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REGIONS CONTAINING EIGENVALUES OF A MATRIX*

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Abstract. In this paper, regions containing eigenvalues of a matrix are obtained in terms of partial absolute deleted row sums and column sums. Furthermore, some sufficient and necessary conditions for H-matrices are derived. Finally, an upper bound for the Perron root of nonnegative matrices is presented. The comparison of the new upper bound with the known ones is supplemented with some examples.

Key words. Eigenvalue, H-matrix, Perron root, Nonnegative matrix.

AMS subject classifications. 15A18, 15A42, 15A57, 65F15, 65F99.

1. Introduction. The Gerschgorin circle theorem gives a region in the complex plane which contains all the eigenvalues of a square complex matrix. It is one of those rare instances of a theorem which is elegant and useful and which has a short, elegant proof (see e.g., [1] or [5]). Moreover, we have Brauer's theorem, Ostrowski's theorem and Brauldi's theorem etc., by which we can estimate the inclusion regions of eigenvalues of a matrix in terms of its entries (see [5]).

Let $M_n(\mathbb{C})$ denote the set of all $n \times n$ complex matrices and $\langle n \rangle = \{1, 2, \ldots, n\}$. Let $A = (a_{ij}) \in M_n(\mathbb{C})$. The comparison matrix $m(A) = (m_{ij})$ of A is defined by

$$m_{ij} = \begin{cases} |a_{ij}|, & \text{if } i = j, \\ -|a_{ij}|, & \text{if } i \neq j. \end{cases}$$

Recall that A is an H-matrix if its comparison matrix $m(A)$ is an M-matrix. It is well known that a square matrix A is an M-matrix if it can be written in the form

$$A = \omega I - P, \quad P \text{ is nonnegative}, \ \omega > \rho(P),$$

$\rho(P)$ is the spectral radius of P. Criteria for judging M-matrices can be found in [2, 4, 5, 8].

In Section 2, in terms of partial absolute deleted row sums and column sums, new results are provided to estimate eigenvalues. Some sufficient and necessary conditions for H-matrices are derived from the new eigenvalues inclusion regions. Moreover, in Section 3, the results obtained will be applied to estimate the upper bound of the Perron root of a nonnegative matrix. Some examples are presented in Section 4.

2. Regions containing eigenvalues. Let $A = (a_{ij})$ be an $n \times n$ complex matrix with $n \geq 2$. Let α and β be nonempty index sets satisfying $\alpha \cup \beta = \langle n \rangle$ and $\alpha \cap \beta = \phi$.

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Define partial absolute deleted row sums and column sums as follows:

$$r_i^{(\alpha)}(A) = \sum_{j\neq i, j\in\alpha} |a_{ij}|, \quad c_i^{(\alpha)}(A) = \sum_{j\neq i, j\in\alpha} |a_{ji}|;$$

$$r_i^{(\beta)}(A) = \sum_{j\neq i, j\in\beta} |a_{ij}|, \quad c_i^{(\beta)}(A) = \sum_{j\neq i, j\in\beta} |a_{ji}|.$$

If α contains a single element, say $\alpha = \{i_0\}$, then we assume, by convention, that $r_{i_0}^{(\alpha)}(A) = 0$. Similarly $r_{i_0}^{(\beta)}(A) = 0$ if $\beta = \{i_0\}$. We will sometimes use $r_i^{(\alpha)}$ ($c_i^{(\alpha)}$, $r_i^{(\beta)}$, $c_i^{(\beta)}$) to denote $r_i^{(\alpha)}(A)$ ($c_i^{(\alpha)}(A)$, $r_i^{(\beta)}(A)$, $c_i^{(\beta)}(A)$, respectively) unless a confusion is caused. Clearly, we have

