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SUBDIRECT SUMS OF S-STRICTLY DIAGONALLY DOMINANT MATRICES*

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Abstract. Conditions are given which guarantee that the k-subdirect sum of S-strictly diagonally dominant matrices (S-SDD) is also S-SDD. The same situation is analyzed for SDD matrices. The converse is also studied: given an SDD matrix C with the structure of a k-subdirect sum and positive diagonal entries, it is shown that there are two SDD matrices whose subdirect sum is C.

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Key words. Subdirect sum, Diagonally dominant matrices, Overlapping blocks.

1. Introduction. The concept of k-subdirect sum of square matrices emerges naturally in several contexts. For example, in matrix completion problems, overlapping subdomains in domain decomposition methods, global stiffness matrix in finite elements, etc.; see, e.g., [1], [2], [5], and references therein.

Subdirect sums of matrices are generalizations of the usual sum of matrices (a k-subdirect sum is formally defined below in section 2). They were introduced by Fallat and Johnson in [5], where many of their properties were analyzed. For example, they showed that the subdirect sum of positive definite matrices, or of symmetric M-matrices, is positive definite or symmetric M-matrices, respectively. They also showed that this is not the case for M-matrices: the subdirect sum of two M-matrices may not be an M-matrix, and therefore the subdirect sum of two H-matrices may not be an H-matrix.

In this paper we show that for a subclass of H-matrices the k-subdirect sum of matrices belongs to the same class. We show this for certain strictly diagonally dominant matrices (SDD) and for S-strictly diagonally dominant matrices (S-SDD), introduced in [4]; see also [3], [9], for further properties and analysis. We also show that the converse holds: given an SDD matrix C with the structure of a k-subdirect sum and positive diagonal entries, then there are two SDD matrices whose subdirect sum is C.

2. Subdirect sums. Let A and B be two square matrices of order n_1 and n_2, respectively, and let k be an integer such that $1 \leq k \leq \min(n_1, n_2)$. Let A and B be partitioned into 2×2 blocks as follows,

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}, \tag{2.1}$$

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where A_{22} and B_{11} are square matrices of order k. Following [5], we call the square matrix of order $n = n_1 + n_2 - k$ given by

$$C = \begin{bmatrix} A_{11} & A_{12} & O \\ A_{21} & A_{22} + B_{11} & B_{12} \\ O & B_{21} & B_{22} \end{bmatrix} \tag{2.2}$$

the k-subdirect sum of A and B and denote it by $C = A \oplus_k B$.

It is easy to express each element of C in terms of those of A and B. To that end, let us define the following set of indices

$$\begin{array}{rcl} S_1 & = & \{1, 2, \ldots, n_1 - k\}, \\ S_2 & = & \{n_1 - k + 1, n_1 - k + 2, \ldots, n_1\}, \\ S_3 & = & \{n_1 + 1, n_1 + 2, \ldots, n\}. \end{array} \tag{2.3}$$

Denoting $C = (c_{ij})$ and $t = n_1 - k$, we can write

$$c_{ij} = \begin{cases} a_{ij} & i \in S_1, \quad j \in S_1 \cup S_2 \\ 0 & i \in S_1, \quad j \in S_3 \\ a_{ij} & i \in S_2, \quad j \in S_1 \\ a_{ij} + b_{i-t,j-t} & i \in S_2, \quad j \in S_2 \\ b_{i-t,j-t} & i \in S_2, \quad j \in S_3 \\ 0 & i \in S_3, \quad j \in S_1 \\ b_{i-t,j-t} & i \in S_3, \quad j \in S_2 \cup S_3. \end{cases} \tag{2.4}$$

Note that $S_1 \cup S_2 \cup S_3 = \{1, 2, \ldots, n\}$ and that $n = t + n_2$; see Figure 2.1.

$$\begin{array}{c} \\ C = \end{array} \begin{array}{cc} \begin{array}{ccc} \overleftrightarrow{\;S_1\;} & \overleftrightarrow{\qquad\qquad S_2 \qquad\qquad} & \overleftrightarrow{\qquad S_3 \qquad} \end{array} & \\ \begin{pmatrix} a_{11} & \cdots & a_{1p} & \cdots & a_{1,n_1} & \cdots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{p,1} & \cdots & a_{p,p} + b_{1,1} & \cdots & a_{p,n_1} + b_{1,n_1-t} & \cdots & b_{1,n-t} \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{n_1,1} & \cdots & a_{n_1,p} + b_{n_1-t,1} & \cdots & a_{n_1,n_1} + b_{n_1-t,n_1-t} & \cdots & b_{n_1-t,n-t} \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \cdots & b_{n-t,1} & \cdots & b_{n-t,n_1-t} & \cdots & b_{n-t,n-t} \end{pmatrix} & \begin{array}{c} S_1 \\ \\ S_2 \\ \\ S_3 \end{array} \end{array}$$

FIG. 2.1. *Sets for the subdirect sum $C = A \oplus_k B$, with $t = n_1 - k$ and $p = t + 1$; cf.* (2.4).

