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AN EIGENVALUE INEQUALITY AND SPECTRUM LOCALIZATION FOR COMPLEX MATRICES*

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Abstract. Using the notions of the numerical range, Schur complement and unitary equivalence, an eigenvalue inequality is obtained for a general complex matrix, giving rise to a region in the complex plane that contains its spectrum. This region is determined by a curve, generalizing and improving classical eigenvalue bounds obtained by the Hermitian and skew-Hermitian parts, as well as the numerical range of a matrix.

Key words. Eigenvalues, Inclusion regions, Numerical range, Hermitian part.

AMS subject classifications. 15A18, 15A60.

1. Introduction. In this article, we consider complex matrices $A \in \mathcal{M}_n(\mathbb{C})$, $n \geq 2$, and obtain an inequality satisfied by the real and imaginary part of every eigenvalue of A (Section 3). In turn, this inequality gives rise to a region in the complex plane that contains the spectrum of A (Section 4). We examine when an eigenvalue lies on the boundary of such a region. The proof of the main inequality in Theorem 3.1 is an adaptation to general matrices of the proof of [7, Theorem 3.1] for almost skew-symmetric matrices. As a consequence, the spectrum localization results for almost skew-symmetric matrices and (hyper)tournaments obtained in [2, 3, 6, 7] follow as special cases. Furthermore, the fact that Theorem 3.1 holds for general matrices allows its application to rotations $e^{\mathbf{i}\,\theta}A$, giving rise to improved localization results for the spectrum. Our concluding remarks outline some related research goals (Section 5).

2. Notation and preliminaries. We begin by settling on the notation to be used. For $A \in \mathcal{M}_n(\mathbb{C})$, its *spectrum* is denoted by $\sigma(A)$ and its *spectral radius* by $\rho(A) = \max\{|\lambda| : \lambda \in \sigma(A)\}$. We write $A = H(A) + S(A)$, where

$$H(A) \;=\; \frac{A + A^*}{2} \quad \text{ and } \quad S(A) \;=\; \frac{A - A^*}{2}$$

are the *Hermitian part* and the *skew-Hermitian part* of A, respectively. The *numerical range* is the set

$$F(A) \;=\; \{v^*Av \in \mathbb{C} : v \in \mathbb{C}^n \text{ with } v^*v = 1\},$$

which is a compact and convex subset of $\mathbb{C}$ that contains the spectrum of A (see [5]). Recall that A is Hermitian if and only if $F(A) \subset \mathbb{R}$, and that if A is normal, then

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$F(A)$ coincides with the convex hull of $\sigma(A)$. It is also well known that

$$\operatorname{Re} F(A) \;=\; F(H(A)) \quad \text{and} \quad \mathbf{i}\operatorname{Im} F(A) \;=\; F(S(A)).$$

When $A \in \mathcal{M}_n(\mathbb{R})$, then $F(A)$ is symmetric with respect to the real axis. Also, any eigenvalue $\lambda \in \sigma(A)$ that belongs to the boundary of the numerical range, $\partial F(A)$, is a *normal eigenvalue* of A; namely, there exists a unitary matrix $U \in \mathcal{M}_n(\mathbb{C})$ such that

$$U^*AU \;=\; \lambda I_k \oplus B,$$

where k is the algebraic multiplicity of λ and $\lambda \notin \sigma(B)$; see [5, Theorem 1.6.6].

Given $A \in \mathcal{M}_n(\mathbb{C})$, let y_1 be a unit eigenvector corresponding to the largest eigenvalue δ_1 of its Hermitian part $H(A)$. We define two quantities to be used in the sequel:

$$\text{(2.1)} \qquad \mathrm{v}(A) \;=\; \|S(A)y_1\|_2^2 \quad \text{and} \quad \mathrm{u}(A) \;=\; \operatorname{Im}(y_1^* S(A) y_1).$$

Notice that $\mathrm{u}(A) = 0$ when A is real. Also if $\mathrm{v}(A) = 0$, then (δ_1, y_1) is an eigenpair for both A and A^*. That is, $\mathrm{v}(A)$ can be thought of as a measure of how close δ_1 is to being a normal eigenvalue of A.