$$r_i(A) = \sum_{j\neq i} |a_{ij}| = r_i^{(\alpha)}(A) + r_i^{(\beta)}(A),$$

$$c_i(A) = \sum_{j\neq i} |a_{ji}| = c_i^{(\alpha)}(A) + c_i^{(\beta)}(A).$$

Define, for all $i \in \alpha$ and $j \in \beta$,

$$G_i^{(\alpha)} = \left\{ z \in \mathbb{C} : |z - a_{ii}| \leq r_i^{(\alpha)}(A) \right\},$$

$$G_j^{(\beta)} = \left\{ z \in \mathbb{C} : |z - a_{jj}| \leq r_j^{(\beta)}(A) \right\},$$

$$G_{ij}^{(\alpha\beta)} = \left\{ z \in \mathbb{C} : z \notin G_i^{(\alpha)} \cup G_j^{(\beta)}, \ \left(|z - a_{ii}| - r_i^{(\alpha)}\right) \left(|z - a_{jj}| - r_j^{(\beta)}\right) \leq r_i^{(\beta)} r_j^{(\alpha)} \right\}.$$

Theorem 2.1. *Each eigenvalue of matrix A of order n is contained in the region*

$$G_{\alpha\beta} \bigcup G^{(\alpha\beta)},$$

where

$$G_{\alpha\beta} := \left(\bigcup_{i\in\alpha} G_i^{(\alpha)} \right) \bigcup \left(\bigcup_{j\in\beta} G_j^{(\beta)} \right) \ \text{ and } \ G^{(\alpha\beta)} := \bigcup_{i\in\alpha, j\in\beta} G_{ij}^{(\alpha\beta)}.$$

Proof. Suppose λ is an eigenvalue of A, then there exists a nonzero vector $x = (x_1, \ldots, x_n)^T$ such that

$$Ax = \lambda x. \tag{2.1}$$

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Denote $|x_p| = \max_{i\in\alpha}\{|x_i|\}$, $|x_q| = \max_{j\in\beta}\{|x_j|\}$. Obviously, at least one of x_p and x_q is nonzero.

i) Suppose $x_p x_q \neq 0$, then the p-th equation in (2.1) implies

$$(\lambda - a_{pp})x_p = \sum_{j\in\alpha, j\neq p} a_{pj}x_j + \sum_{j\in\beta} a_{pj}x_j,$$

$$|\lambda - a_{pp}||x_p| \leq \sum_{j\in\alpha, j\neq p} |a_{pj}||x_j| + \sum_{j\in\beta} |a_{pj}||x_j|$$

$$\leq \sum_{j\in\alpha, j\neq p} |a_{pj}||x_p| + \sum_{j\in\beta} |a_{pj}||x_q|, \tag{2.2}$$

i.e.,

$$|\lambda - a_{pp}| \leq r_p^{(\alpha)} + r_p^{(\beta)}\frac{|x_q|}{|x_p|}. \tag{2.3}$$

Similarly, the q-th equation in (2.1) implies

$$(\lambda - a_{qq})x_q = \sum_{j\in\alpha} a_{qj}x_j + \sum_{j\in\beta, j\neq q} a_{qj}x_j.$$

By the same method we have

$$|\lambda - a_{qq}| \leq r_q^{(\alpha)}\frac{|x_p|}{|x_q|} + r_q^{(\beta)}. \tag{2.4}$$

If the following condition

$$|\lambda - a_{pp}| \leq r_p^{(\alpha)} \quad \text{or} \quad |\lambda - a_{qq}| \leq r_q^{(\beta)}$$

holds, then $\lambda \in G_{\alpha\beta}$. Otherwise, if

$$|\lambda - a_{pp}| - r_p^{(\alpha)} > 0 \quad \text{and} \quad |\lambda - a_{qq}| - r_q^{(\beta)} > 0,$$

then inequalities (2.3) and (2.4) imply that

$$\left(|\lambda - a_{pp}| - r_p^{(\alpha)}\right)\left(|\lambda - a_{qq}| - r_q^{(\beta)}\right) \leq r_p^{(\beta)} r_q^{(\alpha)}.$$

That is $\lambda \in G^{(\alpha\beta)}$.

ii) Suppose $x_q = 0$ ($x_p \neq 0$), then inequality (2.3) implies that

$$|\lambda - a_{pp}| \leq r_p^{(\alpha)}.$$

Hence $\lambda \in G_{\alpha\beta}$.

