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# SPECTRAL PROPERTIES OF SIGN SYMMETRIC MATRICES*

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**Abstract.** Spectral properties of sign symmetric matrices are studied. A criterion for sign symmetry of shifted basic circulant permutation matrices is proven, and is then used to answer the question which complex numbers can serve as eigenvalues of sign symmetric $3 \times 3$ matrices. The results are applied in the discussion of the eigenvalues of $QM$-matrices. In particular, it is shown that for every positive integer $n$ there exists a $QM$-matrix $A$ such that $A^k$ is a sign symmetric $P$-matrix for all $k \leq n$, but not all the eigenvalues of $A$ are positive real numbers.

**Key words.** Spectrum, Sign symmetric matrices, Circulant matrices, Basic circulant permutation matrices, $P$-matrices, $PM$-matrices, $Q$-matrices, $QM$-matrices.

**AMS subject classifications.** 15A18, 15A29.

**1. Introduction.** A square complex matrix $A$ is called a *$Q$-matrix [$Q_0$-matrix]* if the sums of principal minors of $A$ of the same order are positive [nonnegative]. Equivalently, $Q$-matrices can be defined as matrices whose characteristic polynomials have coefficients with alternating signs. A square complex matrix is called a *$P$-matrix [$P_0$-matrix]* if all its principal minors are positive [nonnegative]. A square complex matrix is said to be *stable* if its spectrum lies in the open right half plane.

Let $A$ be an $n \times n$ matrix. For subsets $\alpha$ and $\beta$ of $\{1, ..., n\}$ we denote by $A(\alpha|\beta)$ the submatrix of $A$ with rows indexed by $\alpha$ and columns indexed by $\beta$. If $|\alpha| = |\beta|$ then we denote by $A[\alpha|\beta]$ the corresponding minor. The matrix $A$ is called *sign symmetric* if $A[\alpha|\beta]A[\beta|\alpha] \geq 0$ for all $\alpha, \beta \subset \{1, ..., n\}$ such that $|\alpha| = |\beta|$. The matrix $A$ is called *anti sign symmetric* if $A[\alpha|\beta]A[\beta|\alpha] \leq 0$ for all $\alpha, \beta \subset \{1, ..., n\}$, $\alpha \neq \beta$. A matrix $A$ is called *weakly sign symmetric* if $A[\alpha|\beta]A[\beta|\alpha] \geq 0$ for all $\alpha, \beta \subset \{1, ..., n\}$ such that $|\alpha| = |\beta| = |\alpha \cap \beta| + 1$, that is, if the products of symmetrically located (with respect to the main diagonal) almost principal minors are nonnegative. Note that in some recent papers, the term "sign symmetry" is used for matrices which fulfill the above condition only for minors of size 1, that is, matrices in which $a_{ij}a_{ji} \geq 0$ for all $1 \leq i, j \leq n$, e.g. [2].

The research of the relationship between stability, positivity of principal minors and sign symmetry was motivated by a research problem by Taussky [12] calling for investigation of the common properties of totally positive matrices, nonsingular $M$-matrices and positive definite matrices. Stability, positivity of principal minors and weak sign symmetry are amongst those common properties. This paper deals with spectral properties of general sign symmetric matrices and of sign symmetric matrices having some additional properties.

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A square complex matrix is called a $QM$-matrix if all its powers are $Q$-matrices. A matrix all of whose powers are $P$-matrices is called a $PM$-matrix. A major motivation for our research is a known question of Friedland (see [6]) whether the spectra of $PM$-matrices consist of positive numbers only. This question was answered affirmatively in [6] for matrices of order less than 5, while other cases still remain open. The answer to a similar question, where $PM$-matrices are replaced by $QM$-matrices, is negative, as is demonstrated by the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{1.1}$$

In fact, note that while all powers of $A$ are even sign symmetric Q-matrices, $A$ is not even stable. In view of this example, it is reasonable to ask which additional assumptions can be added for $QM$-matrices such that their eigenvalues are still not necessarily positive numbers. We approach this question using results on $3 \times 3$ circulant matrices, obtained in Sections 2 and 3.

The study of sign symmetric matrices is made quite difficult by the fact that there is no efficient criterion to determine whether a given matrix is sign symmetric or not. Therefore, discussion of certain classes of sign symmetric matrices may be important to the research of spectral properties of sign symmetric matrices in general. In Section 2 we deal with a very special class of shifted basic circulant permutation matrices. We formulate and prove a simple criterion for [anti] sign symmetry of matrices of this class. The $3 \times 3$ sign symmetric matrices of this type serve in the next sections for the characterization of the spectra of sign symmetric $3 \times 3$ matrices and for more general results.

In Section 3 we focus on $3 \times 3$ matrices. First, we use the results of Section 2 in order to give an explicit answer to the question which complex numbers can be eigenvalues of a general sign symmetric $3 \times 3$ matrix. Then we again focus on circulant matrices and analyze the arguments of the complex eigenvalues as a function of the sign of the real eigenvalue. This analysis is used later in Section 4.

In Section 4 we discuss spectra of $PM$-matrices and of $QM$-matrices. We start by examining $3 \times 3$ sign symmetric $Q$-matrices. We use the results of Section 3 in order to determine possible spectra of such matrices, in terms of the arguments of the eigenvalues. Then we use completion results developed in [6] in order to generalize our results to matrices of higher order. Afterwards, we prove that for every positive integer $n$ there exists a $QM$-matrix $A$ such that $A^k$ is a sign symmetric $P$-matrix for all $k \leq n$ but not all the eigenvalues of $A$ are positive real numbers.

The paper is concluded in Section 5 with several open problems.

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ELA
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**2. Sign symmetry of shifted basic circulant permutation matrices.** The research of sign symmetric matrices is made quite difficult by the fact that there is no efficient criterion to determine whether a given matrix is sign symmetric or not. In this section we deal with a very special class of shifted basic circulant permutation matrices. We formulate and prove a simple criterion for [anti] sign symmetry of matrices of this class. The $3 \times 3$ sign symmetric matrices of this type serve in the next sections for the characterization of the spectra of sign symmetric $3 \times 3$ matrices and for more general results.

We start with two definitions and notation.

DEFINITION 2.1. (see [9, p. 26]). Let $n$ be a positive integer. An $n \times n$ matrix of the form

$$C_n = \begin{bmatrix} a_1 & a_2 & a_3 & \cdots & \cdots & a_n \\ a_n & a_1 & a_2 & \cdots & \cdots & a_{n-1} \\ a_{n-1} & a_n & a_1 & \cdots & \cdots & a_{n-2} \\ \vdots & & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & a_2 \\ a_2 & a_3 & a_4 & \cdots & a_n & a_1 \end{bmatrix}$$

is called a *circulant matrix.*

DEFINITION 2.2. (see [9, p. 26]). Let $n$ be a positive integer. The *basic circulant permutation* $n \times n$ *matrix* $C_n$ is defined by

$$(C_n)_{ij} = \begin{cases} 1\,, & j = i+1 \\ 1\,, & i = n,\ j = 1 \\ 0\,, & \text{otherwise,} \end{cases}$$

that is,

$$C_n = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ 1 & 0 & \cdots & \cdots & 0 \end{bmatrix}.$$

Note that the spectrum of $C_n$ consists of the $n$th roots of unity.

NOTATION 2.3. For a positive integer $n$ we denote by $I_n$ the identity matrix of order $n$.

