

THE CAUCHY DOUBLE ALTERNANT AND DIVIDED DIFFERENCES*

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Dedicated to my friend Pier Vittorio Ceccherini on the occasion of his 65th birthday

Abstract. As an extension of Cauchy's double alternant, a general determinant evaluation formula is established. Several interesting determinant identities are derived as consequences by means of divided differences.

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1. Cauchy's Double Alternant and Extension. Cauchy's double alternant reads as

$$\Omega = \det_{0 \leq i, j \leq n} \left[\frac{1}{x_i + y_j} \right] = \frac{\prod_{0 \leq i < j \leq n} (x_i - x_j)(y_i - y_j)}{\prod_{0 \leq i, j \leq n} (x_i + y_j)}.$$

For the subsequent use, we denote a variant of it by

$$\Omega' = \det_{1 \leq i, j \leq n} \left[\frac{1}{x_i + y_j} \right] = \frac{\prod_{1 \leq i < j \leq n} (x_i - x_j)(y_i - y_j)}{\prod_{1 \leq i, j \leq n} (x_i + y_j)}.$$

In this paper, we extend it to the following determinant identities:

THEOREM 1 (Extension of Cauchy's double alternant).

$$M_n := \det_{0 \leq i, j \leq n} \left[w_j + \frac{u_i v_j}{x_i + y_j} \right] = \Omega \{1 + \Theta(x, y; u, v, w)\} \prod_{k=0}^n u_k v_k,$$

where $\Theta(x, y; u, v, w)$ is given by the following double sum:

$$\Theta(x, y; u, v, w) = \sum_{i, j=0}^n \frac{w_j}{u_i v_j (x_i + y_j)} \frac{\prod_{i=0}^n (x_i + y_j) \prod_{j=0}^n (x_i + y_j)}{\prod_{i \neq i} (x_i - x_i) \prod_{j \neq j} (y_j - y_j)}.$$

When $u_0 = 0$, it reduces easily to the following interesting result.

PROPOSITION 2 (Determinant identity).

$$\det_{0 \leq i, j \leq n} \left[w_j + \frac{u_i v_j}{x_i + y_j} \right]_{u_0=0} = \Omega' v_0 \prod_{\kappa=1}^n \frac{u_\kappa v_\kappa (y_0 - y_\kappa)}{(y_0 + x_\kappa)} \sum_{k=0}^n \frac{w_k}{v_k} \frac{\prod_{i=1}^n (x_i + y_k)}{\prod_{j \neq k} (y_k - y_j)}.$$

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Proof of Theorem 1. Consider the extended square matrix of order $(n+2) \times (2+n)$ given explicitly by

$$\begin{bmatrix} 1 & \vdots & w_j & \vdots & (0 \leq j \leq n) \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \vdots & w_j + \frac{u_i v_j}{x_i + y_j} & \vdots & (0 \leq i, j \leq n) \end{bmatrix},$$

whose determinant is obviously equal to the determinant stated in the theorem.

Now subtracting the first row from each other row, we transform the matrix into the following one:

$$\begin{bmatrix} 1 & \vdots & w_j & \vdots & (0 \leq j \leq n) \\ \dots & \dots & \dots & \dots & \dots \\ -1 & \vdots & \frac{u_i v_j}{x_i + y_j} & \vdots & (0 \leq i, j \leq n) \end{bmatrix}.$$

Then the Laplace expansion formula with respect to the first row gives

$$M_n = \det_{0 \leq i, j \leq n} \left[\frac{u_i v_j}{x_i + y_j} \right] + \sum_{j=0}^n (-1)^{3+j} w_j \det_{j \neq j} \left[-1 \vdots \frac{u_i v_j}{x_i + y_j} \right].$$

Expanding further the last determinant with respect to the first column, we get

$$\det_{j \neq j} \left[-1 \vdots \frac{u_i v_j}{x_i + y_j} \right] = \sum_{i=0}^n (-1)^{3+i} \det_{\substack{i \neq i \\ j \neq j}} \left[\frac{u_i v_j}{x_i + y_j} \right],$$

which leads us to the following expression:

$$M_n = \det_{0 \leq i, j \leq n} \left[\frac{u_i v_j}{x_i + y_j} \right] + \sum_{i, j=0}^n (-1)^{i+j} w_j \det_{\substack{i \neq i \\ j \neq j}} \left[\frac{u_i v_j}{x_i + y_j} \right].$$