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The argument is analogous when $x_p = 0$ $(x_q \neq 0)$. $\square$

Since A and A^T have the same eigenvalues, one can obtain a theorem for columns by applying Theorem 2.1 to A^T. Define, for all $i \in \alpha$ and $j \in \beta$,

$$V_i^{(\alpha)} = \left\{ z \in \mathbb{C} : |z - a_{ii}| \leq c_i^{(\alpha)}(A) \right\},$$

$$V_j^{(\beta)} = \left\{ z \in \mathbb{C} : |z - a_{jj}| \leq c_j^{(\beta)}(A) \right\},$$

$$V_{ij}^{(\alpha\beta)} = \left\{ z \in \mathbb{C} : z \notin V_i^{(\alpha)} \cup V_j^{(\beta)},\ \left(|z - a_{ii}| - c_i^{(\alpha)} \right) \left(|z - a_{jj}| - c_j^{(\beta)} \right) \leq c_i^{(\beta)} c_j^{(\alpha)} \right\}.$$

Corollary 2.2. *Each eigenvalue of the matrix A of order n is contained in the region*

$$V_{\alpha\beta} \bigcup V^{(\alpha\beta)},$$

where

$$V_{\alpha\beta} := \left(\bigcup_{i \in \alpha} V_i^{(\alpha)} \right) \bigcup \left(\bigcup_{j \in \beta} V_j^{(\beta)} \right) \ \text{ and } \ V^{(\alpha\beta)} := \bigcup_{i \in \alpha, j \in \beta} V_{ij}^{(\alpha\beta)}.$$

Corollary 2.3. *Let $A = (a_{ij}) \in M_n(\mathbb{C})$. Suppose for all $i \in \alpha$ and $j \in \beta$,*

a) $|a_{ii}| - r_i^{(\alpha)} > 0$ and $|a_{jj}| - r_j^{(\beta)} > 0$;

b) $\left(|a_{ii}| - r_i^{(\alpha)} \right) \left(|a_{jj}| - r_j^{(\beta)} \right) > r_i^{(\beta)} r_j^{(\alpha)}$.

Then A is nonsingular.

Proof. If A is singular, i.e., $\lambda = 0$ is an eigenvalue of A, by applying Theorem 2.1, at least one of conditions a) and b) is invalid. Hence A is nonsingular. $\square$

Corollary 2.4. *Let $A = (a_{ij}) \in M_n(\mathbb{C})$. Suppose for all $i \in \alpha$ and $j \in \beta$,*

a) $|a_{ii}| - r_i^{(\alpha)} > 0$ and $|a_{jj}| - r_j^{(\beta)} > 0$;

b) $\left(|a_{ii}| - r_i^{(\alpha)} \right) \left(|a_{jj}| - r_j^{(\beta)} \right) > r_i^{(\beta)} r_j^{(\alpha)}$.

Then A is an H-matrix.

Proof. Obviously, the comparison matrix $m(A)$ satisfies conditions a) and b) as well as A. Hence $m(A)$ is nonsingular from Corollary 2.3.

