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SPECTRAL GRAPH THEORY AND THE INVERSE EIGENVALUE PROBLEM OF A GRAPH*

LESLIE HOGBEN†

Abstract. Spectral Graph Theory is the study of the spectra of certain matrices defined from a given graph, including the adjacency matrix, the Laplacian matrix and other related matrices. Graph spectra have been studied extensively for more than fifty years. In the last fifteen years, interest has developed in the study of generalized Laplacian matrices of a graph, that is, real symmetric matrices with negative off-diagonal entries in the positions described by the edges of the graph (and zero in every other off-diagonal position).

The set of all real symmetric matrices having nonzero off-diagonal entries exactly where the graph G has edges is denoted by $\mathcal{S}(G)$. Given a graph G, the problem of characterizing the possible spectra of B, such that $B \in \mathcal{S}(G)$, has been referred to as the *Inverse Eigenvalue Problem of a Graph.* In the last fifteen years a number of papers on this problem have appeared, primarily concerning trees.

The adjacency matrix and Laplacian matrix of G and their normalized forms are all in $\mathcal{S}(G)$. Recent work on generalized Laplacians and Colin de Verdière matrices is bringing the two areas closer together. This paper surveys results in Spectral Graph Theory and the Inverse Eigenvalue Problem of a Graph, examines the connections between these problems, and presents some new results on construction of a matrix of minimum rank for a given graph having a special form such as a 0,1-matrix or a generalized Laplacian.

Key words. Spectral Graph Theory, Minimum Rank, Generalized Laplacian, Inverse Eigenvalue Problem, Colin de Verdière matrix.

AMS subject classifications. 05C50, 15A18.

1. Spectral Graph Theory. Spectral Graph Theory has traditionally used the spectra of specific matrices associated with the graph, such as the adjacency matrix, the Laplacian matrix, or their normalized forms, to provide information about the graph. For certain families of graphs it is possible to characterize a graph by the spectrum (of one of these matrices). More generally, this is not possible, but useful information about the graph can be obtained from the spectra of these various matrices. There are also important applications to other fields such as chemistry. Here we present only a very brief introduction to this extensive subject. The reader is referred to several books, such as [10], [9], [11], [6], for a more thorough discussion and lists of references to original papers.

We begin by defining terminology and introducing notation. Throughout this discussion, all matrices will be real and symmetric. The *ordered spectrum* (the list of eigenvalues, repeated according to multiplicity in nondecreasing order) of an $n \times n$ matrix B will be denote denoted $\sigma(B) = (\beta_1, \ldots, \beta_n)$ with $\beta_1 \leq \ldots \leq \beta_n$.

A *graph* G means a simple undirected graph (no loops, no multiple edges), whose vertices are positive integers. The *order* of G is the number of vertices. The *degree* of vertex k, $\deg_G k$, is the number of edges incident with k. The graph G is *regular* of degree r if every vertex has degree r. The graph $G - v$ is the result of deleting

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†Dept of Mathematics, Iowa State University, Ames, IA 50011, USA (lhogben@iastate.edu).

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vertex v and all its incident edges from G. If S is a subset of the vertices of G, the *subgraph induced by* S is the result of deleting the complement of S from G, i.e., $\langle S\rangle = G - \bar{S}$. A *tree* is a connected graph with no cycles. We usually restrict our attention to connected graphs, because each connected component can be analyzed separately.

- P_n is a path on n vertices.
- C_n is a cycle on n vertices.
- K_n is the complete graph on n vertices.
- $K_{1,n}$ is a star on $n{+}1$ vertices, i.e., a complete bipartite graph on sets of 1 and n vertices.
- W_{n+1} is a wheel on $n{+}1$ vertices, i.e., a graph obtained by joining one additional vertex to every vertex of C_n.

Let G be a graph with vertices $\{1, \ldots, n\}$. We will discuss the following matrices associated with G.

- The *adjacency matrix*, $\mathcal{A} = [a_{ij}]$, where $a_{ij} = 1$ if $\{i, j\}$ is an edge of G and $a_{ij} = 0$ otherwise. Let $\sigma(\mathcal{A}) = (\alpha_1, \ldots, \alpha_n)$.
- The *diagonal degree matrix*, $\mathcal{D} = \mathrm{diag}(\deg_G 1, \ldots, \deg_G n)$.
- The *normalized adjacency matrix*, $\widehat{\mathcal{A}} = \sqrt{\mathcal{D}}^{-1}\ \mathcal{A}\ \sqrt{\mathcal{D}}^{-1}$, where $\sqrt{\mathcal{D}} = \mathrm{diag}(\sqrt{\deg_G 1}, \ldots, \sqrt{\deg_G n})$. Let $\sigma(\widehat{\mathcal{A}}) = (\hat{\alpha}_1, \ldots, \hat{\alpha}_n)$.
- The *Laplacian matrix*, $\mathcal{L} = \mathcal{D} - \mathcal{A}$. Let $\sigma(\mathcal{L}) = (\lambda_1, \ldots, \lambda_n)$.
- The *normalized Laplacian matrix*, $\widehat{\mathcal{L}} = \sqrt{\mathcal{D}}^{-1}\ (\mathcal{D} - \mathcal{A})\ \sqrt{\mathcal{D}}^{-1} = I - \widehat{\mathcal{A}}$. Let $\sigma(\widehat{\mathcal{L}}) = (\hat{\lambda}_1, \ldots, \hat{\lambda}_n)$.
- The *signless Laplacian matrix*, $|\mathcal{L}| = \mathcal{D} + \mathcal{A}$. Let $\sigma(|\mathcal{L}|) = (\mu_1, \ldots, \mu_n)$.
- The *normalized signless Laplacian matrix*, $\widehat{|\mathcal{L}|} = \sqrt{\mathcal{D}}^{-1}\ (\mathcal{D} + \mathcal{A})\ \sqrt{\mathcal{D}}^{-1} = I + \widehat{\mathcal{A}}$. Let $\sigma(\widehat{|\mathcal{L}|}) = (\hat{\mu}_1, \ldots, \hat{\mu}_n)$.

Example 1.1. For the wheel on five vertices, shown in Figure 1.1, the matrices $\mathcal{A}$, $\widehat{\mathcal{A}}$, $\mathcal{L}$, $\widehat{\mathcal{L}}$, $|\mathcal{L}|$, $\widehat{|\mathcal{L}|}$ and their spectra are

$$\mathcal{A} = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix}, \qquad \widehat{\mathcal{A}} = \begin{bmatrix} 0 & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} \\ \frac{1}{2\sqrt{3}} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{2\sqrt{3}} & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ \frac{1}{2\sqrt{3}} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{2\sqrt{3}} & \frac{1}{3} & 0 & \frac{1}{3} & 0 \end{bmatrix},$$

$$\sigma(\mathcal{A}) = (-2, 1-\sqrt{5}, 0, 0, 1+\sqrt{5}), \qquad \sigma(\widehat{\mathcal{A}}) = (-\tfrac{2}{3}, -\tfrac{1}{3}, 0, 0, 1),$$

$$\mathcal{L} = \begin{bmatrix} 4 & -1 & -1 & -1 & -1 \\ -1 & 3 & -1 & 0 & -1 \\ -1 & -1 & 3 & -1 & 0 \\ -1 & 0 & -1 & 3 & -1 \\ -1 & -1 & 0 & -1 & 3 \end{bmatrix}, \quad \widehat{\mathcal{L}} = \begin{bmatrix} 1 & \frac{-1}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} \\ \frac{-1}{2\sqrt{3}} & 1 & -\frac{1}{3} & 0 & -\frac{1}{3} \\ \frac{-1}{2\sqrt{3}} & -\frac{1}{3} & 1 & -\frac{1}{3} & 0 \\ \frac{-1}{2\sqrt{3}} & 0 & -\frac{1}{3} & 1 & -\frac{1}{3} \\ \frac{-1}{2\sqrt{3}} & -\frac{1}{3} & 0 & -\frac{1}{3} & 1 \end{bmatrix},$$

$$\sigma(\mathcal{L}) = (0, 3, 3, 5, 5), \qquad \sigma(\widehat{\mathcal{L}}) = (0, 1, 1, \tfrac{4}{3}, \tfrac{5}{3}),$$

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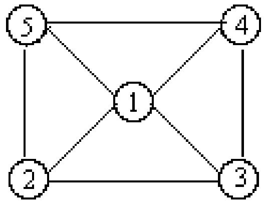

FIG. 1.1. *Wheel on 5 vertices.*

$$|\mathcal{L}| = \begin{bmatrix} 4 & 1 & 1 & 1 & 1 \\ 1 & 3 & 1 & 0 & 1 \\ 1 & 1 & 3 & 1 & 0 \\ 1 & 0 & 1 & 3 & 1 \\ 1 & 1 & 0 & 1 & 3 \end{bmatrix}, \qquad \widehat{|\mathcal{L}|} = \begin{bmatrix} 1 & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} \\ \frac{1}{2\sqrt{3}} & 1 & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{2\sqrt{3}} & \frac{1}{3} & 1 & \frac{1}{3} & 0 \\ \frac{1}{2\sqrt{3}} & 0 & \frac{1}{3} & 1 & \frac{1}{3} \\ \frac{1}{2\sqrt{3}} & \frac{1}{3} & 0 & \frac{1}{3} & 1 \end{bmatrix},$$

$\sigma(|\mathcal{L}|) = (1, \frac{9-\sqrt{17}}{2}, 3, 3, \frac{9+\sqrt{17}}{2})$, $\qquad \sigma(\widehat{|\mathcal{L}|}) = (\frac{1}{3}, \frac{2}{3}, 1, 1, 2)$.