3. Subdirect sums of S-SDD matrices. We begin with some definitions which can be found, e.g., in [4], [9].

DEFINITION 3.1. Given a matrix $A = (a_{ij}) \in \mathbb{C}^{n \times n}$, let us define the ith deleted absolute row sum as

$$r_i(A) = \sum_{j \neq i,\, j=1}^{n} |a_{ij}|, \quad \forall i = 1, 2, \ldots, n,$$

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and the ith deleted absolute row-sum with columns in the set of indices $S = \{i_1, i_2, \ldots\} \subseteq N := \{1, 2, \ldots, n\}$ as

$$r_i^S(A) = \sum_{j \neq i,\, j \in S} |a_{ij}|, \quad \forall i = 1, 2, \ldots, n.$$

Given any nonempty set of indices $S \subseteq N$ we denote its complement in N by $\bar{S} := N \backslash S$. Note that for any $A = (a_{ij}) \in \mathbb{C}^{n \times n}$ we have that $r_i(A) = r_i^S(A) + r_i^{\bar{S}}(A)$.

Definition 3.2. Given a matrix $A = (a_{ij}) \in \mathbb{C}^{n \times n}$, $n \geq 2$ and given a nonempty subset S of $\{1, 2, \ldots, n\}$, then A is an S-strictly diagonally dominant matrix if the following two conditions hold:

$$\left.\begin{array}{lll} i) & |a_{ii}| > r_i^S(A) & \forall i \in S, \\ ii) & (|a_{ii}| - r_i^S(A))\,(|a_{jj}| - r_j^{\bar{S}}(A)) > r_i^{\bar{S}}(A)\, r_j^S(A) & \forall i \in S, \forall j \in \bar{S}. \end{array}\right\} \tag{3.1}$$

It was shown in [4] that an S-strictly diagonally dominant matrix (S-SDD) is a nonsingular H-matrix. In particular, when $S = \{1, 2, \ldots, n\}$, then $A = (a_{ij}) \in C^{n \times n}$ is a strictly diagonally dominant matrix (SDD). It is easy to show that an SDD matrix is an S-SDD matrix for any proper subset S, but the converse is not always true as we show in the following example.

Example 3.3. Consider the following matrix

$$A = \begin{bmatrix} 2.6 & -0.4 & -0.7 & -0.2 \\ -0.4 & 2.6 & -0.5 & -0.7 \\ -0.6 & -0.7 & 2.2 & -1.0 \\ -0.8 & -0.7 & -0.5 & 2.2 \end{bmatrix},$$

which is a $\{1, 2\}$-SDD matrix but is not an SDD matrix. A natural question is to ask if the subdirect sum of S-SDD matrices is in the class, but in general this is not true. For example, the 2-subdirect sum $C = A \oplus_2 A$ gives

$$C = \begin{bmatrix} 2.6 & -0.4 & -0.7 & -0.2 & 0 & 0 \\ -0.4 & 2.6 & -0.5 & -0.7 & 0 & 0 \\ -0.6 & -0.7 & 4.8 & -1.4 & -0.7 & -0.2 \\ -0.8 & -0.7 & -0.9 & 4.8 & -0.5 & -0.7 \\ 0 & 0 & -0.6 & -0.7 & 2.2 & -1.0 \\ 0 & 0 & -0.8 & -0.7 & -0.5 & 2.2 \end{bmatrix}$$

which is not a $\{1, 2\}$-SDD matrix: condition ii) of (3.1) fails for the matrix C for the cases $i = 1, j = 5$ and $i = 2, j = 5$. It can also be observed that C is not an SDD matrix.

This example motivates the search of conditions such that the subdirect sum of S-SDD matrices is in the class of S-SDD matrices (for a fixed set S).

