Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 15, pp. 134-142, April 2006

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ON A LYAPUNOV TYPE EQUATION RELATED TO PARABOLIC SPECTRAL DICHOTOMY*

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Abstract. A Lyapunov type equation characterizing the spectral dichotomy of a matrix by a parabola is proposed and analyzed. The results can be seen as the analogue of the classical Lyapunov stability theorems.

Key words. Eigenvalue, Lyapunov equation, Spectral dichotomy.

AMS subject classifications. 65F15, 93D05.

1. Introduction.

The authors of [7] analyzed the spectral dichotomy of a matrix $A \in \mathbb{C}^{n\times n}$ by the parabola γ of equation

$$y^2 = 2p\,(p/2 - x)\,,\;\; p0. \tag{1.1}$$

The analysis relied upon the function

$$z \mapsto \lambda = \left(z + \sqrt{p/2}\right)^2 \tag{1.2}$$

which, among other properties, maps bijectively the imaginary axis onto the parabola γ. The eigenvalues λ of A and z of

$$\mathcal{A} = \begin{pmatrix} -\sqrt{p/2}I_n & A \\ I_n & -\sqrt{p/2}I_n \end{pmatrix} \tag{1.3}$$

are related by (1.2). Moreover, if

$$\alpha = \sup_{\lambda\in\gamma} \|(\lambda I_n - A)^{-1}\|_2 \;\text{ and }\; \tilde{\alpha} = \sup_{\Re z=0} \|(zI_{2n} - \mathcal{A})^{-1}\|_2, \tag{1.4}$$

then

$$\alpha \le \tilde{\alpha} \le \alpha + \sqrt{\alpha}\sqrt{1+\alpha}. \tag{1.5}$$

The inequalities (1.5) show that the spectral dichotomy problem of the matrix A by the parabola γ is equivalent to that of the matrix $\mathcal{A}$ by the imaginary axis. The latter problem is well known (see [2]) and was used in [7] to derive expressions of the spectral projection of A corresponding to eigenvalues inside (outside) γ from that of $\mathcal{A}$ corresponding to eigenvalues on the left (right) half-plane. This note is concerned with the analysis of a Lyapunov type equation associated with the matrix A and

*Received by the editors 7 January 2006. Accepted for publication 20 April 2006. Handling Editor: Shmuel Friedland.

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the parabola γ. This issue was not addressed in [7]. The equation obtained is the analogue of the standard Lyapunov equation associated with the matrix $\mathcal{A}$ and the imaginary axis (see [5, Chap. 13], [6, Chap. 5], [2, Chap.10]):

$$\mathcal{L}_{\mathcal{A}}(\mathcal{H}) + \mathcal{C} = 0, \tag{1.6}$$

where $\mathcal{C} \in \mathbb{C}^{2n\times 2n}$ and $\mathcal{L}_{\mathcal{A}}$ is the linear operator defined by

$$\mathbb{C}^{2n\times 2n} \ni \mathcal{H} \mapsto \mathcal{L}_{\mathcal{A}}(\mathcal{H}) = \mathcal{A}^*\mathcal{H} + \mathcal{H}\mathcal{A} \in \mathbb{C}^{2n\times 2n}. \tag{1.7}$$

To better understand the goal of this note, we summarize the classical properties of (1.6) in the following "stability" theorem. See [2] and especially Sections 4.3, 10.1 and 10.2 in this reference.

In this theorem and throughout the note, the identity (null) matrix of order n is denoted by I_n (0_n). If B is a square matrix, the notation $B = B^* > 0$ means that B is Hermitian and positive definite.

Theorem 1.1.