Evaluating the last determinant by means of Cauchy's double alternant

$$\det_{\substack{i \neq i \\ j \neq j}} \left[\frac{u_i v_j}{x_i + y_j} \right] = \frac{(-1)^{i+j} \Omega \prod_{i=0}^n (x_i + y_j) \prod_{j=0}^n (x_i + y_j)}{u_i v_j (x_i + y_j) \prod_{i \neq i} (x_i - x_i) \prod_{j \neq j} (y_j - y_j)} \prod_{k=0}^n u_k v_k,$$

we find the determinant identity stated in the theorem. □

2. Divided Differences. In order to make the paper self-contained, we review some basic facts about divided differences. The details can be found in Lascoux [4, Chapter 7], where different notation has been introduced. For a complex function $f(y)$ and uneven spaced grid points $\{x_k\}_{k=0}^n$, the divided differences with respect to

y are defined in succession as follows:

$$\begin{aligned} \Delta[x_0, x_1]f(y) &= \frac{f(x_0) - f(x_1)}{x_0 - x_1}, \\ \Delta[x_0, x_1, x_2]f(y) &= \frac{\Delta[x_0, x_1]f(y) - \Delta[x_1, x_2]f(y)}{x_0 - x_2}, \\ &\dots \vdots \dots \\ \Delta[x_0, x_1, \dots, x_n]f(y) &= \frac{\Delta[x_0, x_1, \dots, x_{n-1}]f(y) - \Delta[x_1, x_2, \dots, x_n]f(y)}{x_0 - x_n}, \end{aligned}$$

which can also be expressed as

$$\Delta[x_0, x_1, \dots, x_n]f(y) = \left\{ \prod_{k=1}^n \Delta[x_k, y]f(y) \right\} \Big|_{y=x_0} \quad (2.1)$$

and the symmetric formula

$$\Delta[x_0, x_1, \dots, x_n]f(y) = \sum_{k=0}^n \frac{f(x_k)}{\prod_{i \neq k} (x_k - x_i)}. \quad (2.2)$$

For variables $X = \{x_0, x_1, \dots, x_n\}$, the elementary and complete symmetric functions in X are defined (cf. Macdonald [5, §1.2]), respectively, by

$$\begin{aligned} \mathbf{e}_0(X) &= 1 \text{ and } \mathbf{e}_m(X) = \sum_{0 \leq k_1 < k_2 < \dots < k_m \leq n} x_{k_1} x_{k_2} \dots x_{k_m} \text{ for } m = 1, 2, \dots; \\ \mathbf{h}_0(X) &= 1 \text{ and } \mathbf{h}_m(X) = \sum_{0 \leq k_1 \leq k_2 \leq \dots \leq k_m \leq n} x_{k_1} x_{k_2} \dots x_{k_m} \text{ for } m = 1, 2, \dots. \end{aligned}$$

Then the divided differences on monomials result in complete symmetric functions.

LEMMA 3 (Sylvester (1839), cf. Bhatnagar [1] and Chu [2]).

$$\begin{aligned} \Delta[x_0, x_1, \dots, x_n]y^m &= \sum_{k=0}^n \frac{x_k^m}{\prod_{i \neq k} (x_k - x_i)} \\ &= \begin{cases} 0, & m = 0, 1, \dots, n-1; \\ \mathbf{h}_{m-n}(x_0, x_1, \dots, x_n), & m = n, n+1, \dots; \\ \frac{(-1)^n}{x_0 x_1 \dots x_n} \mathbf{h}_{-1-m}\left(\frac{1}{x_0}, \frac{1}{x_1}, \dots, \frac{1}{x_n}\right), & m = -1, -2, -3, \dots. \end{cases} \end{aligned}$$

From this lemma, we display a short list of the divided differences for rational

functions, which will be used in the next section for determinant evaluation.