In order to characterize sign symmetry of basic circulant permutation matrices, we prove

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PROPOSITION 2.4. *Let $n$ be a positive integer, and let $\alpha$ and $\beta$ be different nonempty subsets of $\{1,\dots,n\}$ of the same cardinality. The product $C_n[\alpha|\beta]C_n[\beta|\alpha]$ is nonzero if and only if $n$ is even and*

$$\{\alpha,\beta\} = \{\{2,4,\dots,n\},\{1,3,\dots,n-1\}\}. \tag{2.5}$$

*Furthermore, in this case we have $C_n[\alpha|\beta]C_n[\beta|\alpha] = (-1)^{\frac{n}{2}-1}$.*

*Proof.* The product $C_n[\alpha|\beta]C_n[\beta|\alpha]$ is nonzero if and only if

$$C_n[\alpha|\beta] \neq 0 \tag{2.6}$$

and

$$C_n[\beta|\alpha] \neq 0. \tag{2.7}$$

Note that if $i \notin \alpha$ and $i+1 \in \beta$ (where $n+1$ is identified with 1), then $C_n[\alpha|\beta] = 0$. Thus, (2.6) implies that if $i \notin \alpha$, then $i+1 \notin \beta$, which, by (2.7), implies that $i+2 \notin \alpha$. Using repeated argument we obtain

$$\left\{\begin{array}{l} i \notin \alpha \Longrightarrow i+2k \pmod{n} \notin \alpha \\ \\ i \notin \beta \Longrightarrow i+2k \pmod{n} \notin \beta \end{array}\right. \qquad k = 1,2,\dots, \tag{2.8}$$

where $0 \pmod{n}$ is taken as $n$. Since $\alpha$ and $\beta$ are different subsets of $\{1,\dots,n\}$ of the same cardinality, their cardinality is less than $n$. Also, they are nonempty. It thus follows from (2.8) that $n$ is even and (2.5) holds. Furthermore, notice that in this case we have $C_n(1,3,\dots,n-1|2,4,\dots,n) = I_{\frac{n}{2}}$ and $C_n(2,4,\dots,n|1,3,\dots,n-1) = C_{\frac{n}{2}}$, whose determinant is equal to $(-1)^{\frac{n}{2}-1}$. Therefore, the product of the corresponding minors is equal to $(-1)^{\frac{n}{2}-1}$. $\square$

COROLLARY 2.9. *The basic circulant permutation matrix $C_n$ is sign symmetric unless $n = 2k+2$ for some odd positive integer $k$, in which case the matrix $C_n$ is anti sign symmetric.*

REMARK 2.10. Note that by Proposition 2.4, for odd $n$ the matrix $C_n$ is both sign symmetric and anti sign symmetric.

The picture changes if we allow nonzero elements on the main diagonal. In Proposition 2.19 we shall show that for $n > 3$, nonzero scalar shifts of scalar products of basic circulant permutation matrices are all neither sign symmetric nor anti sign symmetric. For $3 \times 3$ matrices we, however, still have

THEOREM 2.11. *Let $x_i, y_i$, $i = 1,2,3$, be real numbers. Then the matrix*

$$A = \begin{bmatrix} x_1 & y_1 & 0 \\ 0 & x_2 & y_2 \\ y_3 & 0 & x_3 \end{bmatrix}$$

*is sign symmetric if and only if $x_j y_1 y_2 y_3 \leq 0$, $j = 1,2,3$, and is anti sign symmetric if and only if $x_j y_1 y_2 y_3 \geq 0$, $j = 1,2,3$.*

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ELA
www.math.technion.ac.il/iic/ela



*Proof.* Since $a_{ij}a_{ji} = 0$ whenever $i \neq j$, all we have to consider are products $A[\alpha|\beta]A[\beta|\alpha]$ when $\alpha$ and $\beta$ are different subsets of $\{1, 2, 3\}$ of cardinality 2. Note that $\alpha \cap \beta = \{k\}$ for some $k \in \{1, 2, 3\}$. It is easy to verify that $A[\alpha|\beta]A[\beta|\alpha] = -x_k y_1 y_2 y_3$ and so our assertion follows. □

As a corollary to Theorem 2.11 we obtain the following characterization of shifted basic circulant permutation $3 \times 3$ matrices.

Corollary 2.12. *Let $x$ and $y$ be real numbers. Then the matrix*

$$A = \begin{bmatrix} x & y & 0 \\ 0 & x & y \\ y & 0 & x \end{bmatrix}$$

*is sign symmetric if and only if $xy \leq 0$ and is anti sign symmetric if and only if $xy \geq 0$.*

Theorem 2.11 and Corollary 2.12 cannot be generalized to matrices of order greater than 3. In order to see it we first prove the following lemma.

Lemma 2.13. *Let $m$ and $n$ be positive integers and let $A$ be the $m \times n$ matrix defined by*

$$a_{ij} = \begin{cases} x\,, & j = i \\ y\,, & j = i+1 \\ 0\,, & \text{otherwise} \end{cases} \qquad i = 1, \ldots, m. \quad j = 1, \ldots, n, \tag{2.14}$$

*where $x, y \neq 0$. Let $\alpha = \{\alpha_1, \ldots, \alpha_k\} \subset \{1, \ldots, m\}$ where $\alpha_1 < \ldots < \alpha_k$ and let $\beta = \{\beta_1, \ldots, \beta_k\} \subset \{1, \ldots, n\}$ where $\beta_1 < \ldots < \beta_k$. The following are equivalent:*
(i) *We have $A[\alpha|\beta] \neq 0$.*
(ii) *We have*

$$\alpha_i \leq \beta_i \leq \alpha_i + 1, \quad i = 1, \ldots, k. \tag{2.15}$$

(iii) *We have $A[\alpha|\beta] = x^p y^{k-p}$, where $p$ is the number of indices $i$ such that $\alpha_i = \beta_i$.*

*Proof.* (i)$\Longrightarrow$(ii). Let $A[\alpha|\beta] \neq 0$. In order to prove (2.15) we first show that

$$\alpha_i \leq \beta_i. \quad i = 1, \ldots, k. \tag{2.16}$$

Assume to the contrary that (2.16) does not hold, and so let $l$ be such that $\beta_l < \alpha_l$. Note that we have $\beta_i < \alpha_j$ whenever $1 \leq i \leq l$ and $l \leq j \leq k$, which, by (2.14), implies that the first $l$ columns of $A(\alpha|\beta)$ may contain nonzero elements only in the first $l - 1$ rows. It thus follows that $A[\alpha|\beta] = 0$. Therefore, $A[\alpha|\beta] \neq 0$ implies (2.16). We now show that

$$\beta_i \leq \alpha_i + 1, \quad i = 1, \ldots, k. \tag{2.17}$$

Assume to the contrary that (2.17) does not hold, and so let $l$ be a positive integer, $l \leq k$, such that $\beta_l > \alpha_l + 1$. Note that we have $\beta_i > \alpha_j + 1$ whenever $l \leq i \leq k$

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ELA
www.math.technion.ac.il/iic/ela



and $1 \le j \le l$, which, by (2.14), implies that the first $l$ rows of $A(\alpha|\beta)$ may contain nonzero elements only in the first $l-1$ columns. It thus follows that $A[\alpha|\beta] = 0$. Therefore, $A[\alpha|\beta] \neq 0$ implies (2.17).