1- *The operator $\mathcal{L}_{\mathcal{A}}$ is nonsingular if and only if $\bar{z}_1 + z_2 \neq 0$ for all eigenvalues z_1 and z_2 of $\mathcal{A}$.*
2- *If the spectrum of $\mathcal{A}$ lies in the left half-plane, then $\mathcal{L}_{\mathcal{A}}$ is nonsingular and for all $\mathcal{C} \in \mathbb{C}^{2n\times 2n}$ the unique solution of (1.6) is given by*

$$\mathcal{H} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \left((itI_{2n} - \mathcal{A})^{-1}\right)^* \mathcal{C}\,(itI_{2n} - \mathcal{A})^{-1}\,dt. \tag{1.8}$$

3- *If $\mathcal{A}$ has no purely imaginary eigenvalues, then for any matrix $\mathcal{C}$, the matrix $\mathcal{H}$ given by (1.8) and*

$$\mathcal{P} = \frac{1}{2}I_{2n} + \frac{1}{2\pi}\int_{-\infty}^{+\infty} (itI_{2n} - \mathcal{A})^{-1}\,dt \tag{1.9}$$

form the unique solution to the system

$$\mathcal{L}_{\mathcal{A}}(\mathcal{H}) + \mathcal{P}^*\mathcal{C}\mathcal{P} - (I_{2n} - \mathcal{P}^*)\,\mathcal{C}\,(I_{2n} - \mathcal{P}) = 0, \tag{1.10}$$

$$\mathcal{P}^2 = \mathcal{P},\ \mathcal{A}\mathcal{P} = \mathcal{P}\mathcal{A},\ \mathcal{H}\mathcal{P} = \mathcal{P}^*\mathcal{H}. \tag{1.11}$$

From (1.8) we see that if $\mathcal{C} = \mathcal{C}^* > 0$ then $\mathcal{H} = \mathcal{H}^* > 0$.
Conversely, if for $\mathcal{C} = \mathcal{C}^* > 0$ there exist matrices $\mathcal{P}$ and $\mathcal{H} = \mathcal{H}^* > 0$ satisfying (1.10) and (1.11), then the matrix $\mathcal{A}$ has no purely imaginary eigenvalues and $\mathcal{P}$ is the spectral projection onto the invariant subspace of $\mathcal{A}$ corresponding to the eigenvalues in the left half-plane given by (1.9) and the matrix $\mathcal{H}$ is given by (1.8).

Note that:

1. The projection $\mathcal{P}$ and the matrix $\mathcal{H}$ can be computed, for example, by the algorithm proposed in [3, Algorithm 2].

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2. The condition $\mathcal{HP} = \mathcal{P}^*\mathcal{H}$ ensures the uniqueness of the solution of system (1.10)-(1.11).

Our immediate aim is to propose a Lyapunov type equation corresponding to the matrix A and the parabola γ and analyze its properties along the lines of Theorem 1.1. Future work will consist in using this equation to derive good and computable estimates analogous to those in [2, p.149], but for matrices whose spectra lie in a parabola.

2. The Lyapunov type equation associated to A and γ. Equation (1.1) can be written

$$-p^2 + 2px + \frac{x^2+y^2}{2} - \frac{x^2-y^2}{2} = 0. \tag{2.1}$$

If we write $\lambda = x + iy \in \mathbb{C}$ then (2.1) becomes

$$-p^2 + p\left(\lambda + \bar{\lambda}\right) + \frac{|\lambda|^2}{2} - \frac{1}{4}\left(\lambda^2 + \bar{\lambda}^2\right) = 0. \tag{2.2}$$

Thus, λ is inside (on) (outside) the parabola γ if and only if the left-hand side of (2.2) is negative (equal to 0) (positive).

Let $C \in \mathbb{C}^{n\times n}$ and consider the matrix equation

$$L_A(H) + C = 0, \tag{2.3}$$

where L_A is the linear operator defined for $H \in \mathbb{C}^{n\times n}$ by

$$L_A(H) = -p^2H + p\left(A^*H + HA\right) + \frac{1}{2}A^*HA - \frac{1}{4}\left((A^*)^2H + HA^2\right). \tag{2.4}$$

Equation (2.3) will play the role of (1.6) when $\mathcal{A}$ and the imaginary axis are respectively replaced by A and the parabola γ. We will examine this equation along the lines of Theorem 1.1.

Proposition 2.1. *The operator L_A is nonsingular if and only if*

$$-p^2 + p(\bar{\lambda} + \mu) + \frac{1}{2}\bar{\lambda}\mu - \frac{1}{4}(\bar{\lambda}^2 + \mu^2) \neq 0 \tag{2.5}$$

for all eigenvalues λ and μ of A.

