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A NOTE ON GENERALIZED PERRON COMPLEMENTS OF Z-MATRICES*

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Abstract. The concept of the Perron complement of a nonnegative and irreducible matrix was introduced by Meyer in 1989 and it was used to construct an algorithm for computing the stationary distribution vector for Markov chains. Here properties of the generalized Perron complement of an $n \times n$ irreducible Z-matrix K are considered. First the result that the generalized Perron complements of K are irreducible Z-matrices is shown, and other properties are presented.

Key words. Nonnegative matrix, Z-matrix, Perron complement, Irreducibility.

AMS subject classifications. 15A48.

1. Introduction. An $n \times n$ real matrix $K = (k_{i,j})$ is called a Z-matrix if the off-diagonal entries are non-positive. A Z-matrix K can be written as

$$K = sI - M, \quad M \text{ is nonnegative.}$$

These matrices arise in many problems in the mathematical and physical sciences. Some of the best-known subclasses of Z-matrices are the class of M-matrices. Since in this paper we often investigate the smallest real eigenvalue, which we denote by $n(K)$, then $n(K) = s - \rho(M)$, where $\rho(\cdot)$ denotes the spectral radius.

Let $K \in R^{n,n}$ be the space of all real $n \times n$ matrices. And let γ and δ be nonempty ordered subsets of $\langle n \rangle := \{1, 2, \cdots, n\}$, both of strictly increasing integers. By $K[\gamma, \delta]$ we shall denote the submatrix of K whose rows and columns are determined by γ and δ, respectively. In the special case when $\gamma = \delta$, we shall use $K[\gamma]$ to denote $K[\gamma, \gamma]$, the principal submatrix of K based on γ.

Suppose that $\beta \subset \langle n \rangle$. If $K[\beta]$ is nonsingular, then the Schur complement of $K[\beta]$ in K is given by

$$\varphi(K/K[\beta]) = K[\alpha] - K[\alpha, \beta](K[\beta])^{-1}K[\beta, \alpha], \ \alpha = \langle n \rangle \setminus \beta.$$

In connection with a divide and conquer algorithm for computing the stationary distribution vector for a Markov chain, Meyer introduced, for an $n \times n$ nonnegative and irreducible matrix K, the notion of the Perron complement. Again, let $\beta \subset \langle n \rangle$, $\alpha = \langle n \rangle \setminus \beta$. Then the Perron complement of $K[\beta]$ in K is given by

$$P(K/K[\beta]) = K[\alpha] + K[\alpha, \beta](\rho(K)I - K[\beta])^{-1}K[\beta, \alpha].$$

Meyer [1] has derived several interesting and useful properties of $P(K/K[\beta])$.

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In this paper we will discuss the generalized Perron complement of a Z-matrix and its properties. In section two we will introduce that if $K = sI - M$ is an irreducible Z-matrix, t satisfies $x < -\rho(M[\alpha]) + t$, then the generalized Perron complement $P_t(K/K[\alpha])$ is an irreducible Z-matrix. And $n(P_t(K/K[\alpha]))$ is a strict decreasing function of t, it also has a close relationship with $n(K)$. For brevity in our proofs, we shall adopt the following notations: if $K \in R^{n,n}$, $\beta \subset \langle n\rangle$, $\alpha = \langle n\rangle \setminus \beta$, then

$$A = K[\alpha], \quad B = K[\alpha, \beta], \quad C = K[\beta, \alpha], \quad D = K[\beta, \beta].$$

2. Generalized Perron complements. Fiedler and Markham [5] introduced the following classification of Z-matrices:

Definition 2.1. Let L_s (for $s = 0, \cdots, n$) denote the class of matrices consisting of real $n \times n$ matrices which have the form

$$K = tI - M, \text{ where } M \geq 0 \text{ and } \rho_s(M) \leq t < \rho_{s+1}(M).$$

Here

$$\rho_s(M) := \max\{\rho(\tilde{M}) : \tilde{M} \text{ is an } s \times s \text{ principal submatrix of } M\},$$

and we set $\rho_0(M) := -\infty$ and $\rho_{n+1}(M) := \infty$.

We now introduce some properties of the Perron complements of nonnegative matrices.