For any $\varepsilon \geq 0$, define $B = (b_{ij}) := m(A) + \varepsilon I$, then we have

$$|b_{ii}| - r_i^{(\alpha)}(B) = (|a_{ii}| + \varepsilon) - r_i^{(\alpha)}(A) > 0,$$

$$|b_{jj}| - r_j^{(\beta)}(B) = (|a_{jj}| + \varepsilon) - r_j^{(\beta)}(A) > 0,$$

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and

$$\begin{aligned}\left(|b_{ii}| - r_i^{(\alpha)}(B)\right)\left(|b_{jj}| - r_j^{(\beta)}(B)\right) &= \left(|a_{ii}| + \varepsilon - r_i^{(\alpha)}(A)\right)\left(|a_{jj}| + \varepsilon - r_j^{(\beta)}(A)\right)\\ &\geq \left(|a_{ii}| - r_i^{(\alpha)}(A)\right)\left(|a_{jj}| - r_j^{(\beta)}(A)\right)\\ &> r_i^{(\beta)}(A) r_j^{(\alpha)}(A)\\ &= r_i^{(\beta)}(B) r_j^{(\alpha)}(B).\end{aligned}$$

By Corollary 2.3, we know that B is nonsingular. Hence $m(A)$ is an M-matrix (see e.g., [4]), which implies that A is an H-matrix. □

LEMMA 2.5. *Let $A = (a_{ij}) \in M_n(\mathbb{C})$, and for all $i \in \alpha$, $j \in \beta$,*

$$\left(|a_{ii}| - r_i^{(\alpha)}(A)\right)\left(|a_{jj}| - r_j^{(\beta)}(A)\right) > r_i^{(\beta)}(A) r_j^{(\alpha)}(A). \tag{2.5}$$

Then the following two conditions are equivalent:

1) For all $i \in \alpha$, $|a_{ii}| - r_i^{(\alpha)}(A) > 0$;

2) $J(A) := \left\{ i : |a_{ii}| > \sum_{j\neq i} |a_{ij}|, i \in \langle n\rangle \right\} \neq \phi$.

Proof. 1)⇒2): Note that condition 1) and (2.5) imply, for all $i \in \alpha$ and $j \in \beta$, that

$$|a_{ii}| - r_i^{(\alpha)}(A) > 0 \text{ and } |a_{jj}| - r_j^{(\beta)}(A) > 0.$$

If $J(A) = \phi$, then for all $i \in \langle n\rangle$, $|a_{ii}| \leq \sum_{j\neq i} |a_{ij}|$, i.e.,

$$0 < |a_{ii}| - r_i^{(\alpha)}(A) \leq r_i^{(\beta)}(A) \text{ and } 0 < |a_{jj}| - r_j^{(\beta)}(A) \leq r_j^{(a)}(A).$$

Thus, we have

$$\left(|a_{ii}| - r_i^{(\alpha)}(A)\right)\left(|a_{jj}| - r_j^{(\beta)}(A)\right) \leq r_i^{(\beta)}(A) r_j^{(\alpha)}(A),$$

which contradicts (2.5). So $J(A) \neq \phi$.

2) ⇒ 1): Since $J(A) \neq \phi$, there exists at least one $i_0 \in \langle n\rangle$ such that $|a_{i_0 i_0}| > \sum_{j\neq i_0} |a_{i_0 j}|$. Without loss of generality, we assume that $i_0 \in \alpha$, then

$$|a_{i_0 i_0}| > \sum_{j\neq i_0} |a_{i_0 j}| \geq r_{i_0}^{(\alpha)}(A).$$

Hence, from (2.5), for all $j \in \beta$, we can derive

$$|a_{jj}| - r_j^{(\beta)}(A) > 0,$$

and then, for all $i \in \alpha$, $|a_{ii}| - r_i^{(\alpha)}(A) > 0$. □

Immediately we have the following consequence from Lemma 2.5 and Corollary 2.4.

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COROLLARY 2.6. *Let A be an $n \times n$ complex matrix. Suppose $J(A) = \{i : |a_{ii}| > \sum_{j \neq i} |a_{ij}|\ , i \in \langle n \rangle\} \neq \phi$ and for all $i \in \alpha$, $j \in \beta$,*

$$\left(|a_{ii}| - r_i^{(\alpha)}(A)\right)\left(|a_{jj}| - r_j^{(\beta)}(A)\right) > r_i^{(\beta)}(A) r_j^{(\alpha)}(A).$$

Then A is an H-matrix.