We now proceed to show our first result. Let A and B be matrices of order n_1 and n_2, respectively, partitioned as in (2.1) and consider the sets S_i defined in (2.3). Then we have the following relations

$$\left.\begin{array}{rcl} r_i^{S_1}(C) & = & r_i^{S_1}(A) \\ r_i^{S_2 \cup S_3}(C) & = & r_i^{S_2}(A) \end{array}\right\}, \quad i \in S_1, \tag{3.2}$$

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which are easily derived from (2.4).

THEOREM 3.4. *Let A and B be matrices of order n_1 and n_2, respectively. Let $n_1 \geq 2$, and let k be an integer such that $1 \leq k \leq \min(n_1, n_2)$, which defines the sets S_1, S_2, S_3 as in* (2.3). *Let A and B be partitioned as in* (2.1). *Let S be a set of indices of the form $S = \{1, 2, \ldots\}$. Let A be S-strictly diagonally dominant, with $card(S) \leq card(S_1)$, and let B be strictly diagonally dominant. If all diagonal entries of A_{22} and B_{11} are positive (or all negative), then the k-subdirect sum $C = A \oplus_k B$ is S-strictly diagonally dominant, and therefore nonsingular.*

Proof. We first prove the case when $S = S_1$. Since A is S_1-strictly diagonally dominant, we have that

$$\left.\begin{array}{lll} i) & |a_{ii}| > r_i^{S_1}(A) & \forall i \in S_1, \\ ii) & (|a_{ii}| - r_i^{S_1}(A))\,(|a_{jj}| - r_j^{S_2}(A)) > r_i^{S_2}(A)\, r_j^{S_1}(A) & \forall i \in S_1, \forall j \in S_2. \end{array}\right\} \tag{3.3}$$

Note that A is of order n_1 and then the complement of S_1 in $\{1, 2, \ldots, n_1\}$ is S_2.

We want to show that C is also an S_1-strictly diagonally dominant matrix, i.e., we have to show that

$$\begin{array}{ll} 1) & |c_{ii}| > r_i^{S_1}(C) \quad \forall i \in S_1, \quad \text{and} \\ 2) & (|c_{ii}| - r_i^{S_1}(C))\,(|c_{jj}| - r_j^{S_2 \cup S_3}(C)) > r_i^{S_2 \cup S_3}(C)\, r_j^{S_1}(C) \;\; \forall i \in S_1,\, \forall j \in S_2 \cup S_3. \end{array} \tag{3.4}$$

Note that since C is of order n, the complement of S_1 in $\{1, 2, \ldots, n\}$ is $S_2 \cup S_3$.

To see that 1) holds we use equations (2.4), (3.2) and part i) of (3.3) (see also Figure 2.1) to obtain

$$|c_{ii}| = |a_{ii}| > r_i^{S_1}(A) = r_i^{S_1}(C), \quad \forall i \in S_1.$$

To see that 2) holds we distinguish two cases: $j \in S_2$ and $j \in S_3$. If $j \in S_2$, from (2.4) we have the following relations (recall that $t = n_1 - k$):

$$r_j^{S_1}(C) = \sum_{j \neq k,\, k \in S_1} |c_{jk}| = \sum_{j \neq k,\, k \in S_1} |a_{jk}| = r_j^{S_1}(A), \tag{3.5}$$

$$\begin{aligned} r_j^{S_2 \cup S_3}(C) &= \sum_{j \neq k,\, k \in S_2 \cup S_3} |c_{jk}| = \sum_{j \neq k,\, k \in S_2} |c_{jk}| + \sum_{j \neq k,\, k \in S_3} |c_{jk}| \\ &= r_j^{S_2}(C) + r_j^{S_3}(C), \end{aligned} \tag{3.6}$$

$$r_j^{S_2}(C) = \sum_{j \neq k,\, k \in S_2} |a_{jk} + b_{j-t,k-t}|, \tag{3.7}$$

$$r_j^{S_3}(C) = \sum_{j \neq k,\, k \in S_3} |b_{j-t,k-t}| = r_{j-t}^{S_3}(B), \tag{3.8}$$

$$c_{jj} = a_{jj} + b_{j-t,j-t}. \tag{3.9}$$

Therefore we can write

$$\begin{aligned} &(|c_{ii}| - r_i^{S_1}(C))\,(|c_{jj}| - r_j^{S_2 \cup S_3}(C)) = \\ &(|a_{ii}| - r_i^{S_1}(A))\,(|a_{jj} + b_{j-t,j-t}| - r_j^{S_2}(C) - r_j^{S_3}(C)),\, \forall i \in S_1,\, \forall j \in S_2, \end{aligned} \tag{3.10}$$