Proof. It is easy to see that (2.5) is satisfied when L_A is nonsingular. Conversely, if (2.5) holds, then Theorem 3.2 in [1, Sec. I.3] shows that L_A is nonsingular. $\square$

Proposition 2.2. *Assume that A has no eigenvalues on the parabola γ. Then the projection P onto the invariant subspace of A corresponding to the eigenvalues inside γ is given by*

$$P = \frac{1}{\pi}\int_{-\infty}^{+\infty} \left(it + \sqrt{p/2}\right)\left(\left(it + \sqrt{p/2}\right)^2 I_n - A\right)^{-1} dt \tag{2.6}$$

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Proof. The projection P is given by (see [4, p. 39]):

$$P = \frac{1}{2i\pi}\int_{\Gamma} (\lambda I_n - A)^{-1}\, d\lambda,$$

where Γ is any positively-oriented contour enclosing the eigenvalues inside γ but excluding the other eigenvalues. Let $\Gamma = \Gamma_{a,p}$ be the closed contour EFG that is depicted in Figure 2.1. The coordinates of E, F and G are $E\left(-a, \sqrt{2p(p/2+a)}\right)$, $F\left(-a, -\sqrt{2p(p/2+a)}\right)$ and $G\,(p/2, 0)$. Assume that the segment joining E and F intersects the real axis at $-a < 0$ with $a > \|A\|_2$. Then $P = \lim_{a\to+\infty} P_{a,p}$ with $P_{a,p} = \frac{1}{2i\pi}\int_{\Gamma_{a,p}} (\lambda I_n - A)^{-1}\, d\lambda$. The change of variables

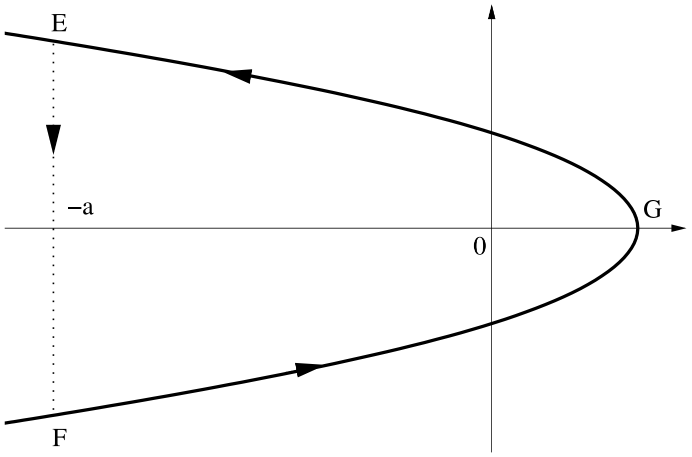

FIG. 2.1. *The contour EFG*

$$\lambda = -a + it \quad \text{with } |t| \le \sqrt{2p(p/2+a)} \quad \text{on the segment EF,}$$

and

$$\lambda = \left(it + \sqrt{p/2}\right)^2 \quad \text{with } |t| \le \sqrt{p/2+a} \quad \text{on the parabola FGE,}$$

leads to $P_{a,p} = Q_1 + Q_2$ with

$$Q_1 = \frac{1}{2\pi}\int_{|t|\le\sqrt{2p(p/2+a)}} ((-a+it)I_n - A)^{-1}\, dt \tag{2.7}$$

and

$$Q_2 = \frac{1}{\pi}\int_{|t|\le\sqrt{p/2+a}} \left(it + \sqrt{p/2}\right)\left(\left(it+\sqrt{p/2}\right)^2 I_n - A\right)^{-1} dt. \tag{2.8}$$

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We have (see e.g. [2, Corollary 11.4.1])

$$\|Q_1\|_2 \le \frac{1}{2\pi}\int_{|t|\le\sqrt{2p(p/2+a)}} \frac{dt}{|-a+it|-\|A\|_2} \le \frac{1}{\pi}\frac{\sqrt{2p(p/2+a)}}{a-\|A\|_2}.$$