(ii)$\Longrightarrow$(iii). Assume that (2.15) holds and let $p$ be the number of indices $i$ such that $\alpha_i = \beta_i$. We prove that $A[\alpha|\beta] = x^p y^{k-p}$ by induction on $k$. If $k = 1$ then the claim is easy. Assume the claim holds for $k < r$ where $r$ is a positive integer, $r > 1$, and let $k = r$. If $p = 0$ then we have $\beta_i = \alpha_i + 1$, $i = 1, \ldots, k$. It follows that $A(\alpha|\beta)$ is a triangular matrix with $y$'s along the main diagonal, and so indeed $A[\alpha|\beta] = y^k = x^p y^{k-p}$. If $p > 0$ then let $l$ be such that $\alpha_l = \beta_l$. Note that we have $\beta_i < \alpha_j$ whenever $1 \le i \le l$ and $l < j \le k$. Also, $\beta_i < \alpha_l$ whenever $1 \le i < l$. By (2.14), it follows that

$$A[\alpha|\beta] = A[\alpha_1, \ldots, \alpha_{l-1}|\beta_1, \ldots, \beta_{l-1}]\, a_{\alpha_l,\alpha_l}\, A[\alpha_{l+1}, \ldots, \alpha_k|\beta_{l+1}, \ldots, \beta_k].$$

We have $a_{\alpha_l,\alpha_l} = x$, and so our assertion follows by applying the inductive assumption to
$A[\alpha_1, \ldots, \alpha_{l-1}|\beta_1, \ldots, \beta_{l-1}]$ and to $A[\alpha_{l+1}, \ldots, \alpha_k|\beta_{l+1}, \ldots, \beta_k]$.

(iii)$\Longrightarrow$(i) is trivial since $x, y \neq 0$. $\square$

Remark 2.18. Note that it follows from (2.15) that

$$\alpha_1 \le \beta_1 \le \alpha_2 \le \beta_2 \le \ldots \le \alpha_k \le \beta_k.$$

It now follows that Corollary 2.12 cannot be generalized to matrices of order greater than 3.

Proposition 2.19. *Let $n$ be a positive integer greater than 3 and let $A$ be the $n \times n$ matrix*

$$\begin{bmatrix} x & y & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & & \ddots & \ddots & y \\ y & 0 & \cdots & 0 & x \end{bmatrix}, \qquad x, y \in \mathbb{R}, \quad x, y \neq 0. \tag{2.20}$$

*Then $A$ is neither sign symmetric nor anti sign symmetric.*

*Proof.* We have

$$A[2, \ldots, n|1, \ldots, n-1] = (-1)^{n-2}\, y\, A[2, \ldots, n-1] = (-1)^{n-2}\, y\, x^{n-2}$$

and

$$A[1, \ldots, n-1|2, \ldots, n] = y^{n-1},$$

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ELA
www.math.technion.ac.il/iic/ela



and so

$$A[2,\ldots,n|1,\ldots,n-1]A[1,\ldots,n-1|2,\ldots,n] = (-xy)^{n-2}\, y^2. \tag{2.21}$$

We also have

$$A[2,\ldots,n-2,n|1,3,\ldots,n-1] = (-1)^{n-3}\, y\, A[2,\ldots,n-2|3,\ldots,n-1], \tag{2.22}$$

and

$$A[1,3,4,\ldots,n-1|2,\ldots,n-2,n] = \begin{cases} y^2\,, & n=4 \\ y\, A[3,\ldots,n-1|3,\ldots,n-2,n]\,, & n>4. \end{cases} \tag{2.23}$$

Note that $A[2,\ldots,n-2|3,\ldots,n-1]$ is a triangular matrix with diagonal elements all equal to $y$. Therefore, we have

$$A[2,\ldots,n-2|3,\ldots,n-1] = y^{n-3}. \tag{2.24}$$

Also, $A[3,\ldots,n-1|3,\ldots,n-2,n]$ is the minor $B[1,\ldots,n-3|1,\ldots,n-4,n-2]$ of the matrix $B = A(3,\ldots,n)$, which is of the form (2.14). Therefore, by Lemma 2.13 we have

$$A[3,\ldots,n-1|3,\ldots,n-2,n] = x^{n-4}y. \tag{2.25}$$

It now follows from (2.22), (2.23), (2.24) and (2.25) that

$$A[2,\ldots,n-2,n|1,3,4,\ldots,n-1]A[1,3,4,\ldots,n-1|2,\ldots,n-2,n] = \tag{2.26}$$

$$= -(-xy)^{n-4}\, y^4.$$

Note that if $xy > 0$, then for an even number $n$ the product of minors (2.21) is positive and the product of minors (2.26) is negative, while for an odd $n$ the product of minors (2.21) is negative and the product of minors (2.26) is positive. If $xy < 0$, then for every $n$ the product of minors (2.21) is positive and the product of minors (2.26) is negative. Our assertion follows. □

We are now able to prove that Theorem 2.11 cannot be generalized to matrices of order greater than 3.

THEOREM 2.27. *Let $n$ be a positive integer greater than 3 and let $x_i, y_i$, $i = 1,\ldots,n$, be nonzero real numbers such that all the $x$'s share the same sign and and such that $\prod_1^n y_i$ is positive in case $n$ is even. Then the $n\times n$ matrix*

$$A = \begin{bmatrix} x_1 & y_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & & \ddots & \ddots & y_{n-1} \\ y_n & 0 & \cdots & 0 & x_n \end{bmatrix}$$

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ELA
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*is neither sign symmetric nor anti sign symmetric.*

*Proof.* Since sign symmetry and anti sign symmetry of a matrix are invariant under multiplication of the matrix by a positive diagonal matrix, it follows that without loss of generality we may assume that $x_1 = \ldots = x_n$. Since $\prod_1^n y_i$ is positive in case $n$ is even, we can define $r = \sqrt[n]{\prod_1^n y_i}$. Let $D$ be the diagonal matrix $diag(d_1, ..., d_n)$ defined by

$$d_1 = \frac{r}{y_n} \qquad and \qquad d_k = \frac{r^k}{y_n y_1 \cdot \ldots \cdot y_{k-1}}, \;\; k = 2, \ldots, n.$$

It is easy to check that the matrix $D^{-1}AD$ is of the form (2.20). Since sign symmetry and anti sign symmetry of a matrix are invariant under diagonal similarity, our claim follows from Proposition 2.19. □

**3. Spectra of sign symmetric** $3 \times 3$ **matrices.** In the previous section we studied sign symmetry of shifted basic circulant permutation matrices. In particular, in Corollary 2.12 we characterized sign symmetry for all shifted basic circulant permutation $3 \times 3$ matrices. In this section we use results of the previous section in order to study the spectra of general sign symmetric matrices, focusing on $3 \times 3$ matrices. This section continues to lay the basis for the results of the next section on $PM$- and $QM$-matrices.

For a nonzero complex number $\lambda$ we shall assume that $-\pi < arg(\lambda) \leq \pi$. We often use the following result due to Kellogg [10].

THEOREM 3.1. ([10, Corollary 1] *Every eigenvalue* $\lambda$ *of an* $n \times n$ $Q$*-matrix satisfies* $|arg(\lambda)| < \pi - \frac{\pi}{n}$. *Every eigenvalue* $\lambda$ *of an* $n \times n$ $Q_0$*-matrix satisfies* $|arg(\lambda)| \leq \pi - \frac{\pi}{n}$.