3. The main eigenvalue inequality. Consider a matrix $A \in \mathcal{M}_n(\mathbb{C})$ and let δ_1 be the largest eigenvalue of its Hermitian part $H(A)$. It is well known that all the eigenvalues of A lie in the closed half-plane $\{z \in \mathbb{C} : \operatorname{Re} z \leq \delta_1\}$. In this section, we obtain a new localization of $\sigma(A)$ by replacing the line $\operatorname{Re} z = \delta_1$ with an appropriate curve. We proceed immediately with the inequality satisfied by every eigenvalue of $A \in \mathcal{M}_n(\mathbb{C})$.

THEOREM 3.1. *Let $A \in \mathcal{M}_n(\mathbb{C})$ and let λ be an eigenvalue of A. Let also $\delta_1 \geq \delta_2$ be the two largest eigenvalues of the Hermitian part of A. Then*

$$\text{(3.1)} \quad (\operatorname{Re}\lambda - \delta_2)\,(\operatorname{Im}\ \lambda - \mathrm{u}(A))^2 \;\leq\; (\delta_1 - \operatorname{Re}\lambda)\ \left[\mathrm{v}(A) - \mathrm{u}(A)^2 + (\operatorname{Re}\lambda - \delta_2)\,(\operatorname{Re}\lambda - \delta_1)\right].$$

Proof. Let $y_1 \in \mathbb{C}^n$ be a unit eigenvector of $H(A)$ corresponding to the eigenvalue δ_1. Then, there exists a unitary $U \in \mathcal{M}_n(\mathbb{C})$, whose first column is y_1, such that

$$U^*H(A)U \;=\; \begin{bmatrix} \delta_1 & 0 \\ 0 & H_1 \end{bmatrix}, \quad \text{where} \quad H_1 \;=\; \operatorname{diag}(\delta_2, \delta_3, \ldots, \delta_n),\ \delta_j \in \sigma(H(A)).$$

Moreover, as $U^*S(A)U$ is skew-Hermitian, we have

$$\text{(3.2)} \qquad U^*S(A)U \;=\; \begin{bmatrix} \mathbf{i}\,\alpha & u^* \\ -u & S_1 \end{bmatrix},$$

where $S_1 \in \mathcal{M}_{n-1}(\mathbb{C})$ is skew-Hermitian, $u \in \mathbb{C}^{n-1}$ and $\alpha \in \mathbb{R}$. Consequently, when $\lambda \in \sigma(A)$,

$$U^*(A - \lambda I)U \;=\; \begin{bmatrix} \delta_1 + \mathbf{i}\,\alpha - \lambda & u^* \\ -u & H_1 + S_1 - \lambda I \end{bmatrix}$$

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is a singular matrix. Note that by assumption,

$$\mathbf{i}\,\alpha = y_1^* S(A) y_1, \quad \text{i.e.,} \quad \alpha = \mathrm{u}(A). \tag{3.3}$$

At this stage, consider the case where

$$\lambda \neq \delta_1 + \mathbf{i}\,\alpha = y_1^*(H(A)+S(A))y_1 = y_1^* A y_1;$$

otherwise (3.1) holds trivially and $\lambda = \delta_1 + \mathbf{i}\,\mathrm{u}(A)$ is a normal eigenvalue of A on the boundary of the numerical range $F(A)$. Then the Schur complement of the leading entry in $U^*(A-\lambda I)U$ is

$$E = (H_1 + S_1 - \lambda I) + \frac{1}{\delta_1 + \mathbf{i}\,\alpha - \lambda}\, uu^*,$$

which must be singular since $A - \lambda I$ is singular [4, p. 21]. The Hermitian and skew-Hermitian parts of E can be readily computed to be

$$H(E) = H_1 - \mathrm{Re}\,\lambda\, I + \frac{\delta_1 - \mathrm{Re}\,\lambda}{(\delta_1 - \mathrm{Re}\,\lambda)^2 + (\alpha - \mathrm{Im}\,\lambda)^2}\, uu^*, \tag{3.4}$$

and

$$S(E) = S_1 - \mathbf{i}\,\mathrm{Im}\,\lambda\, I + \frac{\mathbf{i}\,(\mathrm{Im}\,\lambda - \alpha)}{(\delta_1 - \mathrm{Re}\,\lambda)^2 + (\alpha - \mathrm{Im}\,\lambda)^2}\, uu^*. \tag{3.5}$$