Since $\widehat{\mathcal{L}} = I - \widehat{\mathcal{A}}$ and $\widehat{|\mathcal{L}|} = I + \widehat{\mathcal{A}}$, if the spectrum of any one of $\widehat{\mathcal{A}}$, $\widehat{\mathcal{L}}$, $\widehat{|\mathcal{L}|}$, is known, the spectrum of any of the others is readily computed. If G is regular of degree r then $\widehat{\mathcal{A}} = \frac{1}{r}\,\mathcal{A}$, $\mathcal{L} = rI - \mathcal{A}$, $|\mathcal{L}| = rI + \mathcal{A}$, so if the spectrum of any one of $\mathcal{A}$, $\widehat{\mathcal{A}}$, $\mathcal{L}$, $\widehat{\mathcal{L}}$, $|\mathcal{L}|$, $\widehat{|\mathcal{L}|}$ is known so are the spectra of all of these matrices.

The matrices $\mathcal{A}$, $\widehat{\mathcal{A}}$, $|\mathcal{L}|$, $\widehat{|\mathcal{L}|}$ are all non-negative, and if G is connected, they are all irreducible. The *Perron-Frobenius Theorem* [18] provides the following information about an irreducible non-negative matrix B (where $\rho(B)$ denotes the spectral radius, i.e., maximum absolute value of an eigenvalue of B).

1. $\rho(B) > 0$.
2. $\rho(B)$ is an eigenvalue of B.
3. $\rho(B)$ is algebraically simple as an eigenvalue of B.
4. There is a positive vector x such that $Bx = \rho(B)x$.

Let B be a symmetric non-negative matrix. Eigenvectors for distinct eigenvalues of B are orthogonal. If B has a positive eigenvector x for eigenvalue β, then any eigenvector for a different eigenvalue cannot be positive, and so $\beta = \rho(B)$. Let $e = [1, 1, \ldots, 1]^T$. Then since $\widehat{\mathcal{A}}\,\sqrt{D}e = \sqrt{D}e$, $\rho(\widehat{\mathcal{A}}) = 1$, and $\rho(\widehat{|\mathcal{L}|}) = 2$.

The matrices $\mathcal{A}$, $\mathcal{D}$, $\widehat{\mathcal{A}}$, $\mathcal{L}$, $\widehat{\mathcal{L}}$, $|\mathcal{L}|$, $\widehat{|\mathcal{L}|}$ are also connected via the incidence matrix.

The *(vertex-edge) incidence matrix* $\mathcal{N}$ of graph G with n vertices and m edges is the $n \times m$ 0,1-matrix with rows indexed by the vertices of G and columns indexed by the edges of G, such that the v, e entry of $\mathcal{N}$ is 1 (respectively, 0) if edge e is (respectively, is not) incident with vertex v. Then

$\mathcal{N}\mathcal{N}^T = \mathcal{D} + \mathcal{A} = |\mathcal{L}|$ and $\widehat{|\mathcal{L}|} = (\sqrt{\mathcal{D}}^{-1}\,\mathcal{N})\ (\sqrt{\mathcal{D}}^{-1}\mathcal{N})^T$.

An *orientation* of graph G is the assignment of a direction to each edge, converting edge $\{i, j\}$ to either arc (i, j) or arc (j, i). The *oriented incidence matrix* $\mathcal{N}'$ of an oriented graph G' with n vertices and m arcs is the $n \times m$ 0,1,-1-matrix with rows indexed by the vertices of G and columns indexed by the arcs of G such that the

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$v, (w, v)$-entry of $\mathcal{N}'$ is 1, the $v, (v, w)$-entry of $\mathcal{N}'$ is -1, and all other entries are 0. If G' is any orientation of G and $\mathcal{N}'$ is the oriented incidence matrix then

$\mathcal{N}'\mathcal{N}'^T = \mathcal{D} - \mathcal{A} = \mathcal{L}$ and $\widehat{\mathcal{L}} = (\sqrt{\mathcal{D}}^{-1}\ \mathcal{N}')\ (\sqrt{\mathcal{D}}^{-1}\mathcal{N}')^T$.

So $\mathcal{L}$, $|\mathcal{L}|$, $\widehat{\mathcal{L}}$, $\widehat{|\mathcal{L}|}$ are all positive semidefinite, and so have non-negative eigenvalues. The *inertia* of a matrix B is the ordered triple (i_+, i_-, i_0), where i_+ is the number of positive eigenvalues of B, i_- is the number of negative eigenvalues of B, and i_0 is the number of zero eigenvalues of B. By Sylvester's Law of Inertia [18], the inertia of $\mathcal{L}$ is equal to the inertia of $\widehat{\mathcal{L}}$. Since $\widehat{\mathcal{L}} + \widehat{|\mathcal{L}|} = 2I$, the following facts have been established, provided G is connected.

1. $\sigma(\widehat{|\mathcal{L}|}) \subset [0, 2]$ and $\hat{\mu}_n = 2$ with eigenvector $\sqrt{D}e$.
2. $\sigma(\widehat{\mathcal{A}}) \subset [-1, 1]$ and $\hat{\alpha}_n = 1$ with eigenvector $\sqrt{D}e$.
3. $\sigma(\widehat{\mathcal{L}}) \subset [0, 2]$ and $\hat{\lambda}_1 = 0$ with eigenvector $\sqrt{D}e$.
4. $\lambda_1 = 0$.

If G is not connected, the multiplicity of 0 as an eigenvalue of $\mathcal{L}$ is the number of connected components of G. For each of the matrices $\mathcal{A}$, $\widehat{\mathcal{A}}$, $|\mathcal{L}|$, $\widehat{|\mathcal{L}|}$, $\mathcal{L}$, $\widehat{\mathcal{L}}$ the spectrum is the union of the spectra of the components.

If A is the adjacency matrix of the line graph $L(G)$ of G (cf. [15]), then $\mathcal{N}^T\mathcal{N} = 2I + A$. It follows from the *Singular Value Decomposition Theorem* [18] that the non-zero eigenvalues of $\mathcal{N}\mathcal{N}^T$ and $\mathcal{N}^T\mathcal{N}$ are the same (including multiplicities). Thus the spectrum of $|\mathcal{L}|$ is readily determined from that of the adjacency matrix of $L(G)$. Since $\mathcal{N}^T\mathcal{N}$ is positive semidefinite, the least eigenvalue of the adjacency matrix of $L(G)$ is greater than or equal to -2. See [15] for further discussion of line graphs and graphs with adjacency matrix having all eigenvalues greater than or equal to -2.

We now turn our attention to information about the graph that can be extracted from the spectra of these matrices. This is the approach typically taken in Spectral Graph Theory.

The following parameters of graph G are determined by the spectrum of the adjacency matrix or, equivalently, by its characteristic polynomial $p(x) = x^n + a_{n-2}x^{n-2} + \ldots + a_1x + a_0$ (note $a_{n-1} = 0$ since tr $\mathcal{A} = 0$).

1. the number of edges of G = $-a_{n-2} = \frac{\mathrm{tr}\mathcal{A}^2}{2} = \frac{\sum \alpha_i^2}{2}$
2. the number of triangles of G = $-\frac{a_{n-3}}{2} = \frac{\mathrm{tr}\mathcal{A}^3}{6} = \frac{\sum \alpha_i^3}{6}$.

The first equality in each of these statements is obtained by viewing the coefficient of $p(x)$ as the sum of the principal minors of order k, the second is obtained by considering walks, and the third is obtained by using the fact that a real symmetric matrix is unitarily similar to a diagonal matrix.

Unfortunately these results do not extend cleanly to longer cycles, as can be seen by considering the 4-cycle.

