Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 15, pp. 225-238, August 2006

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http://math.technion.ac.il/iic/ela

SEMITRANSITIVE SUBSPACES OF MATRICES*

SEMITRANSITIVITY WORKING GROUP AT LAW'05, BLED†

Abstract. A set of matrices $\mathcal{S} \subseteq \mathbb{M}_n(\mathbb{F})$ is said to be semitransitive if for any two nonzero vectors $x, y \in \mathbb{F}^n$, there exists a matrix $A \in \mathcal{S}$ such that either $Ax = y$ or $Ay = x$. In this paper various properties of semitransitive linear subspaces of $\mathbb{M}_n(\mathbb{F})$ are studied. In particular, it is shown that every semitransitive subspace of matrices has a cyclic vector. Moreover, if $|\mathbb{F}| \geq n$, it always contains an invertible matrix. It is proved that there are minimal semitransitive matrix spaces without any nontrivial invariant subspace. The structure of minimal semitransitive spaces and triangularizable semitransitive spaces is also studied. Among other results it is shown that every triangularizable semitransitive subspace contains a nonzero nilpotent.

Key words. Semitransitive subspaces of matrices, Triangularizability, Reducibility, Minimality.

AMS subject classifications. 15A30.

1. Introduction. Let $\mathbb{F}$ be an arbitrary field and $\mathbb{M}_n(\mathbb{F})$ the algebra of all $n \times n$ matrices over $\mathbb{F}$, which we will identify with the algebra of all linear operators on the space $\mathcal{V} = \mathbb{F}^n$, when a basis is understood to be fixed. By a well-known definition, a collection $\mathcal{S}$ of operators is said to be transitive on $\mathcal{V}$ if for any given pair of nonzero elements x and y of $\mathcal{V}$ there is an A in $\mathcal{S}$ with $Ax = y$. Transitivity has been studied extensively for sets with various structures, e.g. groups, semigroups, linear spaces and algebras. The frequently-quoted theorem of Burnside, for example, states that when $\mathbb{F}$ is algebraically closed, then $\mathbb{M}_n(\mathbb{F})$ has no proper transitive subalgebra. If the transitive collection $\mathcal{S}$ is merely assumed to be a linear space, then there exist many possibilities. It is known that such an $\mathcal{S}$ has dimension at least $2n - 1$ [1]. If $\mathbb{F}$ is the real or complex field, there is also a natural, topological version of transitivity whereby the collection $\mathcal{S}$ is required only to take x approximately to y: given x and y in $\mathcal{V}$ and positive ε, there is an A in $\mathcal{S}$ such that $\|Ax - y\| < \varepsilon$. (All norms are equivalent in finite dimensions and any one can be fixed for the purpose.) Of course, for linear spaces $\mathcal{S}$ of $\mathbb{M}_n(\mathbb{F})$ the strict and topological notions of transitivity coincide. Note the obviously equivalent definition of transitivity: A set $\mathcal{S}$ is transitive if and only if given nonzero x and y in the underlying space, $\mathcal{S}x$ contains y. Here we write $\mathcal{S}x = \{\ Ax \mid\ A \in \mathcal{S}\ \}$ for a vector $x \in \mathcal{V}$.

A weaker property than transitivity was first proposed by H. Rosenthal and V. Troitsky in [6], where they considered subalgebras $\mathcal{S}$ of bounded operators on a Banach space $\mathcal{V}$ and defined topological and (strict) semitransitivity of $\mathcal{S}$, with strictly

*Received by the editors 5 April 2006. Accepted for publication on 13 August 2006. Handling Editor: Harm Bart.

†The Fourth Linear Algebra Workshop (LAW'05) was held at Bled, Slovenia, May 16-24, 2005. Semitransitivity Working Group was coordinated by Heydar Radjavi and its members were Janez Bernik, Roman Drnovšek, Don Hadwin, Ali Jafarian, Damjana Kokol Bukovšek, Tomaž Košir, Marjeta Kramar Fijavž, Thomas Laffey, Leo Livshits, Mitja Mastnak, Roy Meshulam, Vladimir Müller, Eric Nordgren, Jan Okniński, Matjaž Omladič, Heydar Radjavi, Ahmed Sourour, and Richard Timoney. Please send all the correspondence to Tomaž Košir, Department of Mathematics, University of Ljubljana, Jadranska 19, 1000 Ljubljana, Slovenia (tomaz.kosir@fmf.uni-lj.si). LAW'05 was supported in part by the Ministry of Higher Education, Science and Technology of Slovenia.

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A publication of the International Linear Algebra Society
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semitransitive algebras being the main object of investigation. Topological semitransitivity of $\mathcal{S}$ means that for given x and y in $\mathcal{V}$ and positive ε, either $Ax-y$ or $Ay-x$ is of norm less than ε. Finite-dimensional versions of this concept were considered in [2] for algebras and semigroups. Among other things, minimal semitransitive algebras were characterized in that paper as those simultaneously similar to the algebra of upper triangular Toeplitz operators (that is, the algebra generated by the identity and a nilpotent matrix of index n). Note that semitransitivity of a set $\mathcal{S}$ can be equivalently defined by requiring that for nonzero x and y in $\mathcal{V}$, either $\mathcal{S}x$ should contain y or $\mathcal{S}y$ should contain x. Semitransitivity of linear spaces was studied in [5], where k-fold semitransitivity was also considered and Jacobson's density theorem for rings was extended.

In this paper we prove several results about semitransitive linear spaces of $\mathbb{M}_n(\mathbb{F})$, and construct examples to repudiate certain natural-sounding conjectures. As expected, there are many respects in which transitive and semitransitive spaces differ. For example, it is clear that a transitive space cannot have any nontrivial invariant subspaces, but a semitransitive space can be (simultaneously) triangularizable, as the Toeplitz algebra mentioned above illustrates. Is a minimal semitransitive space necessarily triangularizable (as in the case of algebras)? One of our counterexamples will show that the answer is negative in an extreme way: there are minimal semitransitive spaces which do not have any nontrivial invariant subspaces. Many of the natural questions do not seem to be easy to handle, as our treatment of very small-dimensional cases below will demonstrate. We shall mention some of the unsolved problems at the end of the paper.