Lemma 2.2. ([3]) *If A is a nonnegative irreducible matrix, then for any $t > \rho(A[\beta])$ the generalized Perron complement $P_t(A/A[\beta])$ is also a nonnegative irreducible matrix.*

Lemma 2.3. ([3]) *If A is a nonnegative irreducible matrix, then the Perron root $\rho(P_t(A/A[\beta]))$ of the generalized Perron complement is a strictly decreasing function of t on $(\rho(A[\beta]), \infty)$.*

Lemma 2.4. ([3]) *If A is a nonnegative irreducible matrix, then*

$$\rho(P_t(A/A[\alpha])) \begin{cases} < \rho(A), & if\ t > \rho(A), \\ = \rho(A), & if\ t = \rho(A), \\ > \rho(A), & if\ \rho(A[\alpha]) < t < \rho(A). \end{cases}$$

Now we begin with our first main result:

Theorem 2.5. *Let $K = tI - M$ be an irreducible Z-matrix and $K \in L_s$. Then the generalized Perron complement $P_x(K/K[\alpha])$ is an irreducible Z-matrix for $x < -\rho(M[\alpha]) + t$.*

Proof. Since $K = tI - M$ is an irreducible Z-matrix and $K \in L_s$, then M is a nonnegative irreducible matrix. Let $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$. Then

$$K = tI - M = \begin{bmatrix} tI - A & -B \\ -C & tI - D \end{bmatrix}.$$

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Set $M[\alpha] = A$, we can get that

$$\begin{aligned} P_x(K/tI - A) &= tI - D + (-C)[xI - (tI - A)]^{-1}(-B) \\ &= tI - D - C(tI - xI - A)^{-1}B. \end{aligned}$$

By $x < -\rho(M[\alpha]) + t$,

$$t - x > \rho(M[\alpha]).$$

So

$$P_{t-x}(M/A) = D + C(tI - xI - A)^{-1}B$$

is a nonnegative irreducible matrix. Since irreducibility is independent of the diagonal entries, so

$$\begin{aligned} P_x(K/tI - A) &= tI - D - C(tI - xI - A)^{-1}B \\ &= tI - P_{t-x}(M/A) \end{aligned}$$

is an irreducible Z-matrix. □

From the proof of Theorem 2.5 we can see that, when $K \in L_s$, it means that $\rho_s(M) \le t < \rho_{s+1}(M)$. Under such a condition, if

$$|\alpha| > n - s \ \ (\text{or } |\beta| < s),$$

then

$$\rho(M[\beta]) < \rho_s(M) \le t.$$

It is easy to show that

$$\begin{aligned} \lim_{x \to -\infty} P_{t-x}(M/A) &= \lim_{x \to -\infty} (D + C(tI - xI - A)^{-1}B) \\ &= D = M[\beta]. \end{aligned}$$

Hence there exists an $\tilde{x}$ satisfying

$$\rho(P_{t-\tilde{x}}(M/M[\alpha])) < t.$$

So we can get that, when $x \le \tilde{x}$, the matrix

$$\begin{aligned} P_x(K/tI - A) &= tI - D - C(tI - xI - A)^{-1}B \\ &= tI - P_{t-x}(M/A) \end{aligned}$$

is a nonsingular M-matrix.

Otherwise, if $|\beta| \ge s$, it depends on whether there is $\rho(P_{t-\tilde{x}}(M/M[\alpha])) < t$? In other words, whether $\rho(M[\beta]) \le \rho_s(M)$? Take $|\beta| = s$ for example.

1) If such β satisfies

$$\rho(M[\beta]) = \rho_s(M),$$

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then only when $x \to -\infty$,

$$\begin{aligned} P_x(K/tI-A) &= tI-D-C(tI-xI-A)^{-1}B \\ &= tI-P_{t-x}(M/A) \end{aligned}$$

belongs to M-matrix (which is a singular M-matrix) when $t=\rho_s(M)$. Otherwise if $\rho_s(M)<t<\rho_{s+1}(M)$, it is a nonsingular M-matrix.

2) In another case, if such β satisfies $\rho(M[\beta])<\rho_s(M)$, there exists an $\tilde{x}$ satisfying

$$\rho(P_{t-\tilde{x}}(M/M[\alpha]))<t.$$

So we can get that when $x\le\tilde{x}$, it is a nonsingular M-matrix.

It is also easy to show that $\rho(M[\alpha])$ is different when α has changed. So even we have $x<-\rho(M[\alpha])+t$, it is hard to determine x. For simplicity of computing, we change $x<-\rho(M[\alpha])+t$ to $x<-\rho_{n-1}(M)+t$, then

$$t-x>\rho_{n-1}(M)\ge\rho(M[\alpha]).$$

Now it becomes easier to compute.