Since $0 \in \sigma(E) \subseteq F(E)$ and $F(H(E)) = \mathrm{Re}\, F(E)$ (see [5, Properties 1.2.5, 1.2.6]), it follows that $0 \in F(H(E))$, which in turn implies that there exists unit $x \in \mathbb{C}^n$ such that $x^* H(E) x = 0$; that is,

$$0 = x^* H_1 x - x^* x\, \mathrm{Re}\,\lambda + \frac{\delta_1 - \mathrm{Re}\,\lambda}{(\delta_1 - \mathrm{Re}\,\lambda)^2 + (\alpha - \mathrm{Im}\,\lambda)^2}\, x^* u u^* x, \quad x^* x = 1. \tag{3.6}$$

Notice that since $F(H_1) = [\,\delta_n, \delta_2]$ and since $F(uu^*) = [0, u^*u]$, we respectively have that

$$x^* H_1 x \leq \delta_2 \quad \text{and} \quad x^* u u^* x \leq u^* u.$$

Hence from (3.6) we obtain

$$0 \leq \delta_2 - \mathrm{Re}\,\lambda + \frac{\delta_1 - \mathrm{Re}\,\lambda}{(\delta_1 - \mathrm{Re}\,\lambda)^2 + (\alpha - \mathrm{Im}\,\lambda)^2}\, u^* u. \tag{3.7}$$

Denoting by e_1 the first standard basis vector in $\mathbb{C}^n$, and since unitary matrices preserve the Euclidean norm, it follows that

$$\begin{aligned} \alpha^2 + u^* u &= \left\| \begin{bmatrix} \mathbf{i}\,\alpha \\ -u \end{bmatrix} \right\|_2^2 = \left\| U \begin{bmatrix} \mathbf{i}\,\alpha & u^* \\ -u & S_1 \end{bmatrix} e_1 \right\|_2^2 \\ &= \left\| U \begin{bmatrix} \mathbf{i}\,\alpha & u^* \\ -u & S_1 \end{bmatrix} U^* U e_1 \right\|_2^2 = \| S(A) y_1 \|_2^2 = \mathrm{v}(A). \end{aligned} \tag{3.8}$$

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Equality (3.8) implies that

$$u^* u = \mathrm{v}(A) - \alpha^2. \tag{3.9}$$

The validity of (3.1) now follows from (3.3), (3.7) and (3.9). □

Corollary 3.2. *Let $A \in \mathcal{M}_n(\mathbb{C})$ and let $\delta_1 > \delta_2$ be the two largest eigenvalues of the Hermitian part $H(A)$. If $4(\mathrm{v}(A) - \mathrm{u}(A)^2) < (\delta_1 - \delta_2)^2$, then for every $\lambda \in \sigma(A)$ with $\mathrm{Re}\,\lambda > \delta_2$, we have that $\mathrm{Re}\,\lambda \notin (s,t)$, where*

$$s = \frac{\delta_1 + \delta_2 - \sqrt{(\delta_1 - \delta_2)^2 - 4(\mathrm{v}(A) - \mathrm{u}(A)^2)}}{2},$$
$$t = \frac{\delta_1 + \delta_2 + \sqrt{(\delta_1 - \delta_2)^2 - 4(\mathrm{v}(A) - \mathrm{u}(A)^2)}}{2}.$$

Proof. Consider $\lambda \in \sigma(A)$ such that $\delta_2 < \mathrm{Re}\,\lambda < \delta_1$. Then by (3.1),

$$(\delta_1 - \mathrm{Re}\,\lambda)\left(\frac{\mathrm{v}(A) - \mathrm{u}(A)^2}{\mathrm{Re}\,\lambda - \delta_2} + \mathrm{Re}\,\lambda - \delta_1\right) \geq 0.$$