2. General Properties. Let A^{tr} denote the transpose of a matrix $A \in \mathbb{M}_n(\mathbb{F})$. For a set of matrices $\mathcal{S} \subseteq \mathbb{M}_n(\mathbb{F})$ we denote

$$\mathcal{S}^{\mathrm{tr}} = \left\{ \; A^{\mathrm{tr}} \;\middle|\; A \in \mathcal{S} \; \right\}$$

and for $x, y \in \mathbb{F}^n$ we define

$$x \perp y \Longleftrightarrow x^{\mathrm{tr}} y = 0.$$

THEOREM 2.1. *If $\mathcal{L}$ is a semitransitive subspace of $\mathbb{M}_n(\mathbb{F})$, then so is $\mathcal{L}^{\mathrm{tr}}$.*

Proof. Since $x^{\mathrm{tr}}y$ is a nondegenerate form, it follows that y belongs to a subspace V if and only if $z \perp V$ implies $z \perp y$ for every z. Now take any nonzero x and y and assume that $x \notin \mathcal{L}^{\mathrm{tr}} y$. We want to show that $y \in \mathcal{L}^{\mathrm{tr}} x$. It suffices to show that $z \perp y$ whenever $z \perp \mathcal{L}^{\mathrm{tr}} x$. Now, since $x \notin \mathcal{L}^{\mathrm{tr}} y$, there exists $u \perp \mathcal{L}^{\mathrm{tr}} y$ such that $u \not\perp x$. Next, if $z \perp \mathcal{L}^{\mathrm{tr}} x$, this means that $x \perp \mathcal{L}z$ so that $u \notin \mathcal{L}z$. By semitransitivity of $\mathcal{L}$ we then have that $z \in \mathcal{L}u$. It follows from the above that $y \perp \mathcal{L}u$, so that $y \perp z$ which was to be shown. □

REMARK 2.2. Suppose that $\varphi : \mathbb{F} \to \mathbb{F}$ is a field automorphism. We extend it entrywise to $\mathbb{F}^n$ and $\mathbb{M}_n(\mathbb{F})$ and denote the extension by $\widetilde{\varphi}$. If $\mathcal{L}$ is a subset of $\mathbb{M}_n(\mathbb{F})$, then we write

$$\widetilde{\mathcal{L}} = \left\{ \; \widetilde{\varphi}(A) \mid A \in \mathcal{L} \; \right\}.$$

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Observe that $\widetilde{\mathcal{L}}$ is semitransitive if and only if $\mathcal{L}$ is. □

The next result is implied by Theorem 2.1 and Remark 2.2.

COROLLARY 2.3. *The following are equivalent for a space $\mathcal{L}$ of square complex matrices:*

1. *$\mathcal{L}$ is semitransitive,*
2. *$\mathcal{L}^*$ is semitransitive,*
3. *$\widetilde{\mathcal{L}}$ is semitransitive,*
4. *$\mathcal{L}^{\mathrm{tr}}$ is semitransitive,*

where

$$\widetilde{\mathcal{L}} = \{\ [\overline{a_{ij}}] \mid [a_{ij}] \in \mathcal{L}\ \}\,.$$

The following result is well known. However, for the sake of completeness we give the proof.

PROPOSITION 2.4. *Let $\mathcal{L}$ be a subspace of $\mathbb{M}_n(\mathbb{F})$.*

(a) If $P\mathcal{L}P$ is transitive on the range of P for every projection P of rank 2, then $\mathcal{L}$ is transitive.

(b) If in the case of the complex field $P\mathcal{L}P$ is transitive on the range of P for every orthogonal projection P of rank 2, then $\mathcal{L}$ is transitive.

Proof. Assume that $\mathcal{L}$ is not transitive. Then there exists a nonzero vector x such that $\mathcal{L}x$ is a proper subspace of $\mathbb{F}^n$. So, there exists a nonzero linear functional f such that $f(\mathcal{L}x) = 0$. Choose a basis of $\mathbb{F}^n$ containing x such that the last $n-2$ of its members are contained in the kernel of f. If P is the projection on the first two basis vectors along the rest of them, it follows that $P\mathcal{L}P$ is not transitive proving *(a)*.

To get *(b)* let us modify this proof accordingly. Recall that in the (Hilbert) space $\mathbb{C}^n$ the above functional f can be represented via a nonzero vector $y \in \mathbb{C}^n$, so that $f(u) = y^*u$ for all $u \in \mathbb{C}^n$. The kernels of the functionals $u \mapsto y^*u$ and $u \mapsto x^*u$ have dimension $n-1$ so that their intersection has to be of dimension at least $n-2$. This implies that we may choose an orthogonal basis containing x such that the last $n-2$ of its members are contained in the kernel of f. We may now proceed as above. □

Transitivity of a subspace over $\mathbb{C}$ can thus be verified by orthogonal projections to two-dimensional subspaces only. For semitransitive spaces, however, this no longer holds.

EXAMPLE 2.5. The space

$$\mathcal{L} = \left\{ \begin{pmatrix} a & c & e \\ 0 & d & f \\ b & c & g \end{pmatrix} \;\middle|\; a,b,c,d,e,f,g \in \mathbb{C} \right\}$$

is not semitransitive because $Te_1 \neq e_2$ and $Te_2 \neq e_1$ for every $T \in \mathcal{L}$. Nevertheless, $P\mathcal{L}P$ is semitransitive for every orthogonal projection $P \in \mathbb{M}_3(\mathbb{C})$ of rank two.

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This latter claim is justified as follows. Given a pair $\begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} u \\ v \\ w \end{pmatrix}$ of nonzero vectors, the existence of $T \in \mathcal{L}$ such that $T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$ is equivalent to the existence of a solution to the linear system

$$\begin{pmatrix} x & 0 & z & 0 & 0 & y & 0 \\ 0 & x & 0 & z & 0 & y & 0 \\ 0 & 0 & 0 & 0 & z & 0 & y \end{pmatrix} \begin{pmatrix} a \\ b \\ e \\ g \\ f \\ c \\ d \end{pmatrix} = \begin{pmatrix} u \\ w \\ v \end{pmatrix}.$$

Note that the system has solutions if $z \neq 0$ or if $z = 0$ and $x \neq 0 \neq y$. For $x = z = 0 \neq y$ the system is consistent only if $u = w$, while for $y = z = 0 \neq x$ the system is consistent only if $v = 0$. Hence the system is inconsistent exactly when either

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ is a multiple of } \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \text{ and } v \neq 0$$

or

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ is a multiple of } \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \text{ and } u \neq w.$$

Consequently if for a pair $\begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} u \\ v \\ w \end{pmatrix}$ of nonzero vectors there does not exist $T \in \mathcal{L}$ such that $T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$ or $T \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, then each of the vectors in question must be either a multiple of e_1 or of e_2, and the two vectors must be linearly independent.