One use of spectral graph theory is to assist in determining whether two graphs are isomorphic. If two graphs have different spectra (equivalently, different characteristic polynomials) then clearly they are not isomorphic. However, non-isomorphic graphs can be cospectral. Figure 1.2 shows two graphs having the same spectrum for the adjacency matrix.

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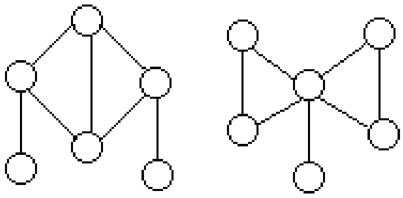

FIG. 1.2. *Cospectral graphs with* $p(x) = -1 + 4\,x + 7\,x^2 - 4\,x^3 - 7\,x^4 + x^6$.

A graph G is called *spectrally determined* if any graph with the same spectrum is isomorphic to G. Of course, one must identify the matrix (e.g., adjacency, Laplacian, etc.) from which the spectrum is taken. Examples of graphs that are spectrally determined by the adjacency matrix [11]:

- Complete graphs.
- Empty graphs.
- Graphs with one edge.
- Graphs missing only 1 edge.
- Regular graphs of degree 2.
- Regular graphs of degree $n-3$, where n is the order of the graph.

However, "most" trees are not spectrally determined, in the sense that as n goes to infinity, the proportion of trees on n vertices that are determined by the spectrum of the adjacency matrix goes to 0 [28]; see also [11]. A recent survey of results on cospectral graphs and spectrally determined graphs can be found in [12].

There are many other graph parameters for which information can be extracted from the spectra of the various matrices associated with a graph. Here we mention only two examples, the vertex connectivity and the diameter.

A *vertex cutset* of G is a subset of vertices of G whose deletion increases the number of connected components of G. The *vertex connectivity* of G, κ_0, is the minimum number of vertices in a vertex cutset (for a graph that is not the complete graph). The second smallest eigenvalue of the Laplacian $\mathcal{L}$, λ_2, is called the algebraic connectivity of G.

THEOREM 1.2. [14]; see also [15] *If G is not K_n, the vertex connectivity is greater than or equal to the algebraic connectivity, i.e.,* $\lambda_2 \leq \kappa_0$.

The *distance* between two vertices in a graph is the length of (i.e., number of edges in) the shortest path between them. The diameter of a graph G, diam(G), is maximum distance between any two vertices of G.

THEOREM 1.3. [5] *The diameter of a connected graph G is less than the number of distinct eigenvalues of the adjacency matrix of G.*

For trees, this result extends to all the matrices whose pattern of nonzero entries is described by the graph, as we shall discuss in the next section. There are also several other diameter results involving the Laplacian and normalized Laplacian, see for example [6].

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2. The Inverse Eigenvalue Problem of a Graph. Spectral Graph Theory originally focused on specific matrices, such as the adjacency matrix or the Laplacian matrix, whose entries are determined by the graph, with the goal of obtaining information about the graph from the matrices. In contrast, the Inverse Eigenvalue Problem of a Graph seeks to determine information about the possible spectra of the real symmetric matrices whose pattern of nonzero entries is described by a given graph.

We need some additional definitions and notation. For a symmetric real $n \times n$ matrix B, the graph of B, $\mathcal{G}(B)$, is the graph with vertices $\{1, \ldots, n\}$ and edges $\{\{i, j\}|\ b_{ij} \neq 0$ and $i \neq j\}$. Note that the diagonal of B is ignored in determining $\mathcal{G}(B)$.

EXAMPLE 2.1. For the matrix $B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 3.1 & -1.5 & 2 \\ 0 & -1.5 & 1 & 1 \\ 0 & 2 & 1 & 0 \end{bmatrix}$, $\mathcal{G}(B)$ is shown in Figure 2.1.

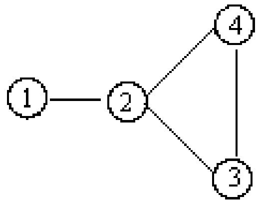

FIG. 2.1. *The graph $\mathcal{G}(B)$ for B in* Example 2.1.

Let S_n be the set of real symmetric $n \times n$ matrices. For G a graph with vertices $\{1, \ldots, n\}$, define $\mathcal{S}(G) = \{B \in S_n|\ \mathcal{G}(B) = G\}$. Note that $\mathcal{A}$, $\widehat{\mathcal{A}}$, $|\mathcal{L}|$, $\widehat{|\mathcal{L}|}$, $\mathcal{L}$, $\widehat{\mathcal{L}} \in \mathcal{S}(G)$. The Inverse Eigenvalue Problem of a Graph is to characterize the possible spectra of matrices in $\mathcal{S}(G)$.

The *multiplicity* of β as an eigenvalue of $B \in S_n$ is denoted by $m_B(\beta)$. The eigenvalue β is *simple* if $m_B(\beta) = 1$. The *maximum multiplicity* of G is $M(G) = \max\{m_B(\beta)|\ \beta \in \sigma(B),\ B \in \mathcal{S}(G)\}$, and the *minimum rank* of G is $\mathrm{mr}(G) = \min\{\mathrm{rank}\, B|\ B \in \mathcal{S}(G)\}$. Since $m_B(0) = \dim\ker B$, it is clear that $\mathrm{M}(G) + \mathrm{mr}(G) = n$. If H is an induced subgraph of G then $\mathrm{mr}(H) \leq \mathrm{mr}(G)$.

If G is not connected, then any matrix $B \in \mathcal{S}(G)$ is block diagonal, with the diagonal blocks corresponding to the connected components of G, and the spectrum of B is the union of the spectra of the diagonal blocks. Thus we usually restrict our attention to connected graphs.

Characterizations of graphs of order n having minimum rank 1, 2, and $n-1$ have been obtained. For any graph G, a matrix $B \in \mathcal{S}(G)$ with rank $B \leq n-1$ can always be obtained by taking $C \in \mathcal{S}(G)$, $\gamma \in \sigma(C)$, and $B = C - \gamma I$. Thus, for any graph G, $\mathrm{mr}(G) \leq n-1$. If $B \in \mathcal{S}(P_n)$, by deleting the first row and last column, we obtain an upper triangular $n-1 \times n-1$ submatrix with nonzero diagonal, so rank $B \geq n-1$.

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Thus $\mathrm{mr}(P_n) = n - 1$.

THEOREM 2.2. [13] *If for all $B \in \mathcal{S}(G)$, all eigenvalues of B are simple, then $G = P_n$. Equivalently, $\mathrm{mr}(G) = n - 1$ implies $G = P_n$.*

For any graph G that has an edge, any matrix in $\mathcal{S}(G)$ has at least two nonzero entries, so $\mathrm{mr}(G) \geq 1$. By examining the rank 1 matrix J (all of whose entries are 1), we see that $\mathrm{mr}(K_n) = 1$. If G is connected, then for any matrix $B \in \mathcal{S}(G)$, there is no row consisting entirely of zeros. Any rank 1 matrix B with no row of zeros has all entries nonzero, and thus $\mathcal{G}(B) = K_n$. Thus, for G a connected graph of order greater than one, $\mathrm{mr}(G) = 1$ is equivalent to $G = K_n$.

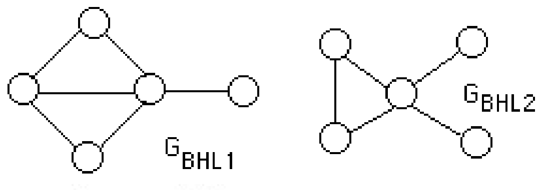

FIG. 2.2. *Forbidden induced subgraphs for* $\mathrm{mr}(G) = 2$.

THEOREM 2.3. [4] *A connected graph G has $\mathrm{mr}(G) \leq 2$ if and only if G does not contain as an induced subgraph any of: P_4, $K_{3,3,3}$ (the complete tripartite graph), G_{BHL1} or G_{BHL2} (shown in* Figure 2.2*).*

Additional characterizations of graphs having minimum rank 2 can be found in [4].

Recall that for any graph G, the diameter of G is less than the number of distinct eigenvalues of the adjacency matrix of G (cf. Theorem 1.3), and the proof extends to show $\mathrm{diam}(G)$ is less than the number of distinct eigenvalues of any non-negative matrix $B \in \mathcal{S}(G)$. If T is a tree and $B \in \mathcal{S}(T)$, it is possible to find a real number γ and a 1,-1-diagonal matrix S such that $STS^{-1} + \gamma I$ is non-negative. Thus, we have the following theorem.