For general sign symmetric matrices we have

THEOREM 3.2. *Let* $n$ *be a positive integer and let* $\lambda$ *be a nonzero complex eigenvalue of a sign symmetric* $n \times n$ *matrix. Then*

$$|arg(\lambda)| \leq \frac{(n-1)\pi}{2n} \qquad or \qquad |arg(\lambda)| \geq \frac{(n+1)\pi}{2n}. \tag{3.3}$$

*Proof.* Let $\lambda$ be a nonzero complex eigenvalue of a sign symmetric $n \times n$ matrix $A$. Since $A$ is sign symmetric, it follows by the Cauchy-Binet formula, see, e.g., [4, p. 9], that for every subset $\alpha$ of $\{1, ..., n\}$ we have

$$A^2[\alpha] = \sum_{\beta \subset \{1,\ldots,n\}, \;\; |\beta| = |\alpha|} A[\alpha|\beta]A[\beta|\alpha] \geq 0,$$

and so it follows that $A^2$ is a $P_0$-matrix. By Theorem 3.1, the eigenvalue $\lambda^2$ of $A^2$ satisfies $|arg(\lambda^2)| \leq \frac{(n-1)\pi}{n}$, which implies (3.3). □

We do not know whether the converse of Theorem 3.2 holds, that is, whether for every $\lambda$ satisfying (3.3) there exists a sign symmetric $n \times n$ matrix with $\lambda$ as an eigenvalue. We can, however, prove such a statement for $3 \times 3$ matrices.

Electronic Journal of Linear Algebra ISSN 1081-3810
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ELA
www.math.technion.ac.il/iic/ela



THEOREM 3.4. *Let $\lambda$ be a nonzero complex number. The following are equivalent:*
(i) $\lambda$ *is an eigenvalue of a sign symmetric shifted basic circulant permutation* $3 \times 3$ *matrix.*
(ii) $\lambda$ *is an eigenvalue of a sign symmetric* $3 \times 3$ *matrix.*
(iii) *We have*

$$|arg(\lambda)| \leq \frac{\pi}{3} \qquad or \qquad |arg(\lambda)| \geq \frac{2\pi}{3}. \tag{3.5}$$

*Proof.* (i)$\Longrightarrow$(ii) is trivial.
(ii)$\Longrightarrow$(iii) is proven in Theorem 3.2.
(iii)$\Longrightarrow$(i). Let $\lambda$ be a nonzero complex number satisfying (3.5). If $\lambda$ is real then we can take a real diagonal matrix with $\lambda$ as a diagonal element. Hence, we have to consider only the case that $\lambda$ is non-real. Since a matrix is sign symmetric if and only if so is its negative, without loss of generality we may assume that $|arg(\lambda)| \geq \frac{2\pi}{3}$. Since $\lambda$ is non-real, it follows that for some $b \geq 0$ the complex number $\mu = \lambda + b$ satisfies $|arg(\mu)| = \frac{2\pi}{3}$. Note that $\mu$ is an eigenvalue of the matrix $|\mu|C_3$, and so $\lambda$ is an eigenvalue of the matrix $A = |\mu|C_3 - bI_3$, which, by Theorem 2.11, is sign symmetric. □

REMARK 3.6. One can show, using Corollary 2.12, that for a non-real complex number $\lambda$ satisfying $|arg(\lambda)| \geq \frac{2\pi}{3}$, the matrix $|\lambda + b|C_3 - bI_3$, where $b$ is the non-negative number such that $|arg(\lambda + b)| = \frac{2\pi}{3}$, is the only sign symmetric shifted basic circulant permutation $3 \times 3$ matrix with $\lambda$ as an eigenvalue.

A real $3 \times 3$ matrix has at least one real eigenvalue. Furthermore, a real circulant matrix

$$A = \begin{bmatrix} x & y & z \\ z & x & y \\ y & z & x \end{bmatrix}$$

has just one real eigenvalue $r$ whenever $y \neq z$. We thus now analyze the argument of the complex eigenvalues of a sign symmetric circulant $3 \times 3$ matrix as a function of the sign of the real eigenvalue. In order to state our results we introduce the following notation.

NOTATION 3.7. Let $\lambda$ be a complex number and let $r$ be a real number. It is observed in [7, Lemma 5.11] that $\left\{\lambda, \overline{\lambda}, r\right\}$ is the spectrum of the circulant matrix

$$A = \begin{bmatrix} x & y & z \\ z & x & y \\ y & z & x \end{bmatrix}, \qquad x, y, z \in \mathbb{R}$$

if and only if

$$x = \frac{r + 2Re(\lambda)}{3}, \qquad y, z = \frac{r - Re(\lambda) \pm \sqrt{3}\, Im(\lambda)}{3}.$$

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ELA
www.math.technion.ac.il/iic/ela



Accordingly, we denote by $C(\lambda, r)$ the circulant matrix $\begin{bmatrix} c_1 & c_2 & c_3 \\ c_3 & c_1 & c_2 \\ c_2 & c_3 & c_1 \end{bmatrix}$, where

$$c_1 = \frac{r + 2Re(\lambda)}{3}, \quad c_2 = \frac{r - Re(\lambda) + \sqrt{3}\, Im(\lambda)}{3}, \quad c_3 = \frac{r - Re(\lambda) - \sqrt{3}\, Im(\lambda)}{3}.$$

REMARK 3.8. Note that $C(\lambda, r)$ and $C(\overline{\lambda}, r) = C(\lambda, r)^T$ are the only $3 \times 3$ circulant matrices with spectrum $\{\lambda, \overline{\lambda}, r\}$. Also, since the product of circulant matrices is also a circulant matrix, it follows from the above observed uniqueness that for a positive integer $k$ we have either $C(\lambda^k, r^k) = C(\lambda, r)^k$ or $C(\lambda^k, r^k) = \left(C(\lambda, r)^k\right)^T$.

THEOREM 3.9. *Let $\lambda$ be a non-real complex number. The following are equivalent:*
(i) *There exists a positive number $r$ such that $C(\lambda, r)$ is sign symmetric.*
(ii) *We have*

$$|arg(\lambda)| < \frac{\pi}{6} \qquad or \qquad |arg(\lambda)| \geq \frac{2\pi}{3}. \tag{3.10}$$

*Proof.* The matrix $C(\lambda, r)$ is sign symmetric if and only if $c_2c_3 \geq 0$ and $(c_2^2 - c_1c_3)(c_3^2 - c_1c_2) \geq 0$. Note that $c_2c_3$ is a quadratic polynomial $p(r)$ in $r$ with leading coefficient 1 and with roots $Re(\lambda) \pm \sqrt{3}\, Im(\lambda)$. Without loss of generality we may assume that $Im(\lambda) > 0$, and so it follows that

$$p(r) \geq 0 \quad \Longleftrightarrow \quad r \geq Re(\lambda) + \sqrt{3}\, Im(\lambda) \quad or \quad r \leq Re(\lambda) - \sqrt{3}\, Im(\lambda). \tag{3.11}$$

The expression $(c_2^2 - c_1c_3)(c_3^2 - c_1c_2)$ is a polynomial $q(r)$ in $r$. If $Re(\lambda)^2 = 3Im(\lambda)^2$, then

$$q(r) = -\frac{24Re(\lambda)^3}{81} r + \frac{16Re(\lambda)^4}{81}.$$

If $Re(\lambda)^2 \neq 3Im(\lambda)^2$, then $q(r)$ is a quadratic polynomial in $r$ with leading coefficient $\frac{Re(\lambda)^2 - 3Im(\lambda)^2}{9}$ and with roots

$$\tilde{r}_1 = \frac{Re(\lambda)^2 + Im(\lambda)^2}{\sqrt{3}\, Im(\lambda) + Re(\lambda)}, \qquad \tilde{r}_2 = -\frac{Re(\lambda)^2 + Im(\lambda)^2}{\sqrt{3}\, Im(\lambda) - Re(\lambda)}.$$

Statement (i) of our theorem is equivalent to the solvability of the system

$$\begin{cases} p(r) \geq 0 \\ q(r) \geq 0 \\ r > 0. \end{cases} \tag{3.12}$$

We distinguish between four cases:

1. $\underline{Re(\lambda)^2 > 3Im(\lambda)^2}$. This is the case in which

Electronic Journal of Linear Algebra ISSN 1081-3810
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Volume 13, pp. 90-110, March 2005

ELA
www.math.technion.ac.il/iic/ela



$$|arg(\lambda)| < \frac{\pi}{6} \qquad or \qquad |arg(\lambda)| > \frac{5\pi}{6}. \tag{3.13}$$

In this case the leading coefficients of both quadratic polynomials $p(r)$ and $q(r)$ are positive, so clearly the system (3.12) is consistent for sufficiently large positive $r$.