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where we have used that $c_{ii} = a_{ii}$, for $i \in S_1$ and equations (3.2), (3.6) and (3.9). Using now that A_{22} and B_{11} have positive diagonal (or both negative diagonal) we have that $|a_{jj} + b_{j-t,j-t}| = |a_{jj}| + |b_{j-t,j-t}|$ and therefore we can rewrite (3.10) as

$$\begin{aligned}&(|c_{ii}| - r_i^{S_1}(C))\,(|c_{jj}| - r_j^{S_2\cup S_3}(C)) = \\ &(|a_{ii}| - r_i^{S_1}(A))\,(|a_{jj}| + |b_{j-t,j-t}| - r_j^{S_2}(C) - r_j^{S_3}(C)),\ \forall i \in S_1,\ \forall j \in S_2.\end{aligned} \tag{3.11}$$

Let us now focus on the second term of the right hand side of (3.11). Observe that from (3.7) and the triangle inequality we have that

$$\begin{aligned} r_j^{S_2}(C) = \sum_{j\neq k,\ k\in S_2} |a_{jk} + b_{j-t,k-t}| \ &\leq \sum_{j\neq k,\ k\in S_2} |a_{jk}| + \sum_{j\neq k,\ k\in S_2} |b_{j-t,k-t}| \\ = r_j^{S_2}(A) + r_{j-t}^{S_2}(B) & \end{aligned} \tag{3.12}$$

and using (3.8), from (3.12) we can write the inequality

$$|a_{jj}| + |b_{j-t,j-t}| - r_j^{S_2}(C) - r_j^{S_3}(C) \geq |a_{jj}| + |b_{j-t,j-t}| - r_j^{S_2}(A) - r_{j-t}^{S_2}(B) - r_{j-t}^{S_3}(B).$$

Since we have $r_{j-t}^{S_2}(B) + r_{j-t}^{S_3}(B) = r_{j-t}^{S_2\cup S_3}(B)$, we obtain

$$|a_{jj}| + |b_{j-t,j-t}| - r_j^{S_2}(C) - r_j^{S_3}(C) \geq |a_{jj}| - r_j^{S_2}(A) + |b_{j-t,j-t}| - r_{j-t}^{S_2\cup S_3}(B),$$

which allows us to transform (3.11) into the following inequality

$$\begin{aligned}&(|c_{ii}| - r_i^{S_1}(C))\,(|c_{jj}| - r_j^{S_2\cup S_3}(C)) \geq \\ &(|a_{ii}| - r_i^{S_1}(A))\,(|a_{jj}| - r_j^{S_2}(A) + |b_{j-t,j-t}| - r_{j-t}^{S_2\cup S_3}(B)),\ \forall i \in S_1,\ \forall j \in S_2,\end{aligned} \tag{3.13}$$

where we have used that $(|a_{ii}| - r_i^{S_1}(A))$ is positive since A is S_1-strictly diagonally dominant. Observe now that $|b_{j-t,j-t}| - r_{j-t}^{S_2\cup S_3}(B)$ is also positive since B is strictly diagonally dominant, and thus we can write

$$|a_{jj}| - r_j^{S_2}(A) + |b_{j-t,j-t}| - r_{j-t}^{S_2\cup S_3}(B) > |a_{jj}| - r_j^{S_2}(A)$$

which jointly with (3.13) leads to the strict inequality

$$(|c_{ii}| - r_i^{S_1}(C))\,(|c_{jj}| - r_j^{S_2\cup S_3}(C)) > (|a_{ii}| - r_i^{S_1}(A))\,(|a_{jj}| - r_j^{S_2}(A)), \tag{3.14}$$

for all $i \in S_1$ and for all $j \in S_2$, Finally, using (ii) of (3.3) (i.e., the fact that A is S_1-strictly diagonally dominant) and equations (3.2) and (3.5) we can write the inequality

$$(|a_{ii}| - r_i^{S_1}(A))\,(|a_{jj}| - r_j^{S_2}(A)) > r_i^{S_2}(A)\, r_j^{S_1}(A) = r_i^{S_2\cup S_3}(C)\, r_j^{S_1}(C)$$

for all $i \in S_1$ and for all $j \in S_2$, which allows to transform equation (3.14) into the inequality

$$(|c_{ii}| - r_i^{S_1}(C))\,(|c_{jj}| - r_j^{S_2\cup S_3}(C)) > r_i^{S_2\cup S_3}(C)\, r_j^{S_1}(C),\ \forall i \in S_1,\ \forall j \in S_2.$$

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Therefore we have proved condition 2) for the case $j \in S_2$.