THEOREM 2.4. [25] *If T is a tree, for any $B \in \mathcal{S}(T)$, the diameter of T is less than the number of distinct eigenvalues of B.*

There are many examples of trees T for which the minimum number of distinct eigenvalues is $\mathrm{diam}(T) + 1$. Barioli and Fallat [1] gave an example of a tree for which the minimum number of distinct eigenvalues is strictly greater than this bound. Since $n - (M(G) - 1) \geq$ the minimum number of distinct eigenvalues, for a tree T, $\mathrm{diam}(T) \leq \mathrm{mr}(T)$.

Most of the progress for trees is based on the Parter-Wiener Theorem and the interlacing of eigenvalues. If B is an $n \times n$ matrix, $B(k)$ is the $(n-1)\times(n-1)$ matrix obtained from B by deleting row and column k. If $B \in \mathcal{S}(G)$, then $B(k) \in \mathcal{S}(G-k)$. Let $B \in S_n$ and $k \in \{1, \ldots, n\}$. If the eigenvalues of B are $\beta_1 \leq \beta_2 \ldots \leq \beta_n$ and the eigenvalues of $B(k)$ are $\theta_1 \leq \theta_2 \leq \ldots \leq \theta_{n-1}$, then by the *Interlacing Theorem* [18] $\beta_1 \leq \theta_1 \leq \beta_2 \leq \theta_2 \ldots \leq \beta_{n-1} \leq \theta_{n-1} \leq \beta_n$.

COROLLARY 2.5. *If $\beta \in \sigma(B)$, $m_{B(k)}(\beta) \in \{m_B(\beta) - 1, m_B(\beta), m_B(\beta) + 1\}$.*

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We say k is a *Parter-Wiener (PW) vertex* of B for eigenvalue β if $m_{B(k)}(\beta) = m_B(\beta) + 1$; k is a *strong PW vertex* of B for β if k is a PW vertex of B for β and β is an eigenvalue of at least three components of $\mathcal{G}(B) - k$. The next theorem is referred to as the *Parter-Wiener Theorem.*

THEOREM 2.6. [27], [29], [22] *If T is a tree, $B \in \mathcal{S}(T)$ and $m_B(\beta) \geq 2$, then there is a strong PW vertex of B for β.*

COROLLARY 2.7. *If T is a tree, $B \in \mathcal{S}(T)$ and $\sigma(B) = (\beta_1, \ldots, \beta_n)$, then β_1 and β_n are simple eigenvalues.*

The set of *high degree vertices* of G is $H(G) = \{k \in V(G) |\ \deg_G k \geq 3\}$. Only high degree vertices can be strong PW vertices.

EXAMPLE 2.8. The star on $n+1$ vertices, $K_{1,n}$, has only one high degree vertex, say vertex 1. Thus this vertex must be the strong PW vertex for any multiple eigenvalue of B with $\mathcal{G}(B) = K_{1,n}$. By choosing the diagonal elements of B for $2, \ldots, n+1$ to be 0, we obtain $m_{B(1)}(0) = n$ and so $m_B(0) = n - 1$ and $\operatorname{mr}(K_{1,n}) = 2$. (In this case the other two eigenvalues are necessarily simple.)

As many people have observed, the Parter-Wiener Theorem need not be true for graphs that are not trees.

EXAMPLE 2.9. For $\mathcal{A}$ the adjacency matrix of C_4, $m_{\mathcal{A}}(0) = 2$ but there is no PW vertex since $C_4 - k$ is P_3 for any vertex k.

We define two more parameters of a graph that are related to maximum multiplicity and minimum rank. The path cover number of G, $P(G)$, is the minimum number of vertex disjoint paths occurring as induced subgraphs of G that cover all the vertices of G, and $\Delta(G) = \max\{p - q \mid \text{there is a set of } q \text{ vertices whose deletion leaves } p \text{ paths}\}$. These parameters are equal for trees, but not for all graphs.

THEOREM 2.10. [19] *For any tree T, $M(T) = P(T) = \Delta(T)$. For any graph G, $\Delta(G) \leq M(G)$.*

THEOREM 2.11. [2] *For any graph G, $\Delta(G) \leq P(G)$.*

The relationship between $M(G)$ and $P(G)$ is less clear.

EXAMPLE 2.12. The wheel on 5 vertices, W_5, shown in Figure 1.1, has $P(W_5) = 2$ by inspection, and $\operatorname{mr}(W_5) = 2$ (by Theorem 2.3), so $M(W_5) = 3 > P(W_5)$.

EXAMPLE 2.13. In [2], it is established that the penta-sun, H_5, shown in Figure 2.3, has $P(H_5) = 3 > M(H_5) = 2$. More information on graphs for which $P(G) > M(G)$ can be found in [2] and [3].

We now consider what combinations of eigenvalues and multiplicities are possible for trees. Since the first and last eigenvalue of a matrix whose graph is a tree must be simple, clearly the order is important in determining which lists of eigenvalues and multiplicities are possible. If the distinct eigenvalues of B are $\breve{\beta}_1 < \ldots < \breve{\beta}_r$ with multiplicities $m_1, \ldots, m_r$, respectively, then $(m_1, \ldots, m_r)$ is called the ordered multiplicity list of B. There has been extensive study of the possible ordered multiplicity lists of trees.

THEOREM 2.14. [13], [20], [21] *The possible ordered multiplicity lists of the following families of trees have been determined. Furthermore, if there is a matrix $B \in \mathcal{S}(G)$ with distinct eigenvalues $\breve{\beta}_1 < \ldots < \breve{\beta}_r$ having multiplicities $m_1, \ldots, m_r$, then for any*

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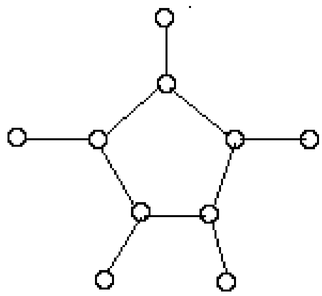

FIG. 2.3. *The penta-sun H_5.*

real numbers $\gamma_1 < \ldots < \gamma_r$, there is a matrix in $\mathcal{S}(G)$ having eigenvalues $\gamma_1, \ldots, \gamma_r$ with multiplicities $m_1, \ldots, m_r$.

- *Paths.*
- *Double Paths.*
- *Stars*
- *Generalized Stars*
- *Double Generalized Stars.*

Thus for any of these graphs, determination of the possible ordered multiplicity lists of the graph is equivalent to the solution of the Inverse Eigenvalue Problem of the graph.

However, Barioli and Fallat [1] established that sometimes there are restrictions on which real numbers can appear as the eigenvalues for an attainable ordered multiplicity list.

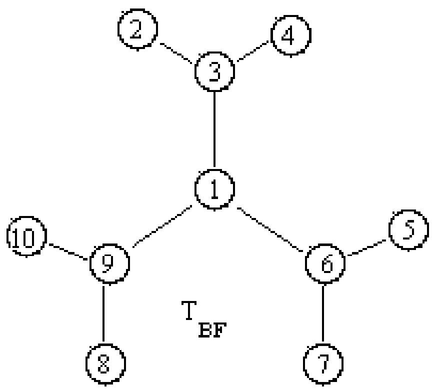

FIG. 2.4. *A tree for which an ordered multiplicity list is possible only for certain real numbers.*

EXAMPLE 2.15. For the tree T_{BF} shown in Figure 2.4, the spectrum of the

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adjacency matrix is $\sigma(\mathcal{A}) = (-\sqrt{5}, -\sqrt{2}, -\sqrt{2}, 0, 0, 0, 0, \sqrt{2}, \sqrt{2}, \sqrt{5})$, so the ordered multiplicity list of $\mathcal{A}$ is $(1, 2, 4, 2, 1)$. But the trace technique in [1] shows that if $B \in \mathcal{S}(T_{BF})$ has the five distinct eigenvalues $\breve{\beta}_1 < \breve{\beta}_2 < \breve{\beta}_3 < \breve{\beta}_4 < \breve{\beta}_5$ with multiplicities $m_B(\breve{\beta}_1) = m_B(\breve{\beta}_5) = 1, m_B(\breve{\beta}_2) = m_B(\breve{\beta}_4) = 2, m_B(\breve{\beta}_3) = 4$, then $\breve{\beta}_1 + \breve{\beta}_5 = \breve{\beta}_2 + \breve{\beta}_4$.