$$\Delta[X] \prod_{k=1}^n (y + u_k) = 1, \quad (2.3)$$

$$\Delta[X] \prod_{k=0}^n (y + u_k) = \sum_{k=0}^n (x_k + u_k), \quad (2.4)$$

$$\Delta[X] \prod_{k=0}^{n+1} (y + u_k) = \sum_{k=0}^n x_k^2 + \mathbf{e}_2(X, U), \quad (2.5)$$

$$\Delta[X] \frac{1}{y + v} = \frac{(-1)^n}{\prod_{k=0}^n (v + x_k)}, \quad (2.6)$$

$$\Delta[X] \frac{\prod_{k=1}^n (y + u_k)}{y + v} = \frac{\prod_{k=1}^n (v - u_k)}{\prod_{k=0}^n (v + x_k)}, \quad (2.7)$$

$$\Delta[X] \frac{\prod_{k=0}^n (y + u_k)}{y + v} = 1 - \frac{\prod_{k=0}^n (v - u_k)}{\prod_{k=0}^n (v + x_k)}, \quad (2.8)$$

$$\Delta[X] \frac{\prod_{k=0}^{n+1} (y + u_k)}{y + v} = \mathbf{e}_1(U, X) - v + \frac{\prod_{i=0}^{n+1} (v - u_i)}{\prod_{j=0}^n (v + x_j)}. \quad (2.9)$$

3. Determinant Identities. Suppose that u_i , v_j and w_j are the three functions given by

$$u_i := u(x_i), \quad v_j := v(y_j) \quad \text{and} \quad w_j := w(y_j).$$

Then we can express the double Θ -sum in Theorem 1 and the sum with respect to k in Proposition 2 in terms of divided differences:

$$\begin{aligned} \Theta(x, y; u, v, w) &= \Delta_x[x_0, x_1, \dots, x_n] \Delta_y[y_0, y_1, \dots, y_n] \\ &\times \left\{ \frac{w(y)}{u(x)v(y)} \frac{\prod_{k=0}^n (x_k + y)(x + y_k)}{x + y} \right\}, \end{aligned} \quad (3.1)$$

$$\sum_{k=0}^n \frac{w_k}{v_k} \frac{\prod_{i=1}^n (x_i + y_k)}{\prod_{j \neq k} (y_k - y_j)} = \Delta_y[y_0, y_1, \dots, y_n] \left\{ \frac{w(y)}{v(y)} \prod_{k=1}^n (x_k + y) \right\}. \quad (3.2)$$

Applying the divided difference formulae displayed in the last section, we can derive without difficulty the following determinant identities.

EXAMPLE 1 ($w_k = v_k = 1$ in Proposition 2).

$$\det_{0 \leq i, j \leq n} \left[\frac{u_i + x_i + y_j}{x_i + y_j} \right]_{u_0=0} = \Omega' \prod_{k=1}^n \frac{u_k(y_0 - y_k)}{(y_0 + x_k)}.$$

EXAMPLE 2 ($w_k = 1$ and $v_k = v + y_k$ in Proposition 2).

$$\det_{0 \leq i, j \leq n} \left[1 + \frac{u_i(v + y_j)}{x_i + y_j} \right]_{u_0=0} = \Omega' \prod_{k=1}^n \frac{u_k(y_0 - y_k)(v - x_k)}{(y_0 + x_k)}.$$

EXAMPLE 3 ($w_k = w + y_k$ and $v_k = 1$ in Proposition 2).

$$\det_{0 \leq i, j \leq n} \left[w + y_j + \frac{u_i}{x_i + y_j} \right]_{u_0=0} = \Omega' \left\{ w - x_0 + \sum_{k=0}^n (x_k + y_k) \right\} \prod_{k=1}^n \frac{u_k (y_0 - y_k)}{(y_0 + x_k)}.$$

EXAMPLE 4 ($w_k = w + y_k$ and $v_k = v + y_k$ in Proposition 2).

$$\det_{0 \leq i, j \leq n} \left[w + y_j + \frac{u_i (v + y_j)}{x_i + y_j} \right]_{u_0=0} = \Omega' \prod_{k=1}^n \frac{u_k (y_0 - y_k) (v - x_k)}{(y_0 + x_k)} \\ \times \left\{ (w - v) + \frac{\prod_{i=0}^n (v + y_i)}{\prod_{j=1}^n (v - x_j)} \right\}.$$

EXAMPLE 5 ($w_k = (w + y_k)(v + y_k)$ and $v_k = 1$ in Proposition 2).