2. $Re(\lambda)^2 = 3Im(\lambda)^2, \quad Re(\lambda) < 0$. This is the case in which

$$|arg(\lambda)| = \frac{5\pi}{6}. \tag{3.14}$$

In this case $p(r) \geq 0$ and $q(r) \geq 0$ whenever $r \geq 0$, and so any $r > 0$ satisfies (3.12).

3. $Re(\lambda)^2 = 3Im(\lambda)^2, \quad Re(\lambda) > 0$. This is the case in which

$$|arg(\lambda)| = \frac{\pi}{6}.$$

In this case $p(r) \geq 0$ whenever $r \geq 2Re(\lambda)$ or $r \leq 0$, and $q(r) \geq 0$ whenever $r \leq \frac{2Re(\lambda)}{3}$. Since $r > 0$, and since in this case we have $Re(\lambda) > 0$, we obtain the contradiction $2Re(\lambda) \leq r \leq \frac{2Re(\lambda)}{3}$.

4. $Re(\lambda)^2 < 3Im(\lambda)^2$. This is the case in which

$$\frac{\pi}{6} < |arg(\lambda)| < \frac{5\pi}{6}. \tag{3.15}$$

In this case the leading coefficient of $q(r)$ is negative and its two roots satisfy $\tilde{r}_1 > 0 > \tilde{r}_2$. Therefore, we have

$$q(r) \geq 0 \iff -\frac{Re(\lambda)^2 + Im(\lambda)^2}{\sqrt{3}\, Im(\lambda) - Re(\lambda)} \leq r \leq \frac{Re(\lambda)^2 + Im(\lambda)^2}{\sqrt{3}\, Im(\lambda) + Re(\lambda)}. \tag{3.16}$$

Under the condition $r > 0$, both (3.11) and (3.16) hold if and only if

$$Re(\lambda) + \sqrt{3}\, Im(\lambda) \leq r \leq \frac{Re(\lambda)^2 + Im(\lambda)^2}{\sqrt{3}\, Im(\lambda) + Re(\lambda)},$$

which is solvable for $r$ if and only if $-Re(\lambda) \geq \frac{Im(\lambda)}{\sqrt{3}}$, that is, $|arg(\lambda)| \geq \frac{2\pi}{3}$. Therefore, in view of (3.15), in this case the system (3.12) is solvable if and only if

$$\frac{2\pi}{3} \leq |arg(\lambda)| < \frac{5\pi}{6}. \tag{3.17}$$

It follows that the system (3.12) is solvable if and only if we have (3.13), (3.14) or (3.17), which together give (3.10). □

REMARK 3.18. In Theorem 3.4 we showed that eigenvalues of a general sign symmetric $3 \times 3$ matrix satisfy (3.5). We then showed in Theorem 3.9 that the

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 13, pp. 90-110, March 2005





requirement that the real eigenvalue of the circulant matrix be positive yields the reduction of the allowed domain for eigenvalues to (3.10). Another interesting case of sign symmetric $3\times3$ matrices whose eigenvalues satisfy $|arg(\lambda)| < \frac{\pi}{6}$ is of $M$-matrices, that is, matrices of the form $\alpha I - B$ where $B$ is an entrywise nonnegative matrix, and where $\alpha$ is greater than or equal to the spectral radius of $B$. It was proven by Ostrowski [11] that if $A$ is an $M$-matrix then every principal submatrix of $A$ is also an $M$-matrix, and also each element of $Adj(A)$ is nonnegative. This implies that all $M$-matrices are weakly sign symmetric, a property that coincides with sign symmetry for $3 \times 3$ matrices. Also, it is known that any eigenvalue $\lambda$ of an $n \times n$ $M$-matrix satisfies $|arg(\lambda - l(A))| < \frac{\pi}{2} - \frac{\pi}{n}$, where $l(A)$ is the minimal real eigenvalue of $A$, e.g. [10, Theorem 1]. It thus follows that the eigenvalues of a $3 \times 3$ $M$-matrix satisfy $|arg(\lambda)| < \frac{\pi}{6}$.

Remark 3.19. In view of Theorem 3.4, it is just natural to ask whether the statements in Theorem 3.9 are also equivalent to statement: *"There exists a sign symmetric shifted basic circulant permutation* $3 \times 3$ *matrix with eigenvalue* $\lambda$ *and a positive eigenvalue"*. The answer to this question is negative, as the latter statement is equivalent to

$$|arg(\lambda)| < \frac{\pi}{6} \qquad or \qquad \frac{2\pi}{3} \leq |arg(\lambda)| < \frac{5\pi}{6}.$$

To see it, note that in view of Corollary 2.12, the matrix $xI_3 + yC_3$, $x, y \in \mathbb{R}$, is a sign symmetric matrix with a positive eigenvalue if and only if $x \leq 0$ and $y > |x|$, or $x > 0$ and $0 \geq y > -x$.

Since $C(\lambda, r)$ is sign symmetric if and only if $-C(\lambda, r) = C(-\lambda, -r)$ is sign symmetric, the following claim follows immediately from Theorem 3.9.

Corollary 3.20. *Let* $\lambda$ *be a non-real complex number. The following are equivalent:*
(i) *There exists a negative number* $r$ *such that* $C(\lambda, r)$ *is sign symmetric.*
(ii) *We have*

$$|arg(\lambda)| \leq \frac{\pi}{3} \qquad or \qquad |arg(\lambda)| > \frac{5\pi}{6}.$$

For the sake of completeness we add here

Theorem 3.21. *Let* $\lambda$ *be a non-real complex number. Then* $C(\lambda, 0)$ *is sign symmetric if and only if we have*

$$|arg(\lambda)| \leq \frac{\pi}{6} \qquad or \qquad |arg(\lambda)| \geq \frac{5\pi}{6}.$$

*Proof.* In this case we have $c_1 = \frac{2Re(\lambda)}{3}$, $c_2 = \frac{-Re(\lambda)+\sqrt{3}\, Im(\lambda)}{3}$ and $c_3 = \frac{-Re(\lambda)-\sqrt{3}\, Im(\lambda)}{3}$. The matrix $C(\lambda, 0)$ is sign symmetric if and only if $c_2c_3 = Re(\lambda)^2 -$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 13, pp. 90-110, March 2005





$3Im(\lambda)^2 \geq 0$ and $(c_2^2 - c_1c_3)(c_3^2 - c_1c_2) = \left(3Re(\lambda)^2 + 3Im(\lambda)^2\right)^2 \geq 0$. The assertion follows. □

Furthermore, by following the proof of Theorem 3.9 we can deduce the following result, to be used in this paper at a later stage.