It follows that $P\mathcal{L}P$ is semitransitive whenever $P \neq \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. On the other hand it is easy to see that $P\mathcal{L}P$ is semitransitive in the case when $P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

This example also demonstrates the possibility that for a given linear space $\mathcal{L}$ of square matrices there may exist a unique pair U, V of one-dimensional subspaces of vectors such that $TU \not\subset V$ and $TV \not\subset U$ for each nonzero $T \in \mathcal{L}$. □

Recall that, given a subspace $\mathcal{L}$ of $\mathbb{M}_n(\mathbb{F})$, a vector $x \in \mathbb{F}^n$ is *cyclic* for $\mathcal{L}$ if $\mathcal{L}x = \mathbb{F}^n$. It is clear from the definition that every nonzero vector is cyclic for a transitive subspace of matrices. The next theorem shows that there exist cyclic vectors for every semitransitive subspace of matrices.

THEOREM 2.6. *Let $\mathcal{L} \subseteq \mathbb{M}_n(\mathbb{F})$ be a semitransitive linear subspace over any field $\mathbb{F}$. Then there exists a cyclic vector for $\mathcal{L}$.*

Proof. We consider two cases:

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(i) $\mathbb{F} = \mathbb{F}_q$ is the finite field of order q. Assume that there is no cyclic vector for $\mathcal{L}$. Then the dimension of $\mathcal{L}x$ is at most $n-1$ for every $x \in \mathbb{F}^n$. Now, semitransitivity of $\mathcal{L}$ implies that

$$(\mathbb{F}^n - \{0\}) \times (\mathbb{F}^n - \{0\}) \subset \bigcup_{0 \neq u \in \mathbb{F}^n} (\{u\} \times \mathcal{L}'u \ \cup \ \mathcal{L}'u \times \{u\}) ,$$

where $\mathcal{L}' = \mathcal{L} - \{0\}$. It follows that

$$(q^n - 1)^2 = |(\mathbb{F}^n - \{0\}) \times (\mathbb{F}^n - \{0\})| \leq$$

$$2 \sum_{0 \neq u \in \mathbb{F}^n} |\mathcal{L}'u| \leq 2(q^n - 1)(q^{n-1} - 1) \ .$$

Hence $q^{n-1} \geq \frac{q^n - 1}{2} + 1$, which is a contradiction.

(ii) $|\mathbb{F}| = \infty$. Let $x_1, \ldots, x_n$ be a basis for $\mathbb{F}^n$. If one of x_i is cyclic, we are done. Otherwise $\mathcal{L}x_i$ is a proper subspace of $\mathbb{F}^n$ for every i. As the field $\mathbb{F}$ is infinite, $\mathbb{F}^n$ is not a union of finitely many proper subspaces. Hence there exists a vector $u \in \mathbb{F}^n$ such that $u \notin \mathcal{L}x_i$, $i = 1, \ldots, n$. By the semitransitivity of $\mathcal{L}$ we have $x_i \in \mathcal{L}u$ for every i, which clearly implies that u is cyclic for $\mathcal{L}$. □

The following immediate corollary extends [5, Corollary 3] to the case of arbitrary fields.

COROLLARY 2.7. *The dimension of every semitransitive subspace of* $\mathbb{M}_n(\mathbb{F})$ *is at least* n.

As the set of vectors x that are not cyclic for a given subspace $\mathcal{L} \subseteq \mathbb{M}_n(\mathbb{F})$ is easily seen to be a closed variety, defined by the condition that $\dim \mathcal{L}x \leq n-1$, we have also the following result.

COROLLARY 2.8. *If* $\mathbb{F}$ *is algebraically closed and if* $\mathcal{L} \subseteq \mathbb{M}_n(\mathbb{F})$ *is a semitransitive subspace, then the set of cyclic vectors for* $\mathcal{L}$ *is open and dense in the Zariski topology of* $\mathbb{F}^n$ *(and thus also in the Euclidean topology of* $\mathbb{C}^n$*).*

We are able to show another interesting property of semitransitive matrix spaces which is known to hold in the transitive case [3].

THEOREM 2.9. *Every semitransitive subspace in* $\mathbb{M}_n(\mathbb{F})$ *contains an invertible element, whenever* $\mathbb{F}$ *contains at least* n *elements.*

Proof. We will prove the theorem by induction on n. For $n = 1$ the assertion is trivial. Assume $n > 1$ and let $\mathcal{L} \subset \mathbb{M}_n(\mathbb{F})$ be semitransitive. Let P be the projection on the first $n-1$ elements of the standard basis of $\mathbb{F}^n$. The compression $P\mathcal{L}P$ is a semitransitive subspace in dimension $n-1$, so by induction it contains an invertible element. Let $B \in \mathcal{L}$ be a matrix such that the compression PBP is invertible. If B is invertible, we are done. Suppose not. Let x be a nonzero vector in the kernel of B.

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We write the matrices in the block partition of P, where x is the n-th basis vector. We have

$$B = \begin{pmatrix} B_1 & 0 \\ * & 0 \end{pmatrix},$$

where B_1 is invertible. Since $\mathcal{L}$ is semitransitive, there exists a matrix $C \in \mathcal{L}$ such that $Cx = x$, so that

$$C = \begin{pmatrix} C_1 & 0 \\ * & 1 \end{pmatrix}.$$

Since $\mathbb{F}$ has at least n elements, there exists $\lambda \in \mathbb{F}$, which is not an eigenvalue of $B_1^{-1}C_1$. Then $C - \lambda B \in \mathcal{L}$ is invertible. □

The following example shows that in the last theorem the assumption on the field cannot be omitted.