$$\det_{0 \leq i, j \leq n} \left[(w + y_j)(v + y_j) + \frac{u_i}{x_i + y_j} \right]_{u_0=0} = \Omega' \prod_{k=1}^n \frac{u_k (y_0 - y_k)}{(y_0 + x_k)} \\ \times \left\{ wv + (w + v) \mathbf{e}_1(X, Y) + \mathbf{e}_2(X, Y) + \sum_{\kappa=0}^n y_\kappa^2 \right\}.$$

EXAMPLE 6 ($u_k = v_k = 1$ and $w_k = 1/w$ in Theorem 1).

$$\det_{0 \leq i, j \leq n} \left[\frac{w + x_i + y_j}{x_i + y_j} \right] = \Omega w^n \left\{ w + \mathbf{e}_1(X, Y) \right\}.$$

EXAMPLE 7 ($u_k = u + x_k$, $v_k = v$ and $w_k = w$ in Theorem 1).

$$\det_{0 \leq i, j \leq n} \left[w + \frac{(u + x_i)v}{x_i + y_j} \right] = \Omega v^n \left\{ v + w - w \prod_{\kappa=0}^n \frac{u - y_\kappa}{v + x_\kappa} \right\} \prod_{\kappa=0}^n (u + x_\kappa).$$

EXAMPLE 8 ($u_k = u$, $v_k = v$ and $w_k = w + y_k$ in Theorem 1).

$$\det_{0 \leq i, j \leq n} \left[w + y_j + \frac{uv}{x_i + y_j} \right] = \Omega (uv)^n \left\{ uv + w \mathbf{e}_1(X, Y) + \mathbf{e}_2(X, Y) + \sum_{\kappa=0}^n y_\kappa^2 \right\}.$$

EXAMPLE 9 ($u_k = u + x_k$, $v_k = v$ and $w_k = u - y_k$ in Theorem 1).

$$\det_{0 \leq i, j \leq n} \left[u - y_j + \frac{(u + x_i)v}{x_i + y_j} \right] = \Omega v^n \left\{ v + \mathbf{e}_1(X, Y) \prod_{\kappa=0}^n \frac{u - y_\kappa}{u + x_\kappa} \right\} \prod_{\kappa=0}^n (u + x_\kappa).$$

EXAMPLE 10 ($u_k = 1/(u + x_k)$, $v_k = v$ and $w_k = w$ in Theorem 1).

$$\det_{0 \leq i, j \leq n} \left[w(u + x_i) + \frac{v}{x_i + y_j} \right] = \Omega v^n w \left\{ \frac{v}{w} + u \mathbf{e}_1(X, Y) + \mathbf{h}_2(X, Y) - \sum_{\kappa=0}^n y_\kappa^2 \right\}.$$

EXAMPLE 11 ($u_k = 1/(u + x_k)$, $v_k = u - y_k$ and $w_k = w$ in Theorem 1).

$$\det_{0 \leq i, j \leq n} \left[(u + x_i)w + \frac{(u - y_j)}{x_i + y_j} \right] = \Omega \left\{ 1 + w \mathbf{e}_1(X, Y) \prod_{\kappa=0}^n \frac{u + x_\kappa}{u - y_\kappa} \right\} \prod_{\kappa=0}^n (u - y_\kappa).$$

EXAMPLE 12 ($u_k = u$, $v_k = v + y_k$ and $w_k = w + y_k$ in Theorem 1).

$$\begin{aligned} \det_{0 \leq i, j \leq n} \left[w + y_j + \frac{u(v + y_j)}{x_i + y_j} \right] &= \Omega u^n \prod_{\kappa=0}^n (v + y_\kappa) \\ &\times \left\{ u - v + w + \mathbf{e}_1(X, Y) - (w - v) \prod_{\kappa=0}^n \frac{v - x_\kappa}{v + y_\kappa} \right\}. \end{aligned}$$

EXAMPLE 13 ($u_k = u + x_k$, $v_k = v$ and $w_k = w + y_k$ in Theorem 1).

$$\begin{aligned} \det_{0 \leq i, j \leq n} \left[w + y_j + \frac{(u + x_i)v}{x_i + y_j} \right] &= \Omega v^n (u + v + w) \prod_{\kappa=0}^n (u + x_\kappa) \\ &\times \left\{ 1 - \frac{u + w + \mathbf{e}_1(X, Y)}{u + v + w} \prod_{\kappa=0}^n \frac{u - y_\kappa}{u + x_\kappa} \right\}. \end{aligned}$$