The proof terminates by letting $a \to +\infty$ in (2.7) and (2.8). ◻

Assume that the matrix $\mathcal{A}$ has no purely imaginary eigenvalues, or equivalently, that A has no eigenvalues on the parabola γ. Then, according to property 3 of Theorem 1.1, for any $\mathcal{C}$, the matrices $\mathcal{H}$ and $\mathcal{P}$ given by (1.8) and (1.9) satisfy the system (1.10)-(1.11). Let us choose

$$\mathcal{C} = \begin{pmatrix} C_1 & 0_n \\ 0_n & C_2 \end{pmatrix} \text{ with } C_1,\ C_2 \in \mathbb{C}^{n\times n} \tag{2.9}$$

and partition $\mathcal{H}$ and $\mathcal{P}$ accordingly

$$\mathcal{H} = \begin{pmatrix} H_1 & H_2 \\ H_3 & H_4 \end{pmatrix} \text{ and } \mathcal{P} = \begin{pmatrix} P_1 & P_2 \\ P_3 & P_4 \end{pmatrix}. \tag{2.10}$$

Then, using the expression of $\mathcal{P}$ in (1.9) we immediately get

$$P_1 = P_4 = \frac{1}{2}I_n + \frac{1}{2}P, \tag{2.11}$$

$$P_3 = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(\left(it+\sqrt{p/2}\right)^2 I_n - A\right)^{-1} dt,\ P_2 = AP_3. \tag{2.12}$$

We mention that other expressions of the projection P and the matrices $P_i,\ i = 1,4$ were derived in [7] using a Jordan-like form of $\mathcal{A}$.

Our aim now is to show that a matrix equation of type (2.3) can be obtained with the help of the relations (2.9-2.12). Using the expressions of $\mathcal{A}$ in (1.3) and the partitioning of $\mathcal{H}$ in (2.10), equation (1.10) translates to

$$-\sqrt{2p}H_1 + H_2 + H_3 + F_1 = 0, \tag{2.13}$$
$$-\sqrt{2p}H_2 + H_4 + H_1A + F_2 = 0, \tag{2.14}$$
$$-\sqrt{2p}H_3 + A^*H_1 + H_4 + F_3 = 0, \tag{2.15}$$
$$A^*H_2 + H_3A - \sqrt{2p}H_4 + F_4 = 0, \tag{2.16}$$

with

$$F_1 = P_1^*C_1P_1 - (I_n - P_1)^*\, C_1\,(I_n - P_1)\,, \tag{2.17}$$
$$F_2 = C_1P_2 + P_3^*C_2,\ F_3 = P_2^*C_1 + C_2P_3, \tag{2.18}$$
$$F_4 = P_1^*C_2P_1 - (I_n - P_1)^*\, C_2\,(I_n - P_1)\,. \tag{2.19}$$

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Adding (2.14) and (2.15) and then using the expression of $H_2 + H_3$ from (2.13), we successively get

$$-\sqrt{2p}\,(H_2 + H_3) + 2H_4 + H_1 A + A^* H_1 + F_2 + F_3 = 0$$

and

$$H_4 = pH_1 - \frac{1}{2}\,(A^* H_1 + H_1 A) - \sqrt{\frac{p}{2}} F_1 - \frac{1}{2}\,(F_2 + F_3)\,. \tag{2.20}$$

Multiplying (2.14) on the left by A^* and (2.15) on the right by A and adding the results, we obtain

$$-\sqrt{2p}\,(A^* H_2 + H_3 A) + 2A^* H_1 A + A^* H_4 + H_4 A + A^* F_2 + F_3 A = 0,$$

which, by (2.16), can be written

$$A^* H_4 + H_4 A - 2pH_4 + 2A^* H_1 A + \sqrt{2p} F_4 + A^* F_2 + F_3 A = 0. \tag{2.21}$$

Now from (2.20) and (2.21), we obtain the equation of type (2.3)

$$L_A(H_1) + F = 0 \tag{2.22}$$

with

$$\begin{aligned} F = {} & \sqrt{\frac{p}{2}} \left(pF_1 + \sqrt{\frac{p}{2}}\,(F_2 + F_3) + F_4 - \frac{1}{2}\,(A^* F_1 + F_1 A) \right) \\ & + \frac{1}{4} A^*\,(F_2 - F_3) - \frac{1}{4}\,(F_2 - F_3)\,A. \end{aligned} \tag{2.23}$$

These calculations will be used to prove the following proposition.