In the case $j \in S_3$, we have from (2.4) that

$$r_j^{S_1}(C) = \sum_{j \neq k,\, k \in S_1} |c_{jk}| = 0.$$

Therefore the condition 2) of (3.4) becomes

$$(|c_{ii}| - r_i^{S_1}(C))\,(|c_{jj}| - r_j^{S_2 \cup S_3}(C)) > 0, \quad \forall i \in S_1, \forall j \in S_3, \tag{3.15}$$

and it is easy to show that this inequality is fulfilled. The first term is positive since, as before, we have that $|c_{ii}| - r_i^{S_1}(C) = |a_{ii}| - r_i^{S_1}(A) > 0$. The second term of (3.15) is also positive since we have that $c_{jj} = b_{j-t,j-t}$ for all $j \in S_3$ and

$$r_j^{S_2 \cup S_3}(C) = \sum_{j \neq k,\, k \in S_2 \cup S_3} |c_{jk}| = \sum_{j \neq k,\, k \in S_2 \cup S_3} |b_{j-t,k-t}| = r_{j-t}^{S_2 \cup S_3}(B),\ \forall j \in S_3,$$

and since B is strictly diagonally dominant we have

$$|b_{j-t,j-t}| - r_{j-t}^{S_2 \cup S_3}(B) > 0,\ \forall j \in S_3.$$

Therefore equation (3.15) is fulfilled and the proof for the case $S = S_1$ is completed.

When $card(S) < card(S_1)$ the proof is analogous. We only indicate that the key point in this case is the subcase $j \in S_1 \backslash S$ for which it is easy to show that a condition similar to 2) for C in (3.4) still holds. □

When $card(S) > card(S_1)$ the preceding theorem is not valid as we show in the following example.

Example 3.5. In this example we show a matrix A that is an S-SDD matrix with $card(S) > card(S_1)$ and a matrix B that is an SDD matrix but the subdirect sum C is not an S-SDD matrix. Let the following matrices A and B be partitioned as

$$A = \left[\begin{array}{r|rrr} 1.0 & -0.3 & -0.4 & -0.5 \\ \hline -0.9 & 1.6 & -0.4 & -0.7 \\ -0.1 & -0.4 & 1.3 & -0.4 \\ -0.1 & -0.9 & -0.1 & 2.0 \end{array}\right] \quad \text{and} \quad B = \left[\begin{array}{rrr|r} 2.0 & 0.2 & -0.3 & -0.1 \\ 0.8 & 2.9 & -0.2 & -0.5 \\ -0.5 & -0.1 & 2.4 & -0.9 \\ \hline -0.6 & -0.8 & -0.8 & 2.3 \end{array}\right].$$

We have from (2.3) that $S_1 = \{1\}$, $S_2 = \{2, 3, 4\}$ and $S_3 = \{5\}$. It is easy to show that A is $\{1, 2\}$-SDD, A is not SDD, and B is SDD. The 3-subdirect sum $C = A \oplus_3 B$

$$C = \left[\begin{array}{r|rrr|r} 1.0 & -0.3 & -0.4 & -0.5 & 0 \\ \hline -0.9 & 3.6 & -0.2 & -1.0 & -0.1 \\ -0.1 & 0.4 & 4.2 & -0.6 & -0.5 \\ -0.1 & -1.4 & -0.2 & 4.4 & -0.9 \\ \hline 0 & -0.6 & -0.8 & -0.8 & 2.3 \end{array}\right]$$

is not a $\{1, 2\}$-SDD: the corresponding condition ii) for C in equation (3.1) fails for $i = 1, j = 5$.

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REMARK 3.6. An analogous result to Theorem 3.4 can be obtained when the matrix B is S-strictly diagonally dominant with $S = \{n_1 + 1, n_1 + 2, \ldots\}$, $card(S) \leq card(S_3)$, and the matrix A is strictly diagonally dominant. The proof is completely analogous, and thus we omit the details.

It is easy to show that if A is a strictly diagonally dominant matrix, then A is also an S_1-strictly diagonally dominant matrix. Therefore we have the following corollary.