The latter inequality and the assumption $(\mathrm{v}(A) - \mathrm{u}(A)^2) < (\delta_1 - \delta_2)^2/4$ imply directly that either

$$\mathrm{Re}\,\lambda \leq \frac{\delta_1 + \delta_2 - \sqrt{(\delta_1 - \delta_2)^2 - 4(\mathrm{v}(A) - \mathrm{u}(A)^2)}}{2}$$

or

$$\mathrm{Re}\,\lambda \geq \frac{\delta_1 + \delta_2 + \sqrt{(\delta_1 - \delta_2)^2 - 4(\mathrm{v}(A) - \mathrm{u}(A)^2)}}{2}. \ \square$$

4. Localization of the spectrum. We will use our results in the previous section to obtain an inclusion region for $\sigma(A)$ determined by a curve $\Gamma(A)$. This curve comprises the points $s + \mathbf{i}\,t \in \mathbb{C}$ that satisfy (3.1) as an equality.

In order to define and plot $\Gamma(A)$, as well as study its position relative to the eigenvalues, it is useful to first rewrite inequality (3.1) by separating the imaginary part from the real part of λ.

Recalling the proof and notation used in Theorem 3.1, notice that (3.1) takes a trivial form when $\mathrm{Re}\,\lambda = \delta_1$ or $\mathrm{Re}\,\lambda \leq \delta_2$. Therefore, let us consider $\lambda \in \sigma(A)$ with $\delta_2 < \mathrm{Re}\,\lambda < \delta_1$. Then (3.1) implies that

$$(\mathrm{Im}\,\lambda - \mathrm{u}(A))^2 \leq (\delta_1 - \mathrm{Re}\,\lambda)\left(\frac{\mathrm{v}(A) - \mathrm{u}(A)^2}{\mathrm{Re}\,\lambda - \delta_2} + \mathrm{Re}\,\lambda - \delta_1\right). \tag{4.1}$$

Definition 4.1. Prompted by the latter inequality, we define a curve $\Gamma(A)$ in the complex plane associated with the matrix $A \in \mathcal{M}_n(\mathbb{C})$ given by

$$\left\{ s + \mathbf{i}\,t \in \mathbb{C} : s,t \in \mathbb{R} \text{ and } (t - \mathrm{u}(A))^2 = (\delta_1 - s)\left(\frac{\mathrm{v}(A) - \mathrm{u}(A)^2}{s - \delta_2} + s - \delta_1\right)\right\}.$$

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Recalling the notation and definitions in (2.1), by Cauchy-Schwarz, we have

$$|y_1^* S(A) y_1| \;\leq\; \|S(A)y_1\|_2 \, \|y_1\|_2.$$

As $\|y_1\|_2 = 1$, the following implications ensue:

$$\mathrm{Re}^2(y_1^* S(A) y_1) + \mathrm{Im}^2(y_1^* S(A) y_1) = |y_1^* S(A) y_1|^2 \;\leq\; \|S(A)y_1\|_2^2 = \mathrm{v}(A) \implies$$

$$\mathrm{Im}^2(y_1^* S(A) y_1) \leq \mathrm{v}(A) - \mathrm{Re}^2(y_1^* S(A) y_1) \leq \mathrm{v}(A) \implies$$

$$\mathrm{u}(A)^2 \leq \mathrm{v}(A). \tag{4.2}$$

We can now make the following observations regarding $\Gamma(A)$.

Observation 4.2.

1. If $\delta_1 = \delta_2$, the defining equation of $\Gamma(A)$ becomes

   $$(t - \mathrm{u}(A))^2 + (s - \delta_1)^2 = -\mathrm{v}(A) + \mathrm{u}(A)^2 = 0;$$

   that is, by (4.2), $\Gamma(A)$ is degenerate, consisting of only one point. Henceforth we assume that $\delta_1 > \delta_2$.

2. The line $\{z \in \mathbb{C} : \mathrm{Re}\, z = \delta_2\}$ is a vertical asymptote of $\Gamma(A)$.

3. $\Gamma(A)$ is symmetric with respect to the horizontal line $\{z \in \mathbb{C} : \mathrm{Im}\, z = \mathrm{u}(A)\}$ which it intercepts at δ_1.