Proposition 2.3. *If the spectrum of A lies inside γ, then L_A is nonsingular and for all $C \in \mathbb{C}^{n\times n}$ the unique solution of (2.3) is given by*

$$H = \frac{1}{\sqrt{2p}\,\pi} \int_{-\infty}^{+\infty} \left(\left(\left(it + \sqrt{p/2} \right)^2 I_n - A \right)^{-1} \right)^* C \left(\left(it + \sqrt{p/2} \right)^2 I_n - A \right)^{-1} dt. \tag{2.24}$$

From (2.24) we see that if $C = C^* > 0$ then $H = H^* > 0$.

Proof. Assume that the spectrum of A lies inside γ and L_A singular. Let λ, μ be two eigenvalues of A such that

$$-p^2 + p(\bar\lambda + \mu) + \frac{1}{2}\bar\lambda\mu - \frac{1}{4}(\bar\lambda^2 + \mu^2) = 0. \tag{2.25}$$

Then

$$-p^2 + p(\lambda + \bar\mu) + \frac{1}{2}\lambda\bar\mu - \frac{1}{4}(\lambda^2 + \bar\mu^2) = 0, \tag{2.26}$$

$$-p^2 + p(\bar\lambda + \lambda) + \frac{1}{2}|\lambda|^2 - \frac{1}{4}(\lambda^2 + \bar\lambda^2) < 0, \tag{2.27}$$

$$-p^2 + p(\bar\mu + \mu) + \frac{1}{2}|\mu|^2 - \frac{1}{4}(\mu^2 + \bar\mu^2) < 0. \tag{2.28}$$

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Equality (2.26) is simply the conjugate of (2.25) and inequalities (2.27) and (2.28) express that λ and μ are inside γ.

Subtracting (2.25)+(2.26) from (2.27) + (2.28) leads to the contradiction

$$|\lambda|^2 + |\mu|^2 - \bar{\lambda}\mu - \lambda\bar{\mu} < 0.$$

Hence L_A is nonsingular.

To prove that H given by (2.24) is the solution of (2.3), note that $P = I_n$ and $\mathcal{P} = I_{2n}$ since the spectrum of A and $\mathcal{A}$ lie respectively inside γ and in the left-half plane. If we choose $C_1 = 0_n$ and $C_2 = C$ in (2.9) and use the partitioning of $\mathcal{H}$ we obtain from (1.8)

$$H_1 = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \left(\left(\left(it + \sqrt{p/2}\right)^2 I_n - A\right)^{-1}\right)^* C \left(\left(it + \sqrt{p/2}\right)^2 I_n - A\right)^{-1} dt.$$

Moreover, from (2.17) - (2.19), we have $F_1 = F_2 = F_3 = 0_n$ and $F_4 = C$. Then (2.22) reduces to (2.3) with $H = \sqrt{2/p}H_1$. $\square$

Remarks

1. The proof of Proposition 2.3 provides a simple way to compute the matrix H which, up to the multiplicative constant $\sqrt{p/2}$, is equal to H_1. The latter can be obtained from $\mathcal{H}$ by applying [3, Algorithm 2] to $\mathcal{A}$.
2. If the spectrum of A lies outside γ, then L_A is not necessarily nonsingular. Indeed, consider for example the parabola γ with $p = 1$ and the matrix $A = \begin{pmatrix} 8 & 0 \\ 0 & 2 \end{pmatrix}$ whose eigenvalues $\lambda = 8$ and $\mu = 2$ are both outside γ and do not satisfy (2.5), i.e., the operator L_A is singular. Moreover, the equation $L_A(H) + C = 0$ with $C = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ has no solution. This point constitutes an important difference with the operator $\mathcal{L}_{\mathcal{A}}$ which remains nonsingular if the spectrum of $\mathcal{A}$ lies either in the left or in the right half plane.
3. We have seen that if A has no eigenvalues on γ, then equation (2.22) is solvable but this equation is not of the type (1.10). This means that property 3 of Theorem 1.1 cannot be generalized, as such, to the parabolic case. However, we still have the following result.