REMARK 2.7. Corollary 2.6 is one of the main results in [2] (see Th. 1), which was used as a criterion for generalized diagonally dominant matrices and M-matrices.

We call A *a generalized doubly diagonally dominant matrix* if $J(A) \neq \phi$ and

$$\left(|a_{ii}| - r_i^{(\alpha)}(A)\right)\left(|a_{jj}| - r_j^{(\beta)}(A)\right) \geq r_i^{(\beta)}(A) r_j^{(\alpha)}(A) \tag{2.6}$$

for all $i \in \alpha$ and $j \in \beta$. See [[3], p. 233] for detail. A is further said to be *a strictly generalized doubly diagonally dominant matrix* if all the strict inequalities in (2.6) hold, and is denoted by $A \in D_n^{(\alpha\beta)}$. It was shown by the authors in [3] that the Schur complement of a generalized doubly diagonally dominant matrix is also a generalized doubly diagonally dominant matrix. Moreover, Gao and Wang [2] showed that a strictly generalized doubly diagonally dominant matrix is an H-matrix.

Here we denote $\tilde{D}_n^{(\alpha\beta)}$ by the set

$$\tilde{D}_n^{(\alpha\beta)} := \left\{A \,\middle|\, AX \in D_n^{(\alpha\beta)}\ ,\ X \text{ is a positive diagonal matrix}\right\}.$$

Clearly we have $D_n^{(\alpha\beta)} \subset \tilde{D}_n^{(\alpha\beta)}$.

Now, we present a sufficient and necessary condition for H-matrices.

COROLLARY 2.8. *A is an H-matrix if and only if $A \in \tilde{D}_n^{(\alpha\beta)}$.*

Proof. $\Rightarrow$: If A is an H-matrix, then there exists a positive diagonal matrix X_1 such that AX_1 is a strictly diagonally dominant matrix, thus we can easily derive $AX_1 \in D_n^{(\alpha\beta)}$, and then $A \in \tilde{D}_n^{(\alpha\beta)}$.

$\Leftarrow$: If $A \in \tilde{D}_n^{(\alpha\beta)}$, then, by definition, there exists a positive diagonal matrix X_2 such that $AX_2 \in D_n^{(\alpha\beta)}$. By Corollary 2.6 we know that AX_2 is an H-matrix, and then there exists a positive diagonal matrix X_3 such that AX_2X_3 is a strictly diagonal dominant matrix. Since X_2X_3 is also a positive diagonal matrix, A is an H-matrix. □

LEMMA 2.9. ([8]) *Let $A = (a_{ij})$ be an $n \times n$ H-matrix. Then $J(A) = \{i : |a_{ii}| > \sum_{j \neq i} |a_{ij}|, i \in \langle n \rangle\} \neq \phi$.*

COROLLARY 2.10. *Let $A = (a_{ij})$ be an $n \times n$ H-matrix. Then for nonempty index sets α and β satisfying $\alpha \cup \beta = \langle n \rangle$ and $\alpha \cap \beta = \phi$, there exists at least one pair (i_0, j_0), $i_0 \in \alpha$ and $j_0 \in \beta$, such that*

$$\left(|a_{i_0 i_0}| - r_{i_0}^{(\alpha)}(A)\right)\left(|a_{j_0 j_0}| - r_{j_0}^{(\beta)}(A)\right) > r_{i_0}^{(\beta)}(A) r_{j_0}^{(\alpha)}(A).$$

Proof. From Lemma 2.9, we have $J(A) \neq \phi$. Suppose for all $i \in \alpha$ and $j \in \beta$,

$$\left(|a_{ii}| - r_i^{(\alpha)}(A)\right)\left(|a_{jj}| - r_j^{(\beta)}(A)\right) \leq r_i^{(\beta)}(A) r_j^{(\alpha)}(A).$$

Then A is not an H-matrix by Theorem 3 of [2], contradicting the assumptions. □

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3. Upper bounds for the Perron root. In this section we shall discuss the upper bound for the Perron root of nonnegative matrices by using the eigenvalues inclusion region obtained in Section 2.