4. As the defining equation of $\Gamma(A)$ for any fixed t is a cubic equation in s, $\Gamma(A)$ may intercept the line $\mathcal{L} = \{z \in \mathbb{C} : \mathrm{Im}\, z = \mathrm{u}(A)\}$ in up to three distinct points. Specifically, the defining equation of $\Gamma(A)$ for $t = \mathrm{u}(A)$, $s \neq \delta_2$, is

   $$(\delta_1 - s)\left(s^2 - (\delta_1 + \delta_2)s + \delta_1\delta_2 + \mathrm{v}(A) - \mathrm{u}(A)^2\right) = 0. \tag{4.3}$$

   Consider then the discriminant of the quadratic factor

   $$\Delta = (\delta_1 - \delta_2)^2 - 4\left(\mathrm{v}(A) - \mathrm{u}(A)^2\right).$$

   ► If $\Delta < 0$, then (4.3) has only one real root, δ_1. Consequently, $\Gamma(A)$ intercepts $\mathcal{L}$ only once, at the point $(\delta_1, \mathrm{u}(A))$ and $\Gamma(A)$ is a simple open curve; see Examples 4.3 and 4.4.

   ► If $\Delta > 0$, then (4.3) has distinct roots

   $$s_1 = \frac{\delta_1 + \delta_2 - \sqrt{\Delta}}{2}, \quad s_2 = \frac{\delta_1 + \delta_2 + \sqrt{\Delta}}{2} \quad \text{and} \quad \delta_1.$$

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In this case $\Gamma(A)$ comprises two branches one of which is a Jordan curve (closed and simple); see the matrix A in Example 4.5. Note that s_1, s_2 coincide with the endpoints of the interval (s, t) in Corollary 3.2.

► If $\Delta = 0$, then (4.3) has a double root $\dfrac{\delta_1 + \delta_2}{2}$ and a simple root δ_1. In this case $\Gamma(A)$ is a folium of Descartes; see the matrix B in Example 4.5.

5. By (4.2) and since for points $s + \mathbf{i}\,t \in \Gamma(A)$ we have

$$(\delta_1 - s)\left(\frac{\mathrm{v}(A) - \mathrm{u}(A)^2}{s - \delta_2} + s - \delta_1\right) \geq 0,$$

$\Gamma(A)$ must lie in the complex region $\{z \in \mathbb{C} : \delta_2 < \operatorname{Re} z \leq \delta_1\}$.

Based on the above observations and inequality (4.1), $\Gamma(A)$ yields a localization for the spectrum of A justified as follows:

- First of all, by items (2) and (5) of Observation 4.2, **every eigenvalue λ of A with $\operatorname{Re}\lambda \leq \delta_2$ lies to the left of** $\Gamma(A)$.

- When $\Gamma(A)$ is a simple open curve, any $\lambda \in \sigma(A)$ with $\delta_2 < \operatorname{Re}\lambda < \delta_1$ must lie in a region W bounded by $\Gamma(A)$ with the following property: By items (2), (3) and (5) of Observation 4.2, as well as the orientation of the inequality in (4.1), for every $w \in W$, $\operatorname{Re} w \leq \delta_1$ and $\operatorname{Re} w + \mathbf{i}\,t \in \Gamma(A)$ for some $t \geq 0$. In other words, **when** $\Gamma(A)$ **is a simple open curve ($\Delta < 0$), all eigenvalues of A lie to the left of** $\Gamma(A)$; this situation is illustrated in Examples 4.3 and 4.4.

- Finally, when $\Gamma(A)$ is not a simple open curve ($\Delta \geq 0$), some eigenvalues λ with $\delta_2 < \operatorname{Re}\lambda < \delta_1$ may lie to the right of the point s_2 in regions formed by $\Gamma(A)$ that satisfy (4.1). When $\Delta > 0$, by Corollary 3.2, no eigenvalue has real part in (s_1, s_2). See the two instances in Example 4.5, as well as our discussion in Section 5.