PROPOSITION 2.4. *If for $C = C^* > 0$ there exist matrices P and $H = H^* > 0$ such that*

$$L_A(H) + P^*CP - (I_n - P^*)C(I_n - P) = 0, \tag{2.29}$$

with

$$AP = PA,\ P^2 = P, \tag{2.30}$$

then the matrix A has no eigenvalues on γ and P is the spectral projection onto the invariant subspace of A corresponding to the eigenvalues inside γ, given by (2.6).

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Proof. Let (λ, u) be an eigenpair of A.

Multiplying (2.29) on the right by Pu and then taking the scalar product with Pu and using (2.30) gives

$$f(\lambda, p)\,(HPu, Pu) = -\,(CPu, Pu)\,, \tag{2.31}$$

with $f(\lambda, p) = -p^2 + p(\bar{\lambda} + \lambda) + \frac{1}{2}|\lambda|^2 - \frac{1}{4}(\bar{\lambda}^2 + \lambda^2)$.
Acting in a similar way with $(I_n - P)u$ gives

$$f(\lambda, p)\,((I_n - P)u)^*\,H\,((I_n - P)u) = ((I_n - P)u)^*\,C\,((I_n - P)u)\,. \tag{2.32}$$

If λ is on γ, then $f(\lambda, p) = 0$, and since $C = C^* > 0$, we obtain $Pu = (I_n - P)u = 0$, which contradicts $u \neq 0$.
If λ is outside γ, then (2.31) shows that $Pu = 0$, and if λ is inside γ, then (2.32) shows that $(I_n - P)u = 0$.

Therefore $\mathrm{Null}\,(P)$ contains the invariant subspace of A associated with the eigenvalues outside γ and $\mathrm{Null}\,(I_n - P)$ contains the invariant subspace of A associated with the eigenvalues inside γ. Since P is a projection, $\mathrm{Null}\,(P)$ and $\mathrm{Null}\,(I_n - P) \equiv \mathrm{Range}\,(P)$ are complementary subspaces in $\in \mathbb{C}^n$. We conclude that P is the projection onto the invariant subspace of A corresponding to the eigenvalues inside γ. ∎

Final remarks

1. The results of this note can be directly extended to translated parabolas:

   $$\gamma_d = \{\lambda = x + iy \mid x + (p/2 - d) + iy \in \gamma\}$$

   of equation $y^2 = 2p(d - x)$ where d is a real parameter.
   The operator L_A defined in (2.4) becomes

   $$L_{A_d}(H) = 2p(p/2 - d)H + L_A(H) \quad \text{with} \quad A_d = A + (p/2 - d)I_n.$$

   Propositions 2.2 - 2.4 remain valid if γ and A are replaced by γ_d and A_d.
2. The main results of this note remain valid for a linear bounded operator A in a Hilbert space. More precisely:
   In Proposition 2.1 the fact that the condition (2.5) holds under the nonsingularity of L_A extends straightforwardly. Conversely, if the condition (2.5) is satisfied, then Theorem 3.2 in [1, Sec. I.3] ensures that L_A is nonsingular. In fact, this theorem ensures that L_A is nonsingular if (2.5) is satisfied for all λ and μ in the spectrum of A (i.e. the complementary in $\mathbb{C}$ of the resolvent set of A). Also, this theorem guarantees the existence and uniqueness of a more general linear operator equation (see page 21 in this reference).
   Propositions 2.2-2.4 remain valid with essentially the same proofs.
3. Using the same technique as in this note, one can easily treat the case where γ is the ellipse of equation $x^2/a^2 + y^2/b^2 = 1$. See also [3] for other results concerning this case.

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Acknowledgment. The authors would like to thank the referee who, among other interesting remarks, drew our attention to reference [1] and to the fact that the main results of this note can be extended to linear bounded operators in a Hilbert space.

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