THEOREM 3.22. *Let $\lambda$ be a non-real complex number. The following are equivalent:*
(i) *There exists a nonnegative number $R$ such that for every $r > R$ the matrix $C(\lambda, r)$ is sign symmetric.*
(ii) *We have*

$$|arg(\lambda)| < \frac{\pi}{6} \qquad or \qquad |arg(\lambda)| \geq \frac{5\pi}{6}.$$

(iii) *We have*

$$|arg(\lambda)| < \frac{\pi}{6} \qquad or \qquad |arg(-\lambda)| \leq \frac{\pi}{6}.$$

*Proof.* (i)$\Longrightarrow$(ii). It is shown in the proof of Theorem 3.9 that if $|arg(\lambda)| = \frac{\pi}{6}$ then there exists no positive $r$ such that $C(\lambda, r)$ is sign symmetric (see Case 3 there), and that if $\frac{\pi}{6} < |arg(\lambda)| < \frac{5\pi}{6}$ then there exists no $r$, $r > \frac{Re(\lambda)^2 + Im(\lambda)^2}{\sqrt{3}\, Im(\lambda) + Re(\lambda)}$, such that $C(\lambda, r)$ is sign symmetric (see Case 4 there). The implication follows.

(ii)$\Longrightarrow$(i). It is shown in the proof of Theorem 3.9 that if $|arg(\lambda)| < \frac{\pi}{6}$ or $|arg(\lambda)| > \frac{5\pi}{6}$ then for $r$ sufficiently large the matrix $C(\lambda, r)$ is sign symmetric (see Case 1 there), and that if $|arg(\lambda)| = \frac{5\pi}{6}$ then $C(\lambda, r)$ is sign symmetric for every $r$, $r > 0$ (see Case 2 there). The implication follows.

(ii)$\Longleftrightarrow$(iii) is clear. □

By applying Theorem 3.22 to the matrix $-C(\lambda, r)$ we obtain

THEOREM 3.23. *Let $\lambda$ be a non-real complex number. The following are equivalent:*
(i) *There exists a nonpositive number $R$ such that for every $r < R$ the matrix $C(\lambda, r)$ is sign symmetric.*
(ii) *We have*

$$|arg(\lambda)| \leq \frac{\pi}{6} \qquad or \qquad |arg(\lambda)| > \frac{5\pi}{6}.$$

(iii) *We have*

$$|arg(\lambda)| \leq \frac{\pi}{6} \qquad or \qquad |arg(-\lambda)| < \frac{\pi}{6}.$$

Electronic Journal of Linear Algebra ISSN 1081-3810
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ELA
www.math.technion.ac.il/iic/ela



**4. Spectra of sign symmetric $Q$-matrices.** In [6] it is asked whether the spectrum of any $PM$-matrix consists of positive numbers only. This question was answered affirmatively in [6] for matrices of order less than 5, while other cases still remain open. In the introduction we gave an example showing that the answer to a similar question, where $PM$-matrices are replaced by $QM$-matrices, is negative. In view of that example, it is reasonable to ask which additional assumptions can be added for $QM$-matrices such that their eigenvalues are still not necessarily positive numbers. The discussion of this question is the main aim of this section. We start by examining $3\times 3$ sign symmetric $Q$-matrices. We use the results of the previous section in order to determine possible spectra of such matrices, in terms of the arguments of the eigenvalues. Then we use completion results developed in [6] in order to generalize our results to general order. Afterwards, we prove that for every positive integer $n$ there exists a $QM$-matrix $A$ such that $A^k$ is a sign symmetric $P$-matrix for all $k \le n$ but not all the eigenvalues of $A$ are positive real numbers.

We start with a combination of Theorems 3.1 and 3.2.

Corollary 4.1. *Let $\lambda$ be a nonzero complex eigenvalue of a sign symmetric $n \times n$ $Q$-matrix. Then*

$$|arg(\lambda)| \le \frac{(n-1)\pi}{2n}$$

*or*

$$\frac{(n+1)\pi}{2n} \le |arg(\lambda)| < \frac{(n-1)\pi}{n}. \tag{4.2}$$

Note that for $n = 3$ the inequality (4.2) is impossible. Therefore, we have

Corollary 4.3. *Let $\lambda$ be a nonzero complex eigenvalue of a sign symmetric $3 \times 3$ $Q$-matrix. Then*

$$|arg(\lambda)| \le \frac{\pi}{3}.$$

The converse of Corollary 4.3 does not necessarily hold, that is, it is not clear that for every choice of a nonzero complex number $\lambda$ such that $|arg(\lambda)| \le \frac{\pi}{3}$ there exists a sign symmetric $3 \times 3$ $Q$-matrix with $\lambda$ as an eigenvalue. In fact, if we restrict ourselves to circulant matrices, then we have the following.

Theorem 4.4. *Let $\lambda$ be a nonzero complex number. The following are equivalent:*
(i) *$\lambda$ is an eigenvalue of a sign symmetric $3 \times 3$ $Q$-matrix of the form*

$$\begin{bmatrix} x & y & 0 \\ 0 & x & y \\ y & 0 & x \end{bmatrix}, \qquad x, y \in \mathbb{R}.$$

Electronic Journal of Linear Algebra ISSN 1081-3810
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ELA
www.math.technion.ac.il/iic/ela



(ii) $\lambda$ *is an eigenvalue of a sign symmetric* $3 \times 3$ *circulant* $Q$*-matrix*

$$\begin{bmatrix} x & y & z \\ z & x & y \\ y & z & x \end{bmatrix}, \qquad x, y, z \in \mathbb{R}. \tag{4.5}$$

(iii) *We have*

$$|arg(\lambda)| < \frac{\pi}{6}. \tag{4.6}$$

*Proof.* (i)$\Longrightarrow$(ii) is trivial.

(ii)$\Longrightarrow$(iii). Let $\lambda$ be an eigenvalue of a sign symmetric $3 \times 3$ matrix $A$ of the form (4.5). If $\lambda$ is real then, since $A$ is a $Q$-matrix, it follows by Theorem 3.1 that $\lambda > 0$ and (4.6) follows. If $\lambda$ is non-real then $A$ has another eigenvalue $r$ which is real. Furthermore, by Theorem 3.1 we have $r > 0$. Our claim now follows from Theorem 3.9 and Corollary 4.3.

(iii)$\Longrightarrow$(i). Assume that (4.6) holds. If $\lambda$ is real, then the matrix $\lambda I_3$ satisfies our requirements. If $\lambda$ is non-real, then let $b > 0$ be such that $|arg(\lambda - b)| = \frac{\pi}{6}$. Since the spectrum of $I_3 - C_3$ is $\left\{0, \sqrt{3}\, e^{i\frac{\pi}{6}}, \sqrt{3}\, e^{-i\frac{\pi}{6}}\right\}$, it follows that $\lambda$ is an eigenvalue of the $Q$-matrix $\frac{|\lambda - b|}{\sqrt{3}}(I_3 - C_3) + bI_3 = \left(\frac{|\lambda-b|}{\sqrt{3}} + b\right) I_3 - \frac{|\lambda-b|}{\sqrt{3}} C_3$, which, by Corollary 2.12, is sign symmetric. □

In the sequel we use the following completion result from [6].

Proposition 4.7. ([6, Proposition 1]) *Let $z$ be a non-real complex number with a negative real part. Then the set* $\left\{ z, \overline{z}, \underbrace{|z|, \ldots, |z|}_{m} \right\}$ *is a spectrum of a $Q$-matrix (that is, the set has positive elementary symmetric functions) whenever* $m > \frac{2\,|Re(z)|}{|z| - |Re(z)|}$.

The following is an interesting application of Theorem 3.9 and Proposition 4.7.