Example 4.3. Let

$$A = \begin{bmatrix} -2 & 1 & -1 & 1 \\ 1 & 0 & -1 & -1 \\ 2 & 0 & -3 & 0 \\ 1 & 1 & -1 & 2 \end{bmatrix}$$

for which

$$\mathrm{v}(A) = 1.7051,\ \mathrm{u}(A) = 0,\ \delta_1 = 2.2944 \quad \text{and} \quad \delta_2 = 0.3536.$$

Here $\Delta < 0$ and so $\Gamma(A)$ is a simple open curve. In Figure 4.1, $\Gamma(A)$, the numerical range (shaded) and the Geršgorin disks of A are displayed. Eigenvalues of A are marked with $+$'s.

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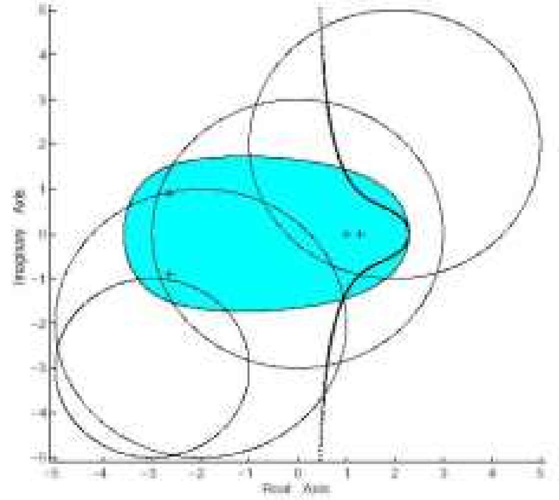

FIG. 4.1. *$\Gamma(A)$ when $\Delta < 0$, the numerical range and the Geršgorin disks of A.*

EXAMPLE 4.4. We have constructed a matrix $A \in \mathcal{M}_4(\mathbb{C})$ with

$$\mathrm{v}(A) = 17.0173,\ \mathrm{u}(A) = 3.3800 \quad \text{and} \quad \delta_1 = 3.5708,\ \delta_2 = -0.2342.$$

As $\Delta < 0$, $\Gamma(A)$ in Figure 4.2 is a simple open curve. A is a non-real matrix and $\Gamma(A)$ is symmetric with respect to the horizontal line $\{z \in \mathbb{C} : \operatorname{Im} z = \mathrm{u}(A) \neq 0\}$. Eigenvalues of A are marked with $+$'s.

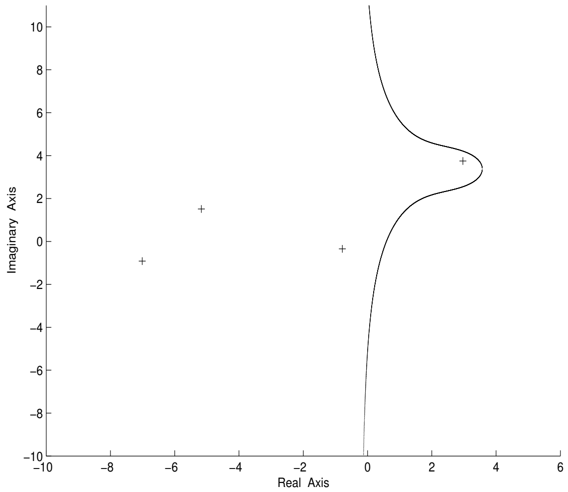

FIG. 4.2. *The curve $\Gamma(A)$ for non-real A and $\Delta < 0$.*

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Example 4.5. We have constructed matrices A and B with the following data:

$$A \in \mathcal{M}_3(\mathbb{R}) \quad \text{with} \quad \mathrm{v}(A) = 0.1063,\ \mathrm{u}(A) = 0,\ \delta_1 = 0.2621,\ \delta_2 = -0.8082;$$

$$B \in \mathcal{M}_5(\mathbb{C}) \quad \text{with} \quad \mathrm{v}(B) = 1,\ \mathrm{u}(B) = 0,\ \delta_1 = 2,\ \delta_2 = 0.$$

In Figure 4.3, the eigenvalues of each matrix are marked with $+$'s.

$\Gamma(A)$ comprises two branches*, one of which is open and the other a Jordan curve ($\Delta > 0$). The closed branch of $\Gamma(A)$ surrounds a real eigenvalue of A and the other branch isolates the rest of the spectrum to its left (since all eigenvalues whose real parts are less than δ_2 must lie to the left of $\Gamma(A)$).