EXAMPLE 2.10. Let $\mathbb{F} = \{\alpha_1, ..., \alpha_q\}$ be a finite field and let

$$\mathcal{L} = \left\{ \begin{pmatrix} a & c_{12} & \cdots & c_{1,q+1} \\ 0 & b+\alpha_1 a & \cdots & c_{2,q+1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b+\alpha_q a \end{pmatrix} \;\middle|\; a, b, c_{ij} \in \mathbb{F} \right\} \subset \mathbb{M}_{q+1}(\mathbb{F}).$$

Then $\mathcal{L}$ is a semitransitive subspace, but it does not contain an invertible element.

3. Minimal semitransitive subspaces. A semitransitive subspace $\mathcal{L} \subseteq \mathbb{M}_n(\mathbb{F})$ is *minimal semitransitive* if it does not contain any proper semitransitive subspace.

EXAMPLE 3.1. The set of upper-triangular Toeplitz matrices

$$\mathcal{L}_0 := \left\{ \begin{pmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ & a_0 & \ddots & \vdots \\ & & \ddots & a_1 \\ & & & a_0 \end{pmatrix} \;\middle|\; a_i \in \mathbb{F} \right\}$$

is a minimal semitransitive subspace in $\mathbb{M}_n(\mathbb{F})$. Furthermore, $\mathcal{L}_0$ is an algebra and is, up to similarity, the unique minimal semitransitive subalgebra of $\mathbb{M}_n(\mathbb{F})$ if $\mathbb{F}$ is algebraically closed (see [2]).

Considering subspaces instead of algebras, the space $\mathcal{L}_0$ of Example 3.1 is no longer unique.

EXAMPLE 3.2. Let $C = (c_{ij}) \in \mathbb{M}_n(\mathbb{F})$ be a fixed matrix with $c_{ij} \neq 0$ for $j \geq i$ and $\mathcal{L}_0$ the subspace of Example 3.1. Then

$$\mathcal{L} := \{\ C * A \mid\ A \in \mathcal{L}_0\ \}$$

(by $*$ we denote the Schur product of matrices) is a minimal semitransitive subspace of $\mathbb{M}_n(\mathbb{F})$. Observe that $\mathcal{L}$ is not similar to $\mathcal{L}_0$ whenever $c_{ii} \neq c_{jj}$ for some i, j.

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Minimal semitransitive spaces are not always triangularizable. Moreover, they are not even always reducible.

THEOREM 3.3. *Let $n \geq 3$. Then there exists a minimal semitransitive subspace $\mathcal{M} \subset \mathbb{M}_n(\mathbb{F})$ without nontrivial invariant subspaces.*

Proof. We construct the example for $n = 3$; for $n \geq 4$ an example can be constructed analogously.

Let $\mathcal{V} = \mathbb{F}^3$ and let e_1, e_2, e_3 be the standard basis for $\mathcal{V}$. Let

$$\mathcal{A} = \left\{ \begin{pmatrix} a & 0 & b \\ b & x_1 & 0 \\ x_2 & x_3 & a \end{pmatrix} \;\middle|\; a, b, x_1, x_2, x_3 \in \mathbb{F} \right\}.$$

We show that $\mathcal{A}$ is semitransitive. Note first that each vector $x = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3$ with $\alpha_1 \neq 0$ is cyclic, i.e. $\mathcal{A}x = \mathcal{V}$. Indeed, considering $a = 1$ ($b = 1$, $x_2 = 1$, respectively) and the remaining parameters equal to 0, we get $\alpha_1 e_1 + \alpha_3 e_3 \in \mathcal{A}x$, $\alpha_3 e_1 + \alpha_1 e_2 \in \mathcal{A}x$ and $\alpha_1 e_3 \in \mathcal{A}x$. Hence $\mathcal{A}x = \mathcal{V}$.

We now show that for each $y = \beta_2 e_2 + \beta_3 e_3$ with $\beta_2 \neq 0$ we have $\mathcal{A}y \supset \vee\{e_2, e_3\}$. Indeed, considering the parameters x_1 and x_3, we get $\beta_2 e_2 \in \mathcal{A}y$ and $\beta_2 e_3 \in \mathcal{A}y$, and so $\mathcal{A}y \supset \vee\{e_2, e_3\}$.

Finally, $\mathcal{A}e_3 \ni e_3$. This implies that $\mathcal{A}$ is semitransitive.

Let $\mathcal{M}$ be a minimal semitransitive subspace of $\mathcal{A}$. Suppose on the contrary that there exists a nontrivial subspace $\mathcal{U} \subset \mathcal{V}$ with $\mathcal{M}\mathcal{U} \subset \mathcal{U}$.

Let $x \in \mathcal{U}$, $x \neq 0$. Write $x = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3$. We show first that there is a nonzero $y \in \mathcal{U} \cap \vee\{e_2, e_3\}$. This is clear if $\alpha_1 = 0$. Let $\alpha_1 \neq 0$. Consider the pair x, e_2. Since $Te_2 \neq x$ for all $T \in \mathcal{A}$, there is an $S \in \mathcal{M}$ such that $Sx = e_2$. So $e_2 \in \mathcal{M}\mathcal{U} \subset \mathcal{U}$. Thus there is a nonzero $y = \beta_2 e_2 + \beta_3 e_3 \in \mathcal{U}$.

We show similarly that $e_3 \in \mathcal{U}$. This is clear if $\beta_2 = 0$. Let $\beta_2 \neq 0$. Consider the pair y, e_3. Since $Te_3 \neq y$ for all $T \in \mathcal{A}$, there is an $S' \in \mathcal{M}$ with $S'y = e_3$. Thus $e_3 \in \mathcal{M}\mathcal{U} \subset \mathcal{U}$.

Consider now the pair e_1, e_2. Since $Te_2 \neq e_1$ for all $T \in \mathcal{A}$, there is an $S'' \in \mathcal{M}$ with $S''e_1 = e_2$. Observe that $S''e_3 = e_1$, and so $e_1 \in \mathcal{M}\mathcal{U} \subset \mathcal{U}$. Further, $e_2 = S''e_1 \in \mathcal{U}$. Hence $\mathcal{U} = \mathcal{V}$, a contradiction.