3. Generalized Laplacians and the Colin de Verdière Number $\mu(G)$. The symmetric matrix $L = [l_{ij}]$ is a generalized Laplacian matrix of G if for all i, j with $i \neq j$, $l_{ij} < 0$ if i and j are adjacent in G and $l_{ij} = 0$ if i and j are nonadjacent. Clearly $\mathcal{L}$ and $\widehat{\mathcal{L}}$ are generalized Laplacians. Note that if L is a generalized Laplacian then $-L$ has non-negative off-diagonal elements, and so there is a real number c such that $cI - L$ is non-negative. Thus, if G is connected, by the Perron-Frobenius Theorem, the least eigenvalue of L is simple. Much recent work in Spectral Graph Theory with generalized Laplacians is based on Colin de Verdière matrices and the Colin de Verdière number $\mu(G)$. This graph parameter was introduced by Colin de Verdière in 1990 ([7] in English). A thorough introduction to this important subject is provided by [17]. Here we list only a few of the definitions and results, following the treatment in [17].

The matrix L is a Colin de Verdière matrix for graph G if

1. L is a generalized Laplacian matrix of G.
2. L has exactly one negative eigenvalue (of multiplicity 1).
3. (Strong Arnold Property) If X is a symmetric matrix such that $LX = 0$ and $x_{i,j} \neq 0$ implies $i \neq j$ and i, j is not an edge of G, then $X = 0$.

The Colin de Verdière number $\mu(G)$ is the maximum multiplicity of 0 as an eigenvalue of a Colin de Verdière matrix. A Colin de Verdière matrix realizing this maximum is called optimal. Note that condition (2) ensures that $\mu(G)$ is the multiplicity of $\lambda_2(L)$ for an optimal Colin de Verdière matrix. The Strong Arnold Property is the requirement that certain manifolds intersect transversally. See [17] for more details.

Clearly $\mu(G) \leq M(G)$, since any Colin de Verdière matrix is in $\mathcal{S}(G)$. There are many examples, such as Example 3.1 below, of matrices in $\mathcal{S}(G)$ where this inequality is strict, due to the failure of the Strong Arnold Property for matrices realizing $M(G)$.

Example 3.1. The star $K_{1,n}$ has $M(K_{1,n}) = n - 1$ and this multiplicity is attained (for eigenvalue 0) by the adjacency matrix. For $n > 3$ (if the high degree vertex is 1), the matrix $X = (e_2 - e_3)(e_4 - e_5)^T + (e_4 - e_5)(e_2 - e_3)^T$ shows that $\mathcal{A}$ does not have the Strong Arnold Property. In fact, $\mu(K_{1,n}) = 2$ (provided $n > 2$) [17].

A *contraction* of the graph G is obtained by identifying two adjacent vertices of G, suppressing any loops or multiple edges that arise in this process. A *minor* of G arises by performing a series of deletions of edges, deletions of isolated vertices, and/or contraction of edges. Colin de Verdière showed that μ is minor-monotone.

Theorem 3.2. [7]; see also [17]. *If H is a minor of G then $\mu(H) \leq \mu(G)$.*

The Strong Arnold Property is essential to this minor-monotonicity, as the following example shows.

Example 3.3. Consider the graph G_{BHL2} shown in Figure 2.2. From [4], $\mathrm{mr}(G_{BHL2}) = 3$, so $M(G_{BHL2}) = 2$, but deletion of the edge that joins the two degree 2 vertices produces $K_{1,4}$ and $M(K_{1,4}) = 3$.

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The Robertson-Seymour theory of graph minors asserts that the family of graphs G with $\mu(G) \leq k$ can be characterized by a finite set of forbidden minors [17]. Colin de Verdière; Robertson, Seymour and Thomas; and Lovász and Schrijver have used this to establish the following characterizations.

THEOREM 3.4. [7], [17]

1. $\mu(G) \leq 1$ *if and only if* G *is a disjoint union of paths.*
2. $\mu(G) \leq 2$ *if and only if* G *is outerplanar.*
3. $\mu(G) \leq 3$ *if and only if* G *is planar.*
4. $\mu(G) \leq 4$ *if and only if* G *is linklessly embeddable.*

A graph is planar if it can be drawn in the plane without crossing edges. A graph is outerplanar if it has such a drawing with a face that contains all vertices. An embedding of a graph G into $\mathbb{R}^3$ is *linkless* if no disjoint cycles in G are linked in $\mathbb{R}^3$. A graph is *linklessly embeddable* if it has a linkless embedding. See [17] for more detail.

THEOREM 3.5. [15] *Let* L *be a generalized Laplacian matrix of the graph* G *with* $\sigma(L) = (\omega_1, \omega_2, \ldots, \omega_n)$. *If* G *is* 2*-connected and outerplanar then* $m_L(\omega_2) \leq 2$. *If* G *is* 3*-connected and planar then* $m_L(\omega_2) \leq 3$.

4. Connections and New Results. Clearly there are close connections between the recent work in Spectral Graph Theory on generalized Laplacians, and the Inverse Eigenvalue Problem of a Graph. Matrices attaining $M(G)$ for eigenvalue 0 are central to this connection. Equivalently, we are concerned with matrices attaining the minimum rank of G. In particular, generalized Laplacian matrices realizing the minimum rank of G are of interest.

We now turn to the question of whether we can realize the minimum rank of G by a matrix in $\mathcal{S}(G)$ having some special form such as:

- the adjacency matrix $\mathcal{A}$,
- a 0,1- matrix,
- $\mathcal{A}+D$ for D a diagonal matrix,
- a generalized Laplacian of G.

OBSERVATION 4.1. *For any graph* G*:*

- *The adjacency matrix* $\mathcal{A}$ *is a 0,1-matrix.*
- *Any* $0,1$*-matrix* $A \in \mathcal{S}(G)$ *is of the form* $\mathcal{A}+D$ *for* D *a diagonal matrix.*
- *For* D *a diagonal matrix,* $-(\mathcal{A}+D)$ *is a generalized Laplacian of* G.
- *Finding a matrix of minimum rank in* $\mathcal{S}(G)$ *with off-diagonal elements nonnegative is equivalent to finding a generalized Laplacian of minimum rank.*

There are very few graphs for which $\mathcal{A}$ realizes minimum rank. The star $K_{1,n}$ is one such graph, but for the path P_n, $\mathcal{A}$ is nonsingular if n is even. We show that for a tree, there is always a 0,1-matrix realizing minimum rank (equivalently, realizing maximal multiplicity for eigenvalue 0). This answers a question raised by Zhongshan Li [26].

THEOREM 4.2. *If* T *is a tree and* $\mathcal{A}$ *is its adjacency matrix, then there exists a* $0,1$*-diagonal matrix* D *such that* $m_{\mathcal{A}+D}(0) = M(T)$*, and thus* $\operatorname{rank}(\mathcal{A}+D) = \operatorname{mr}(T)$.

Proof. By [19], $M(T) = \Delta(T)$, i.e., there exists a set Q of q vertices such that $T-Q$ consists of p disjoint paths and $p-q = M(T)$. For each path, remove alternate interior

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vertices so that the result is isolated vertices (and one path with 2 vertices if the path had an even number of vertices originally). Let Q^* be the set of q^* vertices consisting of the original q vertices and the additional alternate interior vertices deleted. Then Q^* has the property that $T - Q^*$ consists of p^* disjoint paths, each having 1 or 2 vertices, and $p^* - q^* = M(T)$. Choose the diagonal elements of D corresponding to isolated vertices or to deleted vertices to be 0, and choose the diagonal elements of D corresponding to 2-vertex paths to be 1. Then 0 is an eigenvalue of each of the p^* paths and, by interlacing, $m_{\mathcal{A}+D}(0) \geq p^* - q^* = M(T) \geq m_{\mathcal{A}+D}(0)$. Thus $\operatorname{rank}(\mathcal{A} + D) = n - m_{\mathcal{A}+D}(0) = n - M(T) = \operatorname{mr}(T)$. □

By using the algorithm of Johnson and Saiago [24] to produce the initial set Q of vertices to delete, we obtain the following algorithm for producing a 0,1-matrix $\mathcal{A}+D$ of minimum rank among matrices in $\mathcal{S}(T)$. As in [24], $\delta_T(v) = \deg_T v - \deg_{H(T)} v$, where $H(T)$ is the subgraph of T induced by the vertices of degree at least 3 in T.