EXAMPLE 14 ($u_k = u + x_k$, $v_k = v$ and $w_k = (w + y_k)(u - y_k)$ in Theorem 1).

$$\begin{aligned} \det_{0 \leq i, j \leq n} \left[(u - y_j)(w + y_j) + \frac{(u + x_i)v}{x_i + y_j} \right] &= \Omega v^n \prod_{\kappa=0}^n (u + x_\kappa) \\ &\times \left\{ v + \left(\mathbf{e}_2(X, Y, w) + \sum_{\kappa=0}^n y_\kappa^2 \right) \prod_{\kappa=0}^n \frac{u - y_\kappa}{u + x_\kappa} \right\}. \end{aligned}$$

EXAMPLE 15 ($u_k = 1/(u + x_k)$, $v_k = u - y_k$ and $w_k = w + y_k$ in Theorem 1).

$$\begin{aligned} \det_{0 \leq i, j \leq n} \left[(u + x_i)(w + y_j) + \frac{u - y_j}{x_i + y_j} \right] &= \Omega \left\{ 1 - u \mathbf{e}_1(X, Y) - \mathbf{h}_2(X, Y) + \sum_{\kappa=0}^n y_\kappa^2 \right. \\ &\left. + (w + u) \mathbf{e}_1(X, Y) \prod_{\kappa=0}^n \frac{u + x_\kappa}{u - y_\kappa} \right\} \prod_{\kappa=0}^n (u - y_\kappa). \end{aligned}$$

In order to illustrate the method of proof, we prove the determinant identity displayed in Example 3 in detail. According to Proposition 2, we need only to evaluate

(3.2) for $v(y) = 1$ and $w(y) = w + y$. In this case, the corresponding divided differences (3.2) can be evaluated by means of (2.4) as follows:

$$\begin{aligned} & \Delta_y[y_0, y_1, \dots, y_n] \left\{ \frac{w(y)}{v(y)} \prod_{k=1}^n (x_k + y) \right\} \\ &= \Delta_y[y_0, y_1, \dots, y_n] \left\{ (w + y) \prod_{k=1}^n (x_k + y) \right\} \\ &= w - x_0 + \sum_{k=0}^n (x_k + y_k). \end{aligned}$$

Then the determinant identity in Example 3 follows immediately.

In particular, Example 3 implies another interesting determinant identity. Reformulating the general entry of the matrix

$$\frac{(a + u_i + w_j)(c + v_i + w_j)}{c + u_i + v_i + w_j} = a + w_j + \frac{u_i(c - a + v_i)}{c + u_i + v_i + w_j},$$

we derive from Example 3 the following curious formula.

PROPOSITION 4 (Determinant identity).

$$\begin{aligned} \det_{0 \leq i, j \leq n} \left[\frac{(a + u_i + w_j)(c + v_i + w_j)}{c + u_i + v_i + w_j} \right]_{u_0=0} &= \frac{\prod_{1 \leq i < j \leq n} (u_i + v_i - u_j - v_j)(w_i - w_j)}{\prod_{1 \leq i, j \leq n} (c + u_i + v_i + w_j)} \\ &\times \left\{ a + cn + w_0 + \sum_{k=1}^n (u_k + v_k + w_k) \right\} \prod_{k=1}^n \frac{u_k(c - a + v_k)(w_0 - w_k)}{c + u_k + v_k + w_0}. \end{aligned}$$

The very special case $u_k = ku$ of this identity reduces to the determinant evaluation.

COROLLARY 5 (Krattenthaler [3, Eq 5.3]).

$$\begin{aligned} \det_{0 \leq i, j \leq n} \left[\frac{(a + ui + w_j)(c + v_i + w_j)}{c + ui + v_i + w_j} \right] &= \frac{\prod_{1 \leq i < j \leq n} (ui - uj + v_i - v_j)(w_i - w_j)}{\prod_{1 \leq i, j \leq n} (c + ui + v_i + w_j)} \\ &\times n! u^n \left\{ a + cn + u \binom{n+1}{2} + w_0 + \sum_{k=1}^n (v_k + w_k) \right\} \prod_{k=1}^n \frac{(c - a + v_k)(w_0 - w_k)}{c + uk + v_k + w_0}. \end{aligned}$$

Krattenthaler [3, Eq 5.3] discovered this identity by means of the condensation method, which has been the author's primary motivation for this work.

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