For $n \geq 4$ we can take

$$\left\{ \begin{pmatrix} a & 0 & 0 & 0 & \dots & 0 & b \\ b & x_{22} & 0 & 0 & \dots & 0 & 0 \\ x_{31} & x_{32} & x_{33} & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{n-1,1} & x_{n-1,2} & x_{n-1,3} & x_{n-1,4} & \dots & x_{n-1,n-1} & 0 \\ x_{n1} & x_{n2} & x_{n3} & x_{n4} & \dots & x_{n,n-1} & a \end{pmatrix} \;\middle|\; a, b, x_{i,j} \in \mathbb{F} \right\}$$

and proceed as before. □

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4. Triangularizable semitransitive subspaces. In this section we study triangularizable semitransitive subspaces of matrices.

LEMMA 4.1. *Let $\mathcal{L}$ be a semitransitive subspace acting on a vector space $\mathcal{V}$. Assume that a subspace $\mathcal{U}$ of $\mathcal{V}$ is invariant under $\mathcal{L}$ and the action of $\mathcal{L}$ on $\mathcal{V}/\mathcal{U}$ is transitive. Then $\mathcal{L}$ maps the set $\mathcal{V}-\mathcal{U}$ transitively onto $\mathcal{V}$, i.e. for each $v, v' \in \mathcal{V}$ with $v \notin \mathcal{U}$, there is an $L \in \mathcal{L}$ such that $Lv = v'$.*

Proof. Let $\mathcal{W} \leq \mathcal{V}$ be such that $\mathcal{V} = \mathcal{W} \oplus \mathcal{U}$. Let $v = w + u$ and $v' = w' + u'$, with $u, u' \in \mathcal{U}$ and $w, w' \in \mathcal{W}$. We will show that, whenever $w \neq 0$, there exists $L \in \mathcal{L}$ so that $Lv = v'$. Note that if $w \neq 0$ then there exists $S \in \mathcal{L}$ so that $Sw = w' + x$ for some $x \in \mathcal{U}$ (since $\mathcal{L}$ acts transitively on $\mathcal{V}/\mathcal{U}$). Now semitransitivity yields a $T \in \mathcal{L}$ so that $Tv = v' - Sv$ (since $v'' := v' - Sv = u' - x - Su \in \mathcal{U}$ and hence $v \notin \mathcal{L}v''$). This finishes the proof since $(S + T)v = v'$. □

COROLLARY 4.2. *Assume that $\{0\} = \mathcal{V}_0 < \mathcal{V}_1 < \ldots < \mathcal{V}_n = \mathbb{F}^n$ is a triangularizing chain for a linear subspace $\mathcal{L}$ of $\mathbb{M}_n(\mathbb{F})$. The following are equivalent*

1. *$\mathcal{L}$ is semitransitive;*
2. *For each $i = 1, \ldots, n$, $\mathcal{L}$ maps the set $\mathcal{V}_i - \mathcal{V}_{i-1}$ transitively onto $\mathcal{V}_i$.*

If $\mathcal{V}$ is a vector space and $\mathcal{L}$ a vector subspace of $\mathrm{End}(\mathcal{V})$, then the evaluation $\mathrm{ev}_{\mathcal{L}} \colon \mathcal{L} \times \mathcal{V} \to \mathcal{V}$, $\mathrm{ev}_{\mathcal{L}}(L, v) = L(v)$ is a bilinear map and hence gives rise to a linear map $l \colon \mathcal{L} \otimes \mathcal{V} \to \mathcal{V}$. We identify $\mathcal{L}$, l and $\mathrm{ev}_{\mathcal{L}}$. More precisely, if $\mathcal{V} = \mathbb{F}^n$ and $\dim \mathcal{L} = m$, then we view $\mathcal{L}$ as a linear map $l \colon \mathbb{F}^{mn} \to \mathbb{F}^n$, or a bilinear map $\mathrm{ev}_{\mathcal{L}} \colon \mathbb{F}^m \times \mathbb{F}^n \to \mathbb{F}^n$ whenever convenient. If $x \in \mathbb{F}^m$ then we abbreviate $L_x = \mathrm{ev}_{\mathcal{L}}(x, _)$, i.e. for $a \in \mathbb{F}^n$ we have $L_x a = \mathrm{ev}_{\mathcal{L}}(x, a)$. This gives a $\mathbb{F}^m$ parametrization for $\mathcal{L} = \{L_x | x \in \mathbb{F}^m\}$. For $a \in \mathbb{F}^n$ we abbreviate $A_a = \mathrm{ev}_{\mathcal{L}}(_, a)$, i.e. for $x \in \mathbb{F}^m$ we have $A_a x = \mathrm{ev}_{\mathcal{L}}(x, a)$. If $e_1, \ldots, e_n$ denotes the standard basis of $\mathbb{F}^n$, then we also abbreviate $A_i = A_{e_i}$. Furthermore, for the rest of the paper we define $\mathcal{V}_j = \bigvee_{i=1}^{j} e_i$, $j = 1, \ldots, n$, and we denote the range of a matrix A by $\mathcal{R}(A)$.

If $A_1, \ldots, A_n$ are $n \times m$ matrices, then $\mathcal{L}(A_1, \ldots, A_n) = \mathcal{L}(A_i)_i$ denotes the linear space of $n \times n$ matrices, parameterized by $\mathbb{F}^m$ as above, i.e. $\mathcal{L}(A_i)_i = \{L_x | x \in \mathbb{F}^m\}$, where $L_x e_i = A_i x$, or in matrix notation

$$L_x = \begin{pmatrix} A_1 x & A_2 x & \ldots & A_n x \end{pmatrix}.$$

LEMMA 4.3. *If $A_i \colon \mathbb{F}^m \to \mathbb{F}^n$, $i = 1, \ldots, n$ are matrices satisfying $\mathcal{R}(A_j + \sum_{i<j} y_i A_i) \supseteq \mathcal{V}_j$, for all $j \leq n$ and all $y = (y_i)_i \in \mathbb{F}^n$, then the linear space $\mathcal{L}(A_i)_i$ acts semitransitively on $\mathbb{F}^n$.*