ALGORITHM 4.3.

```
1. Set T' = T and Q = ∅.
   Repeat:
   a) Q' = {v | δ_T'(v) ≥ 2},
   b) Set T' = T' − Q'.
   c) Set Q = Q ∪ Q'.
   Until Q' = ∅.
   This determines Q.
2. In each path, remove alternate interior vertices and add these
   vertices to Q to obtain Q*.
3. Choose D = diag(d_1, ..., d_n) where d_k = 1 if k ∉ Q* and the path of
   T − Q* containing k has 2 vertices; otherwise, d_k = 0.
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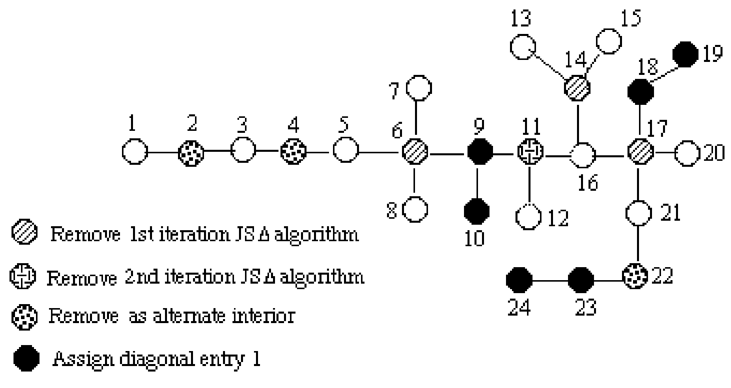

FIG. 4.1. *Applying* Algorithm 4.3 *to a tree* T.

EXAMPLE 4.4. Figure 4.1 illustrates the application of Algorithm 4.3 to the tree T. The vertices $6, 14, 17$ are removed in the first iteration of step 1. The vertex 11 is removed in the second iteration of step 1. The vertices $2, 4, 22$ are removed in step 2 as alternate interior vertices. Thus $Q^* = \{2, 4, 6, 11, 14, 17, 22\}$. The diagonal entries associated with paths having two vertices, i.e., $9, 10, 18, 19, 23, 24$, which are the black

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vertices in Figure 4.1, are assigned 1, and all other diagonal entries (including those corresponding to both shaded and unshaded vertices) are assigned 0. Thus with

$D = \mathrm{diag}(0,0,0,0,0,0,0,0,1,1,0,0,0,0,0,0,0,1,1,0,0,0,1,1)$,

$\mathcal{A}+D$ is a $0,1$-matrix in $\mathcal{S}(T)$ with $\mathrm{rank}(\mathcal{A}+D) = 17 = \mathrm{mr}(T)$.

Algorithm 4.3 will produce the adjacency matrix of T if $\mathcal{A}$ attains the minimum rank of T. Theorem 4.2 need not be true for graphs that are not trees, that is, we cannot always find a 0,1-matrix in $\mathcal{S}(G)$ realizing minimum rank. In particular, it is impossible for the 5-cycle.

Theorem 4.5. *For any n-cycle C_n, $n \geq 3$,* $\mathrm{rank}(\mathcal{A} - 2\cos(\frac{2\pi}{n})I) = n-2 = \mathrm{mr}(C_n)$*. Furthermore, there is a $0,1$-matrix D with* $\mathrm{rank}(\mathcal{A}+D) = \mathrm{mr}(C_n)$ *if and only if $n \neq 5$. Specifically, the diagonal matrices D listed below have this property.*

0. *If $n \equiv 0$ mod 4 then let* $D = \mathrm{diag}(0,0,\ldots,0)$*, i.e., the adjacency matrix realizes minimum rank.*
1. *If $n \equiv 1$ mod 4 and $n \geq 9$ then let* $D = \mathrm{diag}(1,1,1,1,1,1,1,1,1,0,0,\ldots,0)$.
2. *If $n \equiv 2$ mod 4 then let* $D = \mathrm{diag}(1,1,1,1,1,1,0,0,\ldots,0)$.
3. *If $n \equiv 3$ mod 4 then let* $D = \mathrm{diag}(1,1,1,0,\ldots,0)$.

If $n \equiv 0$ mod 3 then $D = I$ also works.

Proof. It is known [10] that the eigenvalues of $\mathcal{A}$ are $2\cos(\frac{2\pi k}{n})$ for $k = 1,\ldots,n$. Since $\cos(\frac{2\pi}{n}) = \cos(\frac{2\pi(n-1)}{n})$, it is clear that $\mathrm{rank}(\mathcal{A} - 2\cos(\frac{2\pi}{n})I) = n-2 \geq \mathrm{mr}(C_n)$. Since P_{n-1} is an induced subgraph of C_n, $\mathrm{mr}(C_n) \geq n-2$, establishing the first statement.

Direct examination of all 32 possibilities for the diagonal shows that all 0,1-matrices in $\mathcal{S}(C_5)$ are nonsingular and thus do not realize $\mathrm{mr}(C_5) = 3$.

For each general case, we exhibit two vectors z_1 and z_2 in the kernel of $\mathcal{A}+D$. Each vector has a repeatable block of 4 entries (or 3 entries for $n \equiv 0$ mod 3) that is used as many times as necessary to obtain a vector of the right length. Such a block will be denoted repeat$[\ldots]$.

Determination of D based on congruence mod 4:

0. Let $n \equiv 0$ mod 4 and $D = \mathrm{diag}(0,0,\ldots,0)$.
   Then $z_1 = (\mathrm{repeat}[-1,0,1,0])^T$ and $z_2 = (\mathrm{repeat}[0,-1,0,1])^T$.
1. Let $n \equiv 1$ mod 4 and $D = \mathrm{diag}(1,1,1,1,1,1,1,1,1,0,0,\ldots,0)$.
   Then $z_1 = (-1,1,0,-1,1,0,-1,1,0,\ \mathrm{repeat}[-1,0,1,0])^T$ and
   $z_2 = (0,-1,1,0,-1,1,0,-1,1,\ \mathrm{repeat}[0,-1,0,1])^T$.
2. Let $n \equiv 2$ mod 4 and $D = \mathrm{diag}(1,1,1,1,1,1,0,0,\ldots,0)$.
   Then $z_1 = (-1,1,0,-1,1,0,\ \mathrm{repeat}[-1,0,1,0])^T$ and
   $z_2 = (0,-1,1,0,-1,1,\ \mathrm{repeat}[0,-1,0,1])^T$.
3. Let $n \equiv 3$ mod 4 and $D = \mathrm{diag}(1,1,1,0,\ldots,0)$.
   Then $z_1 = (-1,1,0,\ \mathrm{repeat}[-1,0,1,0])^T$ and
   $z_2 = (0,-1,1,\ \mathrm{repeat}[0,-1,0,1])^T$.

For $n \equiv 0$ mod 3, $D = I$, let $z_1 = (\mathrm{repeat}[-1,1,0])^T$ and $z_2 = (\mathrm{repeat}[0,-1,1])^T$. □

Corollary 4.6. *For every wheel $W_{n+1}, n \geq 4$, there is a diagonal matrix D such that* $\mathrm{rank}(\mathcal{A}+D) = \mathrm{mr}(W_{n+1})$.

Proof. The wheel W_{n+1} can be constructed by joining vertex $n+1$ to each of the vertices of C_n. Let $\widetilde{\mathcal{A}}$ be the adjacency matrix of C_n. By Theorem 4.5, rank $(\widetilde{\mathcal{A}} - dI_n)$

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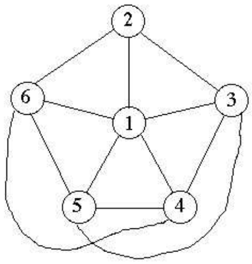
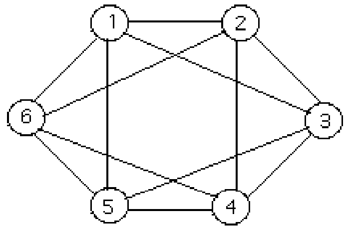


The graph H



The graph $K_{2,2,2}$

FIG. 4.2.

$= \mathrm{mr}(C_n)$ for $d = 2\cos(\frac{2\pi}{n})$. Let e be the vector all of whose entries are 1. Then $\mathrm{mr}(W_{n+1}) \leq \mathrm{rank} \begin{bmatrix} \widetilde{\mathcal{A}} - dI_n & e \\ e^T & \frac{n}{2-d} \end{bmatrix} = \mathrm{rank}(\widetilde{\mathcal{A}} - dI_n) = \mathrm{mr}(C_n) \leq \mathrm{mr}(W_{n+1})$. □

We have seen that for G a cycle or wheel, we can find a diagonal matrix D such that $\mathrm{rank}(\mathcal{A} + D) = \mathrm{mr}(G)$. However, the next example shows that for the graph H shown in Figure 4.2, it is not possible to find a matrix in $\mathcal{S}(H)$ realizing the minimum rank of H that has the form $\mathcal{A} + D$. But for this graph H we are able to exhibit a generalized Laplacian L of H with rank $L = \mathrm{mr}(H)$.

EXAMPLE 4.7. Consider the graph H shown in Figure 4.2. By Theorem 2.3, $\mathrm{mr}(H) = 2$ (this is also demonstrated by the matrix L below). Let