THEOREM 2.6. *Let $K=sI-M$ be an irreducible Z-matrix and $K\in L_s$. Then, for $x<-\rho(M[\alpha])+t$, $n(P_x(K/K[\alpha]))$ is a strictly decreasing function of x on $(-\infty,-\rho(M[\alpha])+t)$.*

Proof. Let $x_1<x_2<-\rho(M[\alpha])+t$. From the proof of Theorem 2.5, we can get

$$P_{x_1}(K/tI-A)=tI-P_{t-x_1}(M/A),$$

$$P_{x_2}(K/tI-A)=tI-P_{t-x_2}(M/A).$$

Since $x_1<x_2<-\rho(M[\alpha])+t$, then

$$t-x_1>t-x_2>\rho(M[\alpha]).$$

So

$$\rho(P_{t-x_1}(M/A))<\rho(P_{t-x_2}(M/A)),$$

$$\begin{aligned} n(P_{x_1}(K/K[\alpha])) &= t-\rho(P_{t-x_1}(M/A)) \\ &> t-\rho(P_{t-x_2}(M/A)) \\ &= n(P_{x_2}(K/K[\alpha])). \end{aligned}$$

That is $n(P_x(K/K[\alpha]))$ is a strictly decreasing function of t on

$$(-\infty,-\rho(M[\alpha])+t). \square$$

THEOREM 2.7. *Let $K=sI-M$ be an irreducible Z-matrix and $K\in L_s$. Then, for $x<-\rho(M[\alpha])+t$,*

$$n(P_x(K/K[\alpha]))\begin{cases} <n(K), & if\ -\rho(M)+t<x<-\rho(M[\alpha])+t, \\ =n(K), & if\ x=-\rho(M)+t, \\ >n(K), & if\ x<-\rho(M)+t. \end{cases}$$

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Proof. Since

$$\begin{aligned} P_x(K/tI-A) &= tI-D-C(tI-xI-A)^{-1}B \\ &= tI-P_{t-x}(M/A), \end{aligned}$$

by $-\rho(M)+t<x<-\rho(M[\alpha])+t$, we have

$$\rho(M[\alpha])<t-x<\rho(M),$$

$$\rho(P_{t-x}(M/A))>\rho(M),$$

$$\begin{aligned} n(P_x(K/tI-A)) &= t-\rho(P_{t-x}(M/A)) \\ &< t-\rho(M)=n(K). \end{aligned}$$

By $x=-\rho(M)+t$,

$$t-x=\rho(M),$$

$$\rho(P_{t-x}(M/A))=\rho(M),$$

$$n(P_x(K/K[\alpha]))=n(K).$$

By $x<-\rho(M)+t$,

$$t-x>\rho(M),$$

$$\rho(P_{t-x}(M/A))<\rho(M),$$

$$\begin{aligned} n(P_x(K/tI-A)) &= t-\rho(P_{t-x}(M/A)) \\ &> t-\rho(M)=n(K). \end{aligned}$$

Hence

$$n(P_x(K/K[\alpha]))\begin{cases} <n(K), & \text{if } -\rho(M)+t<x<-\rho(M[\alpha])+t, \\ =n(K), & \text{if } x=-\rho(M)+t, \\ >n(K), & \text{if} x<-\rho(M)+t. \end{cases} \qquad \square$$

THEOREM 2.8. *Let $K=sI-M$ be an irreducible Z-matrix and $K\in L_s$, satisfy $x<-\rho(M[\alpha])+t$. Then, for $|\alpha|<s$, the following ordering holds between the three matrices $K[\beta]$, $P_x(K/K[\alpha])$ and $\varphi[K/K(\alpha)]$:*

$$\begin{cases} K[\beta]>\varphi(K/K[\alpha])\geq P_x(K/K[\alpha]), & if\ x<0, \\ K[\beta]>P_x(K/K[\alpha])=\varphi(K/K[\alpha]), & if\ x=0, \\ K[\beta]>P_x(K/K[\alpha])\geq\varphi(K/K[\alpha]), & if\ 0<x<-\rho(M[\alpha])+t. \end{cases}$$

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Proof. Since $x < -\rho(M[\alpha]) + t$, $|\alpha| < s$, we have

$$-\rho(M[\alpha]) + t \geq \rho_s(M) - \rho(M[\alpha]) \geq 0,$$

$$K[\beta] = tI - D,$$

$$\varphi(K/K[\alpha]) = tI - D - C(tI - A)^{-1}B,$$

$$P_x(K/K[\alpha]) = tI - D - C(tI - xI - A)^{-1}B.$$

When $x = 0$, it is easy to show that

$$K[\beta] > P_x(K/K[\alpha]) = \varphi(K/K[\alpha]).$$

When $x < 0$, $tI - xI - A \geq tI - A$, we can get that

$$(tI - A)^{-1} \geq (tI - xI - A)^{-1},$$

so

$$K[\beta] > P_x(K/K[\alpha]) \geq \varphi(K/K[\alpha]).$$

When $0 < x < -\rho(M[\alpha]) + t$, $tI - xI - A \leq tI - A$, it follows that

$$(tI - A)^{-1} \leq (tI - xI - A)^{-1},$$

so

$$K[\beta] > \varphi(K/K[\alpha]) \geq P_x(K/K[\alpha]). \square$$

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