Proof. Choose nonzero vectors $y = (y_i)$ and $y' = (y'_i)$ in $\mathbb{F}^n$. Without any loss of generality we assume that $\max\limits_{y'_i \neq 0} i \leq \max\limits_{y_i \neq 0} i$. By the assumption, we have $y' \in \mathcal{R}(\sum_i y_i A_i)$. Let $x \in \mathbb{F}^m$ be such that $(\sum_i y_i A_i)x = y'$ and note that $L_x y = y'$. □

THEOREM 4.4. *Let $\mathcal{L} = \mathcal{L}(A_i)_i$. If $A_i \colon \mathbb{F}^m \to \mathbb{F}^n$ are such that $\mathcal{R}(A_i) \subseteq \mathcal{V}_i$ (that is, $\{0\} < \mathcal{V}_1 < \ldots < \mathcal{V}_n$ is a triangularizing chain for $\mathcal{L}$), then the following assertions are equivalent*

1. *$\mathcal{L}$ is semitransitive;*

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A publication of the International Linear Algebra Society
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2. $\mathcal{R}(A_j + \sum_{i<j} y_i A_i) = \mathcal{V}_j$*, for all* $j = 1, \ldots, n$ *and all* $y = (y_i)_i$*.*

Proof. Combine Lemma 4.3 and Corollary 4.2. □

THEOREM 4.5. *Every triangularizable semitransitive linear space in* $\mathbb{M}_n(\mathbb{F})$ *contains a nonzero nilpotent.*

Proof. Suppose $\mathcal{L}$ is an upper triangular semitransitive linear space in $\mathbb{M}_n(\mathbb{F})$. By Corollary 2.7, $\mathcal{L}$ is at least n-dimensional. Define a linear map $Diag : \mathcal{L} \longrightarrow \mathbb{F}^n$ by declaring that the i-th entry of $Diag(T)$ is $T[i,i]$, so that $Diag(T)$ is simply the vector appearing on the diagonal of T. If $Diag$ is not bijective, then it has a non-trivial kernel, and any matrix in that kernel is nilpotent, so that the proof is complete in such a case. Therefore from now on assume that $\dim \mathcal{L} = n$ and $Diag$ is bijective.

For each m, let L_m be the unique matrix in $\mathcal{L}$ such that $Diag(L_m) = e_m$. Obviously $L_1, \ldots, L_n$ is a basis of $\mathcal{L}$. We shall show that this leads to a contradiction.

Using the notation introduced above let $A_1, A_2, \ldots, A_n$ be $n \times n$ matrices such that

$$L_i = \begin{pmatrix} A_1 e_i & A_2 e_i & \ldots & A_n e_i \end{pmatrix}.$$

Observe that the j-th row of A_j is equal to e_j^{tr} and the rows below the j-th row in A_j are zero. Now we apply Theorem 4.4. Since $\mathcal{L}$ is semitransitive, the matrix $A(y_1, y_2, \ldots, y_{n-1}) = A_n + \sum_{j=1}^{n-1} y_j A_j$ has to be invertible for all values $y_j \in \mathbb{F}$, $j = 1, 2, \ldots, n-1$. The matrix $A = A(y_1, y_2, \ldots, y_{n-1})$ has the following form

$$\begin{pmatrix} y_1 + r_{11} & r_{12} & \ldots & r_{1\,n-1} & r_{1\,n} \\ r_{21} & y_2 + r_{22} & \ldots & r_{2\,n-1} & r_{2\,n} \\ r_{31} & r_{32} & \ldots & r_{3\,n-1} & r_{3\,n} \\ \vdots & \vdots & & \vdots & \vdots \\ r_{n-2\,1} & r_{n-2\,2} & \ldots & r_{n-2\,n-1} & r_{n-2\,n} \\ r_{n-1\,1} & r_{n-1\,2} & \ldots & y_{n-1} + r_{n-1\,n-1} & r_{n-1\,n} \\ 0 & 0 & \ldots & 0 & 1 \end{pmatrix},$$

where $r_{ij} = r_{ij}(y_{i+1}, \ldots, y_{n-1})$ are linear polynomials in the indicated variables and $r_{n-1\,j}$ are constants. Observe that for each m, $m = 1, 2, \ldots, n-1$, the (m,m)-th entry of A is equal to $y_m + r_{m\,m}(y_{m+1}, \ldots, y_{n-1})$ and no other entry in the $(n-m+1) \times (n-m+1)$ principle south-east compression of A is dependent on $y_1, \ldots, y_m$. It follows that there exist $y_1, y_2, \ldots, y_{n-1}$ such that for each $1 \leq k \leq n-1$ the determinant of the $k \times k$ principle south-east compression of A is 1, and the determinant of A is zero. This contradicts the invertibility of A, and the proof is complete. □

Both minimal subspaces $\mathcal{L}_0$ and $\mathcal{L}$ of Examples 3.1 and 3.2 have dimension n. Is this always the case? Before we answer this question for $n = 2$ we need the following auxiliary result.

LEMMA 4.6. *Assume that* $\mathbb{F}$ *is an algebraically closed field. If* $\mathcal{L}$ *is a transitive subspace of* $\mathbb{M}_2(\mathbb{F})$*, then there is a two-dimensional semitransitive subspace contained in* $\mathcal{L}$*.*

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A publication of the International Linear Algebra Society
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Proof. A minimal transitive subspace in $\mathbb{M}_2(\mathbb{F})$ is of dimension 3. (See [1, Sect. 4] and note that the proof given there works over any algebraically closed field.) If $\mathcal{L} = \mathbb{M}_2(\mathbb{F})$ then the statement is obvious. Assume that $\dim \mathcal{L} = 3$. Then it follows that $\mathcal{L}$ is of the form $A^\perp = \{\ B \in \mathbb{M}_2(\mathbb{F}) \mid \operatorname{trace}(BA^{tr}) = 0\ \}$ for some nonzero matrix $A \in \mathbb{M}_2(\mathbb{F})$. Since $\mathcal{L}$ is transitive, the corresponding matrix A is invertible: Observe that if A is of rank 1 then $A = vu^{tr}$ for some nonzero $u, v \in \mathbb{F}^2$. Then $\operatorname{trace}(Buv^{tr}) = v^{tr}Bu = 0$ for all $B \in \mathcal{L}$, and so the set $\{Bu | B \in \mathcal{L}\}$ is one-dimensional contradicting the transitivity.