COROLLARY 3.7. *Let A and B be matrices of order n_1 and n_2, respectively, and let k be an integer such that $1 \leq k \leq \min(n_1, n_2)$. Let A and B be partitioned as in* (2.1). *If A and B are strictly diagonally dominant and all diagonal entries of A_{22} and B_{11} are positive, then the k-subdirect sum $C = A \oplus_k B$ is strictly diagonally dominant, and therefore nonsingular.*

REMARK 3.8. In the general case of successive k-subdirect sums of the form

$$(A_1 \oplus_{k_1} A_2) \oplus_{k_2} A_3 \oplus \cdots$$

when A_1 is S-SDD with $card(S) \leq n_1 - k_1$ and A_2, A_3, ..., are SDD matrices, we have that all the subdirect sums are S-SDD matrices, provided that in each particular subdirect sum the quantity $card(S)$ is no larger than the corresponding overlap, in accordance with Theorem 3.4.

4. Overlapping SDD matrices. In this section we consider the case of square matrices A and B of order n_1 and n_2, respectively, which are principal submatrices of a given SDD matrix, and such that they have a common block with positive diagonals. This situation, as well as a more general case outlined in Theorem 4.1 later in this section, appears in many variants of additive Schwarz preconditioning; see, e.g., [2], [6], [7], [8]. Specifically, let

$$M = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix}$$

be an SDD matrix of order n, with $n = n_1 + n_2 - k$, and with M_{22} a square matrix of order k, such that its diagonal is positive. Let us consider two principal submatrices of M, namely

$$A = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}, \quad B = \begin{bmatrix} M_{22} & M_{23} \\ M_{32} & M_{33} \end{bmatrix}.$$

Therefore the k-subdirect sum of A and B is given by

$$C = A \oplus_k B = \begin{bmatrix} M_{11} & M_{12} & O \\ M_{21} & 2M_{22} & M_{23} \\ O & M_{32} & M_{33} \end{bmatrix}. \tag{4.1}$$

Since A and B are SDD matrices, according to Corollary 3.7 the subdirect sum given by equation (4.1) is also an SDD matrix. This result can clearly be extended to the sum of p overlapping submatrices of a given SDD matrix with positive diagonal entries. We summarize this result formally as follows; cf. a similar result for

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M-matrices in [1]. Here, we consider consecutive principal submatrices defined by consecutive indices of the form $\{i, i+1, i+2, \ldots\}$.

THEOREM 4.1. *Let M be an SDD matrix with positive diagonal entries. Let A_i, $i = 1, \ldots, p$, be consecutive principal submatrices of M of order n_i, and consider the $p-1$ k_i-subdirect sums given by*

$$C_i = C_{i-1} \oplus_{k_i} A_{i+1}, \quad i = 1, \ldots, p-1$$

in which $C_0 = A_1$, and $k_i < \min(n_i, n_{i+1})$. Then each of the k_i-subdirect sums C_i is an SDD matrix, and in particular

$$C_{p-1} = A_1 \oplus_{k_1} A_2 \oplus_{k_2} \cdots \oplus_{k_p} A_p \tag{4.2}$$

is an SDD matrix.

5. SDD matrices with the structure of a subdirect sum. We address the following question. Let C be square of order n, an SDD matrix with positive diagonal entries, and having the structure of a k-subdirect sum. Can we find matrices A and B with the same properties such that $C = A \oplus_k B$? We answer this in the affirmative in the following result.

PROPOSITION 5.1. *Let*

$$C = \begin{bmatrix} C_{11} & C_{12} & O \\ C_{21} & C_{22} & C_{23} \\ O & C_{32} & C_{33} \end{bmatrix},$$

with the matrices C_{ii} of order $n_1 - k$, k, $n_2 - k$, for $i = 1, 2, 3$, respectively, and C an SDD matrix with positive diagonal entries. Then, we can find square matrices A and B of order n_1 and n_2 such that they are SDD matrices with positive diagonal entries, and such that $C = A \oplus_k B$. In other words, we have

$$A = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_{22} & C_{23} \\ C_{23} & C_{33} \end{bmatrix}$$

such that $C_{22} = A_{22} + B_{22}$.

The proof of this proposition resembles that of [5, Proposition 4.1], where a similar question was studied for M-matrices, and we do not repeat it here. We mention that it is immediate to generalize Proposition 5.1 to a matrix C with the structure of a subdirect sum of several matrices such as that of (4.2) of Theorem 4.1.

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