Theorem 4.8. *Let $\lambda$ be a nonzero complex number. The following are equivalent:*
(i) *There exists a positive number $r$ and a nonnegative number $M$ such for every nonnegative integer $m$, $m \geq M$, the matrix* $C(\lambda, r) \oplus diag(\underbrace{|\lambda|, \ldots, |\lambda|}_{m})$ *is a sign symmetric $Q$-matrix.*
(ii) *We have*

$$|arg(\lambda)| < \frac{\pi}{6} \qquad \textit{or} \qquad \pi > |arg(\lambda)| \geq \frac{2\pi}{3}.$$

*Proof.* (i)$\Longrightarrow$(ii). The sign symmetry of $C(\lambda, r) \oplus diag(\underbrace{|\lambda|, \ldots, |\lambda|}_{m})$ yields the sign symmetry of $C(\lambda, r)$, and so by Theorem 3.9 we have $|arg(\lambda)| < \frac{\pi}{6}$ or $|arg(\lambda)| \geq \frac{2\pi}{3}$.

Electronic Journal of Linear Algebra ISSN 1081-3810
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Volume 13, pp. 90-110, March 2005

ELA
www.math.technion.ac.il/iic/ela



Since $\lambda$ is a nonzero eigenvalue of a $Q$-matrix, it follows from Theorem 3.1 that we also have $\pi > |arg(\lambda)|$.

(ii)$\Longrightarrow$(i). By Theorem 3.9 there exists a positive number $r$ such that $C(\lambda, r)$ is a sign symmetric matrix. If $|arg(\lambda)| < \frac{\pi}{6}$, then $C(\lambda, r)$ is also a $Q$-matrix, and the implication follows with $M = 0$. If $\pi > |arg(\lambda)| \geq \frac{2\pi}{3}$ then, by Proposition 4.7, the set $\left\{\lambda, \overline{\lambda}, r\right\}$ can be completed to be a spectrum of a $Q$-matrix by adding to it $m$ copies of $|\lambda|$, where $m > M = \frac{2\,|Re(\lambda)|}{|\lambda| - |Re(\lambda)|}$. The implication follows. ◻

Remark 4.9. It follows from Theorem 4.8 that a sign-symmetric $Q$-matrix is not necessarily stable. This shows a difference between $P$-matrices and $Q$-matrices, since sign symmetric $P$-matrices are proven by Carlson [1] to be stable. In fact, (1.1) is an example of a matrix which is not stable although all of its powers are sign-symmetric $Q$-matrices.

If in Theorem 4.8 we replace "$Q$-matrix" by "$QM$-matrix", we obtain the following.

Theorem 4.10. *Let $\lambda$ be a nonzero complex number. The following are equivalent:*
(i) *There exists a positive number $r$ and a nonnegative number $M$ such for every nonnegative integer $m$, $m \geq M$, the matrix $C(\lambda, r) \oplus diag(\underbrace{|\lambda|, \ldots, |\lambda|}_{m})$ is a sign symmetric $QM$-matrix.*
(ii) *$\lambda$ is an odd root of a positive number, satisfying*

$$|arg(\lambda)| < \frac{\pi}{6} \qquad or \qquad \pi > |arg(\lambda)| \geq \frac{2\pi}{3}.$$

*Proof.* (i)$\Longrightarrow$(ii). In view of the corresponding implication in Theorem 4.8, all we have to show is that $\lambda$ is an odd root of a positive number. Since, by Theorem 3.1, every eigenvalue $\mu$ of an $n \times n$ $Q$-matrix $A$ satisfies $|arg(\mu)| \leq \frac{(n-1)\pi}{n}$, and since by Kronecker's theorem, e.g. [5, Theorem 4.38, p. 375], every complex number whose argument is an irrational multiple of $\pi$ has some power with argument close to $\pi$ as much as we wish, it follows that every eigenvalue $\lambda$ of a $QM$-matrix (of any order) has an argument which is a rational multiple of $\pi$. Furthermore, $\lambda$ cannot be an even root of a positive number, since then it would have a power which is a negative number. It thus follows that $\lambda$ must be an odd root of a positive number.

(ii)$\Longrightarrow$(i). By Theorem 3.9 there exists a positive number $r$ such that $C(\lambda, r)$ is a sign symmetric matrix. Since $\lambda$ is an odd root of a positive number, it follows that all powers of $\lambda$ are either positive numbers or non-real complex numbers. By Proposition 4.7, the set $\left\{\lambda, \overline{\lambda}, r, \underbrace{|\lambda|, \ldots, |\lambda|}_{m}\right\}$ is a spectrum of a $QM$-matrix whenever

$$m > M = \max_{k=1,\ldots,s-1} \frac{2\,|Re(\lambda^k)|}{|\lambda|^k - |Re(\lambda^k)|},$$

Electronic Journal of Linear Algebra ISSN 1081-3810
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Volume 13, pp. 90-110, March 2005

ELA
www.math.technion.ac.il/iic/ela



where $s$ is the smallest positive integer such that $\lambda^s$ is a positive number. The assertion follows. $\square$

Our aim now is to show that for every positive integer $n$ there exists a sign symmetric $QM$-matrix $A$ such that $A^k$ is a sign symmetric $P$-matrix for all $k \leq n$ but not all the eigenvalues of $A$ are positive real numbers. We use the following corollary to Theorem 3.22.

THEOREM 4.11. *Let $\lambda$ be a non-real complex number, and let $n$ be a positive integer. The following are equivalent:*
(i) *There exists a nonnegative number $R$ such that for every $r > R$ the matrices $C(\lambda, r)^k$, $k = 1, \dots, n$, are all sign symmetric.*
(ii) *We have*

$$|arg(\lambda)| < \frac{\pi}{6n} \qquad or \qquad |arg(-\lambda)| \leq \frac{\pi}{6n}. \tag{4.12}$$

*Proof.* (i)$\Longrightarrow$(ii). By Theorem 3.22 we have

$$|arg(\lambda)| < \frac{\pi}{6} \qquad or \qquad |arg(-\lambda)| \leq \frac{\pi}{6}.$$

Assume first that

$$|arg(\lambda)| < \frac{\pi}{6}.$$

Without loss of generality we may assume that $arg(\lambda) \geq 0$, and so

$$0 \leq arg(\lambda) < \frac{\pi}{6}. \tag{4.13}$$

If $n = 1$, then there is nothing to prove. So, we assume that $n > 1$ and we shall show that

$$arg(\lambda) < \frac{\pi}{6n}. \tag{4.14}$$

Assume to the contrary that

$$arg(\lambda) \geq \frac{\pi}{6n}. \tag{4.15}$$

In view of (4.13) and (4.15), let $k$ be the minimal positive integer, $1 \leq k < n$, such that

$$\frac{\pi}{6(k+1)} \leq arg(\lambda) < \frac{\pi}{6k}. \tag{4.16}$$

The eigenvalue $\lambda^{k+1}$ of the matrix $C(\lambda, r)^{k+1}$ thus satisfies

$$\frac{\pi}{6} \leq (k+1)arg(\lambda) = arg(\lambda^{k+1}).$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 13, pp. 90-110, March 2005

ELA
www.math.technion.ac.il/iic/ela



Since the matrix $C(\lambda, r)^{k+1}$ is sign symmetric for every $r > R$, it follows from Theorem 3.22 that we necessarily have

$$(k+1)arg(\lambda) = arg(\lambda^{k+1}) \geq \frac{5\pi}{6}. \tag{4.17}$$

Note that by (4.16) we have

$$k\, arg(\lambda) < \frac{\pi}{6}. \tag{4.18}$$

It now follows from (4.17) and (4.18) that $arg(\lambda) > \frac{2\pi}{3}$, in contradiction to (4.13). Our assumption that (4.15) holds is thus false, and so we have (4.14).