Since nonzero multiples of A determine the same subspace $A^\perp$ and since it is enough to consider A up to similarity, we need to consider two cases only: $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}$, $\alpha \neq 0$, and $A_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Then we find that

$$A_1^\perp = \left\{ \begin{pmatrix} \alpha a & b \\ c & -a \end{pmatrix} \;\middle|\; a, b, c \in \mathbb{F} \right\} \text{ and } A_2^\perp = \left\{ \begin{pmatrix} a & -a-c \\ b & c \end{pmatrix} \;\middle|\; a, b, c \in \mathbb{F} \right\}.$$

Now $\mathcal{L}_1 = \left\{ \begin{pmatrix} \alpha a & b \\ 0 & -a \end{pmatrix} \;\middle|\; a, b \in \mathbb{F} \right\}$ and $\mathcal{L}_2 = \left\{ \begin{pmatrix} a & 0 \\ b & -a \end{pmatrix} \;\middle|\; a, b \in \mathbb{F} \right\}$ are two-dimensional semitransitive subspaces in $A_1^\perp$ and $A_2^\perp$, respectively. □

PROPOSITION 4.7. *Let $\mathbb{F}$ be algebraically closed.*

(a) Every semitransitive subspace of $\mathbb{M}_2(\mathbb{F})$ which is not transitive is triangularizable.

(b) Every minimal semitransitive subspace of $\mathbb{M}_2(\mathbb{F})$ is similar to

$$\left\{ \begin{pmatrix} a & b \\ 0 & ca \end{pmatrix} \;\middle|\; a, b \in \mathbb{F} \right\}$$

for a fixed $c \neq 0$.

Proof. (a) If $\mathcal{L}x = \mathbb{F}^2$ for all nonzero $x \in \mathbb{F}^2$, then $\mathcal{L}$ is transitive. Hence, we may assume that there exists a nonzero $x \in \mathbb{F}^2$ such that $\mathcal{L}x$ (necessarily containing x) is not equal to $\mathbb{F}^2$. Choose this x for the first basis vector to get $\mathcal{L}$ upper triangular.

(b) Let $\mathcal{L} \subset \mathbb{M}_2(\mathbb{F})$ be minimal semitransitive. By Lemma 4.6, $\mathcal{L}$ cannot be transitive. By (a) we may assume that $\mathcal{L}$ is upper triangularizable. Observe that $\dim \mathcal{L} = 2$ (if $\dim \mathcal{L} = 3$ then $\mathcal{L}$ consists of all upper triangular matrices and is clearly not minimal semitransitive). By Theorem 4.5, $\mathcal{L}$ contains a nilpotent and hence also an element of the form $\operatorname{diag}(1, c)$. □

The following example shows that in Proposition 4.7 we cannot omit the assumption that $\mathbb{F}$ is algebraically closed.

EXAMPLE 4.8. The space

$$\left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \;\middle|\; a, b \in \mathbb{R} \right\}$$

is a minimal semitransitive subspace of $\mathbb{M}_2(\mathbb{R})$, but it has no nonzero nilpotents.

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A publication of the International Linear Algebra Society
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By above, if $\mathbb{F}$ is an algebraically closed field, every minimal semitransitive subspace in $\mathbb{M}_2(\mathbb{F})$ is triangularizable and 2-dimensional. However, neither of this holds for $n \geq 3$.

EXAMPLE 4.9. The space

$$\mathcal{S} := \left\{ \begin{pmatrix} a & b & c \\ 0 & d & a \\ 0 & 0 & d \end{pmatrix} \;\middle|\; a, b, c, d \in \mathbb{F} \right\}$$

is a minimal semitransitive subspace of $\mathbb{M}_3(\mathbb{F})$ but $\dim(\mathcal{S}) = 4$.

In order to show semitransitivity of $\mathcal{S}$, take the basis matrices

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},\ B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ C = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and a nonzero vector $u = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. If $z \neq 0$, u is cyclic for $\mathcal{S}$, since

$$\frac{1}{z}Cu = e_1,\ \frac{1}{z}Au - \frac{x}{z}e_1 = e_2, \text{ and } \frac{1}{z}Du - \frac{y}{z}e_2 = e_3.$$

Furthermore, if $z = 0$ and $y \neq 0$ we have

$$\frac{1}{y}Bu = e_1 \text{ and } \frac{1}{y}Du = e_2,$$

while $Au = u$ in the case when $y = z = 0$ and $x \neq 0$.

Now let us show that $\mathcal{S}$ is minimal semitransitive. Assume the contrary that there exists a semitransitive subspace $\mathcal{S}_0 \subsetneq \mathcal{S}$. By semitransitivity, there are matrices in $\mathcal{S}_0$ taking the basis vector e_3 to e_1, e_2 and e_3, respectively. Therefore the following linearly independent matrices

$$\begin{pmatrix} 0 & b_1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & b_2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 0 & b_3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

are contained in $\mathcal{S}_0$ for some $b_1, b_2, b_3 \in \mathbb{F}$. Since $\dim(\mathcal{S}_0) = 3$, these matrices span $\mathcal{S}_0$. Let $v = \begin{pmatrix} 0 \\ -1 \\ b_1 \end{pmatrix}$ if $b_1 \neq 0$, and $v = \begin{pmatrix} b_2 \\ -1 \\ 0 \end{pmatrix}$ otherwise. Now note that no matrix from $\mathcal{S}_0$ maps either of vectors v and e_1 into another. This contradiction proves the minimality of $\mathcal{S}$.

We would now like to describe all triangularizable semitransitive n-dimensional subspaces of $\mathbb{M}_n(\mathbb{F})$ for $n = 2, 3$. Let $\mathcal{T}_0 \subset \mathbb{M}_n(\mathbb{F})$ denote the set of all upper triangular matrices and for every $i = 1, \ldots, n-1$ let $\mathcal{T}_i \subset \mathcal{T}_0$ be the set of all matrices having 0's beneath the i-th superdiagonal.

PROPOSITION 4.10. *Let $\mathbb{F}$ be algebraically closed and $n \in \{2, 3\}$. Every triangularizable semitransitive n-dimensional subspace of $\mathbb{M}_n(\mathbb{F})$ is similar to*

$$\vee \{T_0, T_1, \ldots, T_{n-1}\}$$

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A publication of the International Linear Algebra Society
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for some nonzero $T_i \in \mathcal{T}_i$ of rank $n-i$.