$B = \begin{bmatrix} d_1 & 1 & 1 & 1 & 1 & 1 \\ 1 & d_2 & 1 & 0 & 0 & 1 \\ 1 & 1 & d_3 & 1 & 1 & 0 \\ 1 & 0 & 1 & d_4 & 1 & 1 \\ 1 & 0 & 1 & 1 & d_5 & 1 \\ 1 & 1 & 0 & 1 & 1 & d_6 \end{bmatrix}$. Assume that it is possible to find d_i such that rank $B = 2$. Let r_i denote the ith row of B. Since r_3 and r_4 are clearly independent, every other row must be expressible as a linear combination of these two. Applying this to r_5 yields $d_4 = d_5 = 1$. Applying this to r_6 yields $d_3 = d_6 = 0$. With these values of d_3 and d_4, attempting to express r_1 as a linear combination of r_3 and r_4 produces a contradiction. However, we can realize minimum rank with the generalized Laplacian matrix $L = \begin{bmatrix} -15 & -1 & -3 & -4 & -4 & -3 \\ -1 & 1 & -1 & 0 & 0 & -1 \\ -3 & -1 & 0 & -1 & -1 & 0 \\ -4 & 0 & -1 & -1 & -1 & -1 \\ -4 & 0 & -1 & -1 & -1 & -1 \\ -3 & -1 & 0 & -1 & -1 & 0 \end{bmatrix}$, for which $\sigma(L) = (-8 - 6\sqrt{3}, 0, 0, 0, 0, -8 + 6\sqrt{3})$.

The next example shows that it is not always possible to find a generalized Laplacian L of that realizes minimum rank. This answers a question raised by the author

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[16].

EXAMPLE 4.8. Consider the graph $K_{2,2,2}$ shown in Figure 4.2.

The matrix $B = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & -3 & -2 & 1 & 0 & -1 \\ 1 & -2 & -1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 3 & 2 \\ 1 & -1 & 0 & 1 & 2 & 1 \end{bmatrix} \in \mathcal{S}(K_{2,2,2})$ has rank $B = 2$.

Now let matrix $M \in \mathcal{S}(K_{2,2,2})$. $\mathrm{Det}[1,2,3;4,5,6] = m_{15}m_{26}m_{34} + m_{16}m_{24}m_{35}$, so if M is non-negative, $\mathrm{Det}[1,2,3;4,5,6] > 0$ and rank $M \geq 3$. Thus the minimum rank of $K_{2,2,2}$ can not be realized by a matrix in $\mathcal{S}(K_{2,2,2})$ with non-negative off-diagonal entries. Equivalently, the minimum rank of $K_{2,2,2}$ can not be realized by a generalized Laplacian.

Even though it is not always possible to find a matrix realizing minimum rank that has one of the special forms adjacency matrix (i.e., 0,1-matrix with all diagonal entries 0), 0,1-matrix, adjacency matrix + diagonal matrix, generalized Laplacian matrix, there are many graphs for which it is possible. The next two theorems offer ways to obtain matrices of special form realizing minimum rank for graphs that have a *cut-vertex*, that is the vertex in a singleton cutset. As in [2], the *rank spread* of v in G, $r_v(G) = \mathrm{mr}(G) - \mathrm{mr}(G-v)$. Let v be a cut-vertex of G and let $\widetilde{G}_i, i = 1, \ldots, h$ be the components of $G - v$. Let G_i be the subgraph of G induced by v and the vertices of $\widetilde{G}_i$. G is called the *vertex-sum* of $G_1, \ldots, G_h$ and it is shown in [2] that $r_v(G) = \min\left\{\sum_{i=1}^{h} r_v(G_i),\ 2\right\}$.

THEOREM 4.9. *Suppose the graph G is the vertex-sum at v of the graphs G_i, for $i = 1, \ldots, h$, and $r_v(G) = 2$. Let $\widetilde{G}_i = G_i - v$. If there exist matrices $\widetilde{A}_1, \ldots, \widetilde{A}_h$ such that for $i = 1, \ldots, h$,*

- $\widetilde{A}_i \in \mathcal{S}(\widetilde{G}_i)$,
- rank $\widetilde{A}_i = \mathrm{mr}(\widetilde{G}_i)$,
- $\widetilde{A}_i$ *is of type* $\mathcal{X}$,

then there exists a matrix A with

- $A \in \mathcal{S}(G)$,
- rank $A = \mathrm{mr}(G)$,
- A *is of type* $\mathcal{X}$

for $\mathcal{X}$ any of the following types of matrix: adjacency matrix, 0,1-matrix, sum of a diagonal matrix and the adjacency matrix, generalized Laplacian matrix.

Proof. Without loss of generality, assume $v = 1$, the vertices of $\widetilde{G}_1$ are next, followed by the vertices of $\widetilde{G}_2$, etc. Since $r_1(G) = 2$, $\mathrm{mr}(G) = \mathrm{mr}(G-1) + 2 = \sum \mathrm{mr}(\widetilde{G}_i) + 2$. Let $\widetilde{A} = \widetilde{A}_1 \oplus \ldots \oplus \widetilde{A}_h$. Clearly $\widetilde{A} \in \mathcal{S}(G-1)$, rank $\widetilde{A} = \mathrm{mr}(G-1)$, and if $\widetilde{A}_i$ is of type $\mathcal{X}$ for all i, then so is $\widetilde{A}$. Define $A = \begin{bmatrix} 0 & b^T \\ b & \widetilde{A} \end{bmatrix}$ where $b_i = t$ if $\{i, 1\}$ is an edge of G and $b_i = 0$ otherwise. Choose $t = 1$ if $\mathcal{X}$ is the class of adjacency matrices, 0,1-matrices or sums of a diagonal matrix with an adjacency matrix, and $t = -1$ if $\mathcal{X}$ is the class of generalized Laplacians. Clearly $A \in \mathcal{S}(G)$ and A is of type $\mathcal{X}$. Further, rank $A \leq$ rank $\widetilde{A} + 2 = \mathrm{mr}(G-1) + 2 = \mathrm{mr}(G) \leq \mathrm{rank}\, A$, so rank $A =$

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mr(G). □

We illustrate this theorem in the next two examples.

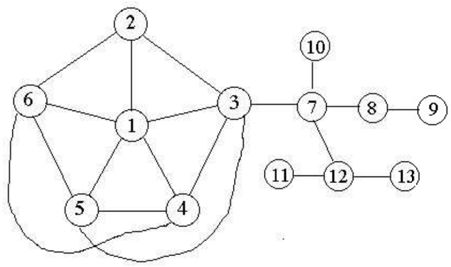

FIG. 4.3. *A graph to which* Theorem 4.9 *can be applied to produce a generalized Laplacian of minimum rank.*

EXAMPLE 4.10. For the graph G shown in Figure 4.3, vertex 7 is a cut-vertex. The components of $G-7$ are the induced subgraphs $\langle 1,2,3,4,5,6\rangle$, $\langle 8,9\rangle$, $\langle 10\rangle$, and $\langle 11,12,13\rangle$, having minimum ranks $2,1,0$, and 2, respectively. Since $\mathrm{mr}(\langle 7,10\rangle) = 1$ and $\mathrm{mr}(\langle 10\rangle) = 0$, $r_7(\langle 7,10\rangle) = 1$. Similarly, $r_7(\langle 7,8,9\rangle) = 1$. Since $r_v(G) = \min\left\{\sum_{i=1}^{h} r_v(G_i),\ 2\right\}$, $r_7(G) = 2$. Thus $\mathrm{mr}(G) = (2+1+0+2)+2 = 7$. We can apply Theorem 4.9, Example 4.7, and Algorithm 4.3 to produce a generalized Laplacian L having rank $L = 7$. The matrix produced is

$$L = \begin{bmatrix} -15 & -1 & -3 & -4 & -4 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -3 & -1 & 0 & -1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -4 & 0 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -4 & 0 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -3 & -1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}.$$

EXAMPLE 4.11. For the graph G shown in Figure 4.4, vertex 8 is a cut-vertex of G. We show $r_8(G) = 2$: Let $G_1 = \langle 8,10,11,12,13,14,15\rangle$; $\mathrm{mr}(G_1) = 5$ because $G_1 \neq P_7$ and P_6 is an induced subgraph of G_1. The minimum rank of the six cycle $\langle 10,11,12,13,14,15\rangle$ is 4, so $r_8(G_1) = 1$. Since $r_8(\langle 8,9\rangle) = 1$, $r_8(G) = 2$.