Let $A = (a_{ij})$ be an $n \times n$ nonnegative matrix with $n \geq 2$. It is known that for the Perron root $\rho(A)$, i.e., the spectral radius of A, the following inequality holds (see e.g., [4] or [7]):

$$\min_{1\leq i\leq n} \sum_{j=1}^{n} a_{ij} \leq \rho(A) \leq \max_{1\leq i\leq n} \sum_{j=1}^{n} a_{ij}. \tag{3.1}$$

Another known bound of $\rho(A)$ belongs to Brauer and Gentry (see [6]):

$$\min_{i\neq j} M_A(i,j) \leq \rho(A) \leq \max_{i\neq j} M_A(i,j), \tag{3.2}$$

where

$$M_A(i,j) = \frac{1}{2}\left\{a_{ii} + a_{jj} + \left[(a_{ii} - a_{jj})^2 + 4\sum_{k\neq i} |a_{ik}| \sum_{k\neq j} |a_{jk}|\right]^{\frac{1}{2}}\right\}.$$

For simplicity, (3.1) and (3.2) are called Frobenius' bound and Brauer-Gentry's bound, respectively.

Theorem 3.1. *Let A be an $n \times n$ nonnegative matrix. Then we have*

$$\rho(A) \leq \max_{i\in\alpha, j\in\beta} Q_A(i,j),$$

where

$$Q_A(i,j) = \frac{1}{2}\left\{a_{ii} + r_i^{(\alpha)} + a_{jj} + r_j^{(\beta)} + \left[\left(a_{ii} + r_i^{(\alpha)} - a_{jj} - r_j^{(\beta)}\right)^2 + 4r_i^{(\beta)} r_j^{(\alpha)}\right]^{\frac{1}{2}}\right\}.$$

Proof. Note that $\rho(A)$ is an eigenvalue of A by the well known Perron-Frobenius theory of nonnegative matrices (see e.g., [4]). So if we define

$$B := \rho(A)I - A,$$

then B is singular.

For all $i \in \alpha$ and $j \in \beta$, if $|b_{ii}| - r_i^{(\alpha)}(B) > 0$ and

$$\left(|b_{ii}| - r_i^{(\alpha)}(B)\right)\left(|b_{jj}| - r_j^{(\beta)}(B)\right) > r_i^{(\beta)}(B) r_j^{(\alpha)}(B),$$

i.e.,

$$\begin{cases} |\rho(A) - a_{ii}| - r_i^{(\alpha)}(A) > 0, \\ \left(|\rho(A) - a_{ii}| - r_i^{(\alpha)}(A)\right)\left(|\rho(A) - a_{jj}| - r_j^{(\beta)}(A)\right) > r_i^{(\beta)}(A) r_j^{(\alpha)}(A). \end{cases}$$

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Then, by Corollary 2.3, we can derive B is nonsingular. So if B is singular, there exists at least one pair (i_0, j_0), $i_0 \in \alpha$ and $j_0 \in \beta$, such that

$$|\rho(A) - a_{i_0 i_0}| - r_{i_0}^{(\alpha)}(A) \leq 0$$

or

$$\left(|\rho(A) - a_{i_0 i_0}| - r_{i_0}^{(\alpha)}(A)\right)\left(|\rho(A) - a_{j_0 j_0}| - r_{j_0}^{(\beta)}(A)\right) \leq r_{i_0}^{(\beta)}(A) r_{j_0}^{(\alpha)}(A).$$