Proof. For $n = 2$, the result follows from Proposition 4.7. Now, let $n = 3$. Assume that $\mathcal{L}$ is a semitransitive subspace of $\mathbb{M}_3(\mathbb{F})$ of dimension 3 and that $\{0\} = \mathcal{V}_0 < \mathcal{V}_1 < \mathcal{V}_2 < \mathcal{V}_3 = \mathbb{F}^3$ is a triangularizing chain for $\mathcal{L}$. Let $\mathcal{L} = \mathcal{L}(A_1, A_2, A_3)$. Note that if $X \in \mathbb{M}_3(\mathbb{F})$ is invertible, then $\mathcal{L} = \mathcal{L}(A_1X, A_2X, A_3X)$. We make use of Theorem 4.4. The condition $\mathcal{R}(A_3) = \mathcal{V}_3 = \mathbb{F}^3$ shows that A_3 is invertible and hence we can assume, without any loss of generality, that $A_3 = I$. Note that any linear combination of A_1 and A_2 is nilpotent, since the range of $A_3 + a_2A_2 + a_1A_1$ is $\mathbb{F}^3$ (for all a_1, a_2). Since $\mathcal{R}(A_1) = \mathcal{V}_1$ and $\mathcal{R}(A_2) = \mathcal{V}_2$, we must have

$$A_1 = \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} a & b & e \\ c & d & f \\ 0 & 0 & 0 \end{pmatrix},$$

and the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + a_1 \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}$$

is nilpotent for every choice of a_1. Observe that we must have $x = 0$. Indeed, if $x \neq 0$, then we must have $a = d = c = 0$ and $\mathcal{R}(A_2 - x^{-1}bA_1)$ is 1-dimensional; a contradiction. Now inspect the condition $\mathcal{R}(A_2 + a_1A_1) = \mathcal{V}_2$ again and note that for every choice of a_1, the vector

$$\begin{pmatrix} e \\ f \end{pmatrix} + a_1 \begin{pmatrix} y \\ 0 \end{pmatrix}$$

is not in the range of the nilpotent matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

But this is only possible if $f \neq 0$, $a = c = d = 0$ and $b \neq 0$. Hence

$$A_1 = \begin{pmatrix} 0 & 0 & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & b & e \\ 0 & 0 & f \\ 0 & 0 & 0 \end{pmatrix}, A_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

with y, b, f nonzero. Now note that

$$\mathcal{L}(A_1, A_2, A_3) = \bigvee \left\{ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & b & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} y & e & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}. \square$$

In the last result we cannot omit the assumption that $\mathbb{F}$ is algebraically closed.

EXAMPLE 4.11. The linear space

$$\left\{ \begin{pmatrix} a & b & c \\ 0 & -c & b \\ 0 & 0 & a \end{pmatrix} \;\middle|\; a, b, c \in \mathbb{R} \right\}$$

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A publication of the International Linear Algebra Society
Volume 15, pp. 225-238, August 2006

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is semitransitive over $\mathbb{R}$ but not over $\mathbb{C}$.

Using similar methods as in the proof of Proposition 4.10 and considerably more computational efforts one can describe the structure of all 4-dimensional triangularizable semitransitive spaces.

PROPOSITION 4.12. *Every 4-dimensional triangularizable semitransitive subspace in the space of all 4×4 matrices over an algebraically closed field $\mathbb{F}$ is similar either to the space*

$$\mathcal{L}_1 = \bigvee \left\{ \begin{pmatrix} a & b & c & 0 \\ 0 & d & e & 0 \\ 0 & 0 & f & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & g & h & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \right.$$

$$\left. \begin{pmatrix} 0 & 0 & j & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\},$$

where $b, c, e, h \in \mathbb{F}$ are arbitrary elements and $a, d, f, g, i, j \in \mathbb{F}$ are nonzero elements, or to the space

$$\mathcal{L}_2 = \bigvee \left\{ \begin{pmatrix} a & b & c & 0 \\ 0 & df & e & 0 \\ 0 & 0 & f & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -dg - dh^2ij & gj & 0 \\ 0 & dghij & -ghij^2 & 0 \\ 0 & 0 & ghij & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \right.$$

$$\left. \begin{pmatrix} 0 & -dh^2i & g & 0 \\ 0 & dghi & -ghij & 1 \\ 0 & 0 & ghi & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -dghi & 0 & 1 \\ 0 & dg^2i & -g^2ij & 0 \\ 0 & 0 & g^2i & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\},$$

where $b, c, f, h, j \in \mathbb{F}$ are arbitrary elements and $a, d, e, g, i \in \mathbb{F}$ are nonzero elements.

In conclusion, we would like to state four questions. To motivate the first one observe that if $\mathcal{L}$ is a semitransitive subspace of $\mathbb{M}_n(\mathbb{C})$, then $\mathcal{L} + \mathcal{L}^*$ is a transitive subspace. To prove the statement, assume that $\mathcal{L} + \mathcal{L}^*$ is not transitive. So there exists a nonzero vector x in the underlying space such that $\mathcal{L}x + \mathcal{L}^*x$ is a proper subspace. Let y be a nonzero vector perpendicular to this proper subspace. Then y is not in $\mathcal{L}x$ and x is not in $\mathcal{L}y$ contradicting semitransitivity of $\mathcal{L}$. Observe that this implies again that the minimal possible dimension of a semitransitive subspace in $\mathbb{M}_n(\mathbb{C})$ is n (cf. Corollary 2.7), because the minimum for the dimension of a transitive space is $2n - 1$.

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A publication of the International Linear Algebra Society
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QUESTION 4.13. *What additional conditions (given that $\mathcal{L} + \mathcal{L}^*$ is transitive) make $\mathcal{L}$ semitransitive?*

QUESTION 4.14. *Are there minimal semitransitive subspaces over algebraically closed fields that are transitive?*

QUESTION 4.15. *Do semitransitive subspaces over algebraically closed fields always contain nilpotents?*

QUESTION 4.16. *Is every n-dimensional semitransitive subspace over an algebraically closed field triangularizable?*

The last two of these questions have recently been answered affirmatively by part of the authors of this paper and their proofs will appear in a subsequent paper.

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