obtaining (5.1),

$$x_1 = \xi_1,$$
$$x_2 = \xi_2 - \frac{1}{2}\xi_2^2,$$
$$\nu = \mu,$$

which are the inverses of (5.3).

ALGORITHM 5.3. *Computation of discrete quadratic Brunovsky form.*
Step 1 *Compute $\sum_{i=1}^{n-1} \boldsymbol{X}_i F_i A + \frac{1}{2}G$ and decompose it into $L + D + U$.*
Step 2 *Compute $\bar{G} = 2(L + D)$.*
Step 3 *Compute the non-diagonal elements of P_1 from (4.12) and (4.16).*
Step 4 *Compute the diagonal elements of P_1 from (4.8).*
Step 5 *Compute P_i $(i = 2, \cdots, n)$ and Q from from (4.6) and (4.7).*

EXAMPLE 5.4. *Find the Brunovsky form of the following discrete system:*

$$\xi_1(t+1) = \xi_2(t) + \xi_1^2(t) + \xi_2^2(t) + \mu^2(t) + O(\xi,\mu)^3,$$
$$\xi_2(t+1) = \mu(t) + \mu^2(t) + O(\xi,\mu)^3.$$

Solutions. In this system, n=2 and coefficients are

$$F_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 13, pp. 40-55, February 2005

ELA
www.math.technion.ac.il/iic/ela

$$\begin{aligned} F_2 &= 0, \\ G &= 0, \\ h &= \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \end{aligned}$$

Following Algorithm 5.3, we find the Brunovsky form,

$$\begin{aligned} x_1(t+1) &= x_2(t) + O(x,\nu)^3, \\ x_2(t+1) &= \nu(t) + O(x,\nu)^3, \end{aligned}$$

and the corresponding transformations

$$\begin{aligned} \xi_1(t) &= x_1(t) + 2x_1^2(t) + x_2^2(t), \\ \xi_2(t) &= x_2(t) - x_1^2(t) + x_2^2(t), \\ \mu(t) &= \nu(t) - x_2^2(t). \end{aligned}$$

Thus, this system is linearized.

6. Conclusions. In this paper, we proposed a method for computing Brunovsky forms of both continuous and discrete quadratic control systems, which are linearly controllable with a single input. Our approach is constructive in nature and computes Brunovsky forms explicitly in a three-step manner, with the introduction of linear operators $\mathbf{L}$ and $\mathbf{X}_i$ $(i = 0, \cdots, n-1)$: (i) we first derived relationships between the coefficients of original systems, those of the quadratically equivalent systems, and the corresponding transformation matrices; (ii) after simplifying these relationships, we obtained a necessary condition for all equivalent systems; (iii) from the necessary conditions, we finally computed the Brunovsky forms and the corresponding transformations. However, the linear operators $\mathbf{L}$ and $\mathbf{X}_i$ and, therefore, the Brunovsky forms are fundamentally different for continuous and discrete systems. For a continuous quadratic system, which has at most $n(n+1)^2/2$ nonlinear terms, there are two types of Brunovsky forms with $n(n-1)/2$ nonlinear terms each; for a discrete quadratic system, there is only one type, and the numbers of nonlinear terms of a general system and its Brunovsky form are $n(n+1)^2/2+n$ and $n(n+1)/2$, respectively.

For continuous systems, we used the same transformations as in [3] and found that the inclusion of transformation vector r yields multiple solutions of a type of Brunovsky forms. Therefore, we suggested to set $r = 0$ in order to maintain the uniqueness. That is, under transformations defined by (1.8) for both continuous and discrete systems, the Brunovsky forms and the corresponding quadratic transformations always exist and are uniquely determined by the original systems. In this sense, our study can be viewed as a constructive proof of the existence and uniqueness of Brunovsky forms.

Finally, the method proposed in this paper can be extended for studying quadratic control systems, not linearly controllable or with multiple input. In addition, the method could be useful for applications involving analysis and control of quadratic systems.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 13, pp. 40-55, February 2005

ELA
www.math.technion.ac.il/iic/ela

REFERENCES

[1] P. Brunovsky. A classification of linear controllable systems. *Kybernetika*, 3:173–187, 1970.
[2] T. Kailath. *Linear Systems*. Prentice-Hall, Englewood Cliffs, New Jersey, 1980.
[3] W. Kang and A. J. Krener. Extended quadratic controller normal form and dynamic state feedback linearization of nonlinear systems. *SIAM Journal on Control and Optimization*, 30:1319–1337, 1992.