The components of $G-8$ are $G_2 = \langle 1,2,3,4,5,6,7\rangle$, $\langle 9\rangle$, and the six cycle. Since we can produce $0,1$-matrices of minimum rank for the the six cycle (by Theorem 4.5, we can use all 1s on the diagonal) and $\langle 9\rangle$ (use 0), if we can find a $0,1$-matrix for G_2, we can apply Theorem 4.9. Vertex 1 is a cut-vertex of G_2. Since $r_1(\langle 1,6,7\rangle) = 1$ and $r_1(\langle 1,5\rangle) = 1$, $r_1(G_2) = 2$, so we can apply Theorem 4.9 to find a $0,1$-matrix of minimum rank for G_2. The diagonal entries chosen are $0,1,1,1,0,1,1$.

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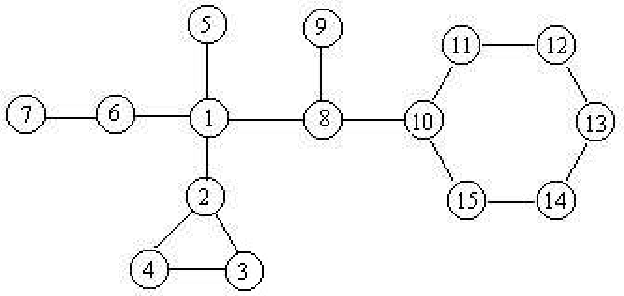

Fig. 4.4. *A graph to which* Theorem 4.9 *can be applied to produce a* $0,1$*-matrix of minimum rank.*

When we apply Theorem 4.9 to the sum at vertex 8, the diagonal matrix produced is $D = \mathrm{diag}(0,1,1,1,0,1,1,0,0,1,1,1,1,1,1)$.

We can check that for this D, $\mathcal{A}+D$ is a minimum rank matrix by verifying $\mathrm{rank}(\mathcal{A}+D) = 10$ and showing this is the minimum rank of G. Since the components of $G_2 - 1$ are $\langle 2,3,4\rangle$, $\langle 5\rangle$, and $\langle 6,7\rangle$, having minimum ranks $1,0$, and 1, respectively, and $r_1(G_2) = 2$, $\mathrm{mr}(G_2) = 4$. Since the components of $G - 8$ are G_2, $\langle 9\rangle$, and the six cycle, having minimum ranks $4,0$, and 4, respectively, and $r_8(G) = 2$, $\mathrm{mr}(G) = 10$.

Theorem 4.12. *Suppose the graph G is the vertex-sum at v of the graphs G_i, for $i = 1,\dots,h$ and $r_v(G) = 0$. If there exist matrices $A_1,\dots,A_h$ such that for $i = 1,\dots,h$,*

- *$A_i \in \mathcal{S}(G_i)$,*
- *rank $A_i = \mathrm{mr}(G_i)$,*
- *A_i is of type $\mathcal{X}$,*

then there exists a matrix A with

- *$A \in \mathcal{S}(G)$,*
- *rank $A = \mathrm{mr}(G)$,*
- *A is of type $\mathcal{X}$*

for $\mathcal{X}$ one of the classes: adjacency matrix, sum of a diagonal matrix and the adjacency matrix, generalized Laplacian matrix.

Proof. Since by [2], $r_v(G) = \min\{\sum r_v(G_i), 2\}$, and $r_v(G) = 0$, clearly $r_v(G_i) = 0$ for $i = 1,\dots,h$. Thus $\mathrm{mr}(G_i) = \mathrm{mr}(G_i - v)$ for $i = 1,\dots,h$. Then $\mathrm{mr}(G) = \mathrm{mr}(G - v) + 0 = \sum \mathrm{mr}(G_i - v) = \sum \mathrm{mr}(G_i)$.

For $i = 1,\dots,h$, let $\breve{A}_i$ be the $n \times n$ matrix obtained from A_i by embedding it in the appropriate place (setting all other entries 0). Let $A = \breve{A}_1 + \ldots + \breve{A}_h$. If $A_i \in \mathcal{X}$ for all i, then so is A (the only entry that is the sum of more than one nonzero entry is the v,v-entry). And rank $A \le \sum$rank $A_i = \sum \mathrm{mr}(G_i) = \mathrm{mr}(G) \le$ rank A, so rank $A = \mathrm{mr}(G)$. ☐

We illustrate this theorem in the next example.

Example 4.13. The graph G shown in Figure 4.5 is the vertex-sum at 4 of the induced subgraphs $\langle 1,2,3,4\rangle$ and $G_1 = \langle 4,5,6,7,8,9,10,11,12\rangle$. Since $\mathrm{mr}(\langle 1,2,3,4\rangle) = 2 = \mathrm{mr}(\langle 1,2,3\rangle)$, $r_4(\langle 1,2,3,4\rangle) = 0$. To show $r_4(G_1) = 0$, we compute $\mathrm{mr}(G_1)$ and

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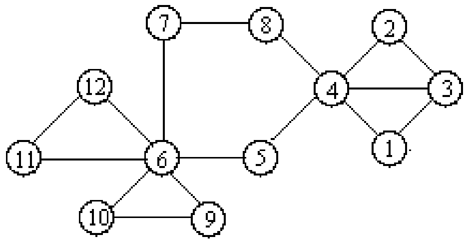

FIG. 4.5. *A graph to which* Theorem 4.12 *can be applied to produce a matrix of minimum rank that is the sum of the adjacency matrix and a diagonal matrix.*

$\mathrm{mr}(G_1 - 4)$.

The graph G_1 is the vertex sum at 6 of $\langle 4,5,6,7,8\rangle$, $\langle 6,9,10\rangle$, and $\langle 6,11,12\rangle$. Since $\mathrm{mr}(\langle 4,5,6,7,8\rangle) = \mathrm{mr}(\langle 4,5,7,8\rangle) = 3$, $r_6(\langle 4,5,6,7,8\rangle) = 0$. Similarly, $\mathrm{mr}(\langle 6,9,10\rangle) = 1$, $r_6(\langle 6,9,10\rangle) = 0$, $\mathrm{mr}(\langle 6,11,12\rangle) = 1$, and $r_6(\langle 6,11,12\rangle) = 0$. Thus, $r_6(G_1) = 0$ and $\mathrm{mr}(G_1) = 5$.

Likewise, $G_1 - 4$ is the vertex sum at 6 of $\langle 5,6\rangle$, $\langle 6,7,8\rangle$, $\langle 6,9,10\rangle$, and $\langle 6,11,12\rangle$. Since $\mathrm{mr}(\langle 5,6\rangle) = 1$ and $\mathrm{mr}(\langle 5\rangle) = 0, r_6(\langle 5,6\rangle) = 1$. Similarly, $r_6(\langle 6,7,8\rangle) = 1$, so $r_6(G_1 - 4) = 2$. Thus, $\mathrm{mr}(G_1 - 4) = \mathrm{mr}(\langle 5\rangle) + \mathrm{mr}(\langle 7,8\rangle) + \mathrm{mr}(\langle 9,10\rangle) + \mathrm{mr}(\langle 11,12\rangle) + r_6(G_1 - 4) = 5$. Since $\mathrm{mr}(G_1) = \mathrm{mr}(G_1 - 4)$, $r_4(G_1) = 0$.

Thus, $\mathrm{mr}(G) = \mathrm{mr}(G_1 - 4) + \mathrm{mr}(\langle 1,2,3\rangle) + r_4(G) = 7$. We can apply Theorem 4.12 (twice) and Theorem 4.5 to produce a diagonal matrix D such that rank $\mathcal{A}+D = 7$. The matrix $\mathcal{A} + D$ is built from

$$A_1 = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \in \mathcal{S}(\langle 1,2,3,4\rangle),$$

$$A_2 = \begin{bmatrix} \frac{1-\sqrt{5}}{2} & 1 & 0 & 0 & 1 \\ 1 & \frac{1-\sqrt{5}}{2} & 1 & 0 & 0 \\ 0 & 1 & \frac{1-\sqrt{5}}{2} & 1 & 0 \\ 0 & 0 & 1 & \frac{1-\sqrt{5}}{2} & 1 \\ 1 & 0 & 0 & 1 & \frac{1-\sqrt{5}}{2} \end{bmatrix} \in \mathcal{S}(\langle 4,5,6,7,8\rangle),$$

$$\text{and } A_3 = A_4 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \in \mathcal{S}(\langle 6,9,10\rangle) = \mathcal{S}(\langle 6,11,12\rangle).$$

Note A_1 and A_2 will overlap on vertex 4, and A_2, A_3 and A_4 overlap on vertex 6. The diagonal matrix produced is
$D = (0,0,1,1+\frac{1-\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}, 2+\frac{1-\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}, 1,1,1,1)$.

Just as we were able to apply a theorem more than once (Theorem 4.9 in Example 4.11, and Theorem 4.12 in Example 4.13), we could apply both theorems (successively) to a single example, perhaps more than once.

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