Moreover, by the fact that

$$\rho(A) \geq \max_{i \in \langle n \rangle} a_{ii}$$

(see e.g., [4] or [5]), we have

$$\rho(A) - a_{i_0 i_0} - r_{i_0}^{(\alpha)}(A) \leq 0 \tag{3.3}$$

or

$$\left(\rho(A) - a_{i_0 i_0} - r_{i_0}^{(\alpha)}(A)\right)\left(\rho(A) - a_{j_0 j_0} - r_{j_0}^{(\beta)}(A)\right) \leq r_{i_0}^{(\beta)}(A) r_{j_0}^{(\alpha)}(A). \tag{3.4}$$

From (3.3) and (3.4), we can easily derive

$$\rho(A) \leq Q_A(i_0, j_0).$$

Thus,

$$\rho(A) \leq \max_{i \in \alpha, j \in \beta} Q_A(i, j). \quad \square$$

For a complex square matrix A, it follows that $\rho(A) \leq \rho(|A|)$ (see e.g., [4] or [5]), where $|A| = (|a_{ij}|)$ is a nonnegative matrix. Immediately we obtain the following result:

Corollary 3.2. *Let $A = (a_{ij}) \in M_n(\mathbb{C})$. Then,*

$$\rho(A) \leq \max_{i \in \alpha, j \in \beta} \frac{1}{2} \Big\{ |a_{ii}| + r_i^{(\alpha)} + |a_{jj}| + r_j^{(\beta)} + \left[\left(|a_{ii}| + r_i^{(\alpha)} - |a_{jj}| - r_j^{(\beta)} \right)^2 + 4 r_i^{(\beta)} r_j^{(\alpha)} \right]^{\frac{1}{2}} \Big\}.$$

4. Examples.

In this section, some numerical examples are provided to compare our bounds with some known ones.

Example 1. Consider the nonnegative matrix

$$A = \begin{bmatrix} 6 & 1 & 2 & 1 \\ 3 & 5 & 1 & 1 \\ 0 & 3 & 3 & 1 \\ 0 & 0 & 3 & 3 \end{bmatrix}, \quad \rho(A) = 8.6716.$$

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Using the upper bounds of Frobenius and Brauer-Gentry, we can obtain the same result $\rho(A) \leq 10.0000$. For $\alpha = \{1,2\}$ and $\beta = \{3,4\}$, by Theorem 3.1, we have $\rho(A) \leq 9.1623$.

Example 2 ([7]). Consider the nonnegative matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 & 3 & 3 \\ 1 & 2 & 3 & 4 & 4 & 4 \\ 1 & 2 & 3 & 4 & 5 & 5 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{bmatrix}, \quad \rho(A) = 17.2069.$$

By Frobenius' bound (3.1), Brauer-Gentry's bound (3.2), Ledermann's bound [[9], Chapter 2, Theorem 1.3], Ostrowski's bound [[9], Chapter 2, Theorem 1.4] and Brauer's bound [[9], Chapter 2, Theorem 1.5], we have

$$\rho(A) \leq 21.0000 \text{ (by Frobenius' bound)},$$
$$\rho(A) \leq 20.5083 \text{ (by Brauer-Gentry's bound)},$$
$$\rho(A) \leq 20.9759 \text{ (by Ledermann's bound)},$$
$$\rho(A) \leq 20.5000 \text{ (by Ostrowski's bound)},$$
$$\rho(A) \leq 20.2596 \text{ (by Brauer's bound)}.$$

For $\alpha = \{1,2,3\}$ and $\beta = \{4,5,6\}$, by Theorem 3.1, we have $\rho(A) \leq 19.1168$.

Example 3. Consider the complex matrix

$$A = \begin{bmatrix} 4+3i & 2 & 3i & 0 \\ 2 & 4 & 1 & 2i \\ 2 & 0 & 3 & 1 \\ 2 & 1 & 3 & 3 \end{bmatrix}, \quad \rho(A) = 7.3783.$$

Let $\alpha = \{1,2\}$ and $\beta = \{3,4\}$. By Corollary 3.2, we have $\rho(A) \leq 9.5414$.

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