Assume now that

$$|arg(-\lambda)| \leq \frac{\pi}{6}.$$

Without loss of generality we may assume that $arg(-\lambda) \geq 0$, and so

$$0 \leq arg(-\lambda) \leq \frac{\pi}{6}. \tag{4.19}$$

If $n = 1$ then there is nothing to prove. So, we assume that $n > 1$ and we shall show in a very similar manner to what we did above that

$$arg(-\lambda) \leq \frac{\pi}{6n}. \tag{4.20}$$

Assume to the contrary that

$$arg(-\lambda) > \frac{\pi}{6n}. \tag{4.21}$$

In view of (4.19) and (4.21), let $k$ be the minimal positive integer, $1 \leq k < n$, such that

$$\frac{\pi}{6(k+1)} < arg(-\lambda) \leq \frac{\pi}{6k}. \tag{4.22}$$

The eigenvalue $(-\lambda)^{k+1}$ of the matrix $(-C(\lambda, r))^{k+1}$ thus satisfies

$$\frac{\pi}{6} < (k+1)arg(-\lambda) = arg((-\lambda)^{k+1}).$$

As is observed in Remark 3.8, the matrix $(-C(\lambda, r))^{k+1}$ is either $C((-\lambda)^{k+1}, (-r)^{k+1})$ or $\left(C((-\lambda)^{k+1}, (-r)^{k+1})\right)^T$. Since it is sign symmetric for every $r > R$, it follows from Theorem 3.23 that we necessarily have

$$(k+1)arg(-\lambda) = arg((-\lambda)^{k+1}) > \frac{5\pi}{6}. \tag{4.23}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 13, pp. 90-110, March 2005

ELA
www.math.technion.ac.il/iic/ela



Note that by (4.22) we have

$$k\,arg(-\lambda) \leq \frac{\pi}{6}. \tag{4.24}$$

It now follows from (4.23) and (4.24) that $arg(-\lambda) > \frac{2\pi}{3}$, in contradiction to (4.19). Our assumption that (4.21) holds is thus false, and so we have (4.20).

(ii)$\Longrightarrow$(i) follows immediately by Theorem 3.22. □

LEMMA 4.25. *A $3 \times 3$ circulant matrix is a $P$-matrix if and only if it is a $Q$-matrix.*

*Proof.* The principal minors of a $3 \times 3$ circulant matrix of the same order are all equal. Therefore, their sum is positive if and only if each one is positive. □

THEOREM 4.26. *Let $\lambda$ be a nonzero complex number and let $n$ be a positive integer. The following are equivalent:*
(i) *There exist nonnegative numbers $R$ and $M$ such that for every $r > R$ and for every nonnegative integer $m$, $m \geq M$, the matrix $A = C(\lambda, r) \oplus diag(\underbrace{|\lambda|, \ldots, |\lambda|}_{m})$ is a $QM$-matrix. Furthermore, the matrices $A^k$ are sign symmetric $P$-matrices for $k = 1, \ldots, n$.*
(ii) *$\lambda$ is an odd root of a positive number, satisfying $|arg(\lambda)| < \frac{\pi}{6n}$.*

*Proof.* (i)$\Longrightarrow$(ii). Note that if $A^k$ is sign symmetric, then $(C(\lambda, r))^k$ is sign symmetric. By Theorem 4.11 we have (4.12). Since $\lambda$ is an eigenvalue of the $P$-matrix $A$, it follows by Theorem 3.1 that $|arg(-\lambda)| > \frac{\pi}{n}$, and together with (4.12) we have $|arg(\lambda)| < \frac{\pi}{6n}$. The fact that $\lambda$ is an odd root of a positive number follows from Theorem 4.10.

(ii)$\Longrightarrow$(i). By Theorem 4.11, there exists a nonnegative number $R$ such that for every $r > R$ the matrices $C(\lambda, r)^k$, $k = 1, \ldots, n$, are all sign symmetric. Note that the matrices $C(\lambda, r)^k$, $k = 1, \ldots, n$, are also $Q$-matrices, since their eigenvalues are the positive number $r^k$ and the conjugate pair $\lambda^k$, $\overline{\lambda}^k$, where $|arg(\lambda^k)| < \frac{k\pi}{6n} \leq \frac{\pi}{6}$. By Lemma 4.25 it follows that these matrices are $P$-matrices as well. By Proposition 4.7, the set $\left\{\lambda, \overline{\lambda}, r, \underbrace{|\lambda|, \ldots, |\lambda|}_{m}\right\}$ is a spectrum of a $QM$-matrix whenever

$$m > M = \max_{k=1,\ldots,s-1} \frac{2\,|Re(\lambda^k)|}{|\lambda|^k - |Re(\lambda^k)|},$$

where $s$ is the smallest positive integer such that $\lambda^s$ is a positive number. The proof of the implication is thus complete. □

**5. Open problems.** The class of sign symmetric matrices has not been studied extensively. In this section we outline some fundamental questions regarding this class to be investigated. We mainly focus on problems related to the results presented in the previous sections.

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The first problem refers to efficient characterization of sign symmetric matrices. A question that can be raised about sign symmetric matrices is to find some "nice" criterion to determine whether a matrix belongs to this class. In [7] the following theorem was proven: *A matrix $A$ is sign symmetric if and only if for every positive diagonal matrix $D$, the matrix $(DA)^2$ is a $P_0$-matrix.* This criterion is, however, very hard to check, especially if the matrix is not sparse. Thus, we pose

PROBLEM 5.1. Find a "nice" efficient criterion to determine whether a given matrix is sign symmetric.

In view of the study in Section 2, in which we characterized those complex numbers that can serve as eigenvalues of sign symmetric $3 \times 3$ matrices, we pose

PROBLEM 5.2. For a general positive integer $n$, characterize those complex numbers that serve as eigenvalues of sign symmetric $n \times n$ matrices.

Similar questions can be asked regarding sign symmetric matrices having additional properties. For example, one may ask

PROBLEM 5.3. Characterize those complex numbers that serve as eigenvalues of sign symmetric $P$-matrices (or $Q$-matrices, or $PM$-matrices, or $QM$-matrices).

Note that in the case of P-matrices, Carlson [1] proved that an eigenvalue $\lambda$ of a sign symmetric $P$-matrix satisfies $|arg(\lambda)| < \frac{\pi}{2}$. We even showed in Corollary 4.1 that

$$|arg(\lambda)| \leq \frac{(n-1)\pi}{2n}. \tag{5.4}$$

However, even in the case $n = 3$ we do not know whether every number $\lambda$ satisfying (5.4) belongs to the spectrum of some sign symmetric $n \times n$ P-matrix.

Another question related to our results is the following. In Section 2 we formulated and proved a simple criterion for (anti-) sign symmetry of shifted basic circulant permutation matrices, which form a subclass of the greater class of circulant matrices. In Theorem 4.4 we showed that for matrices of order 3, belonging to the spectrum of a circulant sign symmetric $Q$-matrix is equivalent to being in the spectrum of some shifted basic circulant permutation sign symmetric $Q$-matrix. This leads us to ask whether our results in Section 2 can be generalized to general circulant matrices. We pose

QUESTION 5.5. What is the relation, in general, between the spectra of circulant matrices and the spectra of shifted basic circulant permutation matrices?

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