$\Gamma(B)$ intersects itself like a folium of Descartes ($\Delta = 0$) whose loop encloses an eigenvalue λ with $\mathrm{Re}\,\lambda > \frac{\delta_1+\delta_2+\sqrt{\Delta}}{2} = s_2$. All other eigenvalues lie to the left of $\Gamma(B)$ for the same reason as above.

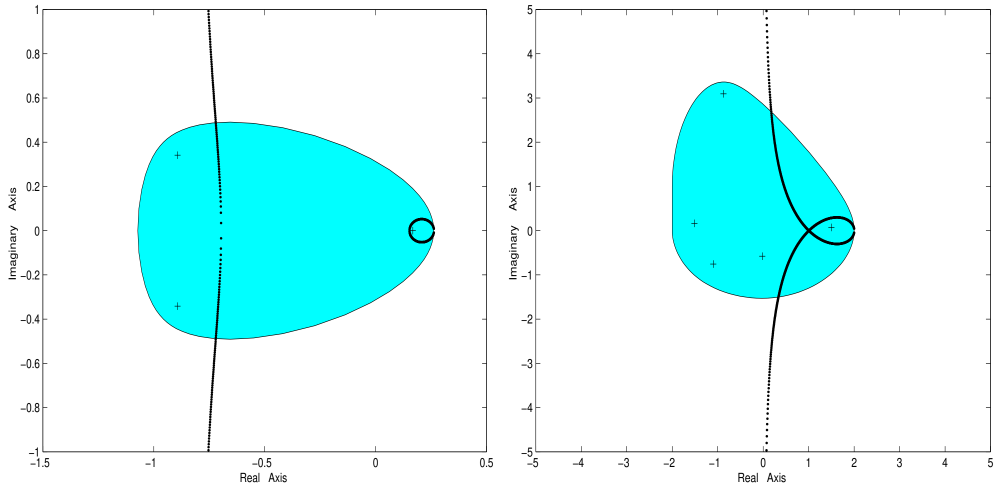

Fig. 4.3. *$\Gamma(A)$ has a simple closed branch and $\Gamma(B)$ is not simple.*

Note that when A is a normal matrix, by the definitions and the fact that left and right eigenvectors of A coincide, it follows that there exists $\lambda \in \sigma(A)$ with $\mathrm{Re}\,\lambda = \delta_1$ and $\mathrm{v}(A) = \mathrm{Im}^2\lambda$, $\mathrm{u}(A) = \mathrm{Im}\,\lambda$. In particular, if $\mathrm{Im}\,\lambda = 0$ (e.g., when A is Hermitian), then $\delta_1 \in \sigma(A)$, namely, an eigenvalue of A lies on $\Gamma(A)$.

Our next goal is to identify eigenvalues on $\Gamma(A)$. For that purpose, denote the circle centered at $\alpha \in \mathbb{C}$ with radius $r \geq 0$ by

$$C(\alpha, r) \;=\; \{s + \mathbf{i}\,t : s, t \in \mathbb{R} \;\text{ and }\; (s - \mathrm{Re}\,\alpha)^2 + (y - \mathrm{Im}\,\alpha)^2 = r^2\}.$$

*Some gaps appearing in the graphs are due to resolution issues.

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THEOREM 4.6. *Let $A \in \mathcal{M}_n(\mathbb{C})$ and λ be an eigenvalue of A such that $\lambda \in \Gamma(A)$. Let also $\delta_1 > \delta_2$ be the two largest eigenvalues of the Hermitian part of A. Then either λ is a real root of the equation*

$$\lambda^3 - (2\delta_1 + \delta_2)\lambda^2 + (\delta_1^2 + 2\delta_1\delta_2 + \mathrm{v}(A))\lambda + \mathrm{u}(A)^2(\delta_1 - \delta_2) - \delta_1(\mathrm{v}(A) + \delta_1\delta_2) = 0,$$

or $\lambda = x + \mathbf{i}\,y \notin C(\delta_1, \mathrm{u}(A))$ *satisfies the equation*

$$2x^3 - (5\delta_1 + \delta_2)x^2 + [2\delta_1\delta_2 + 4\delta_1^2 + 2y^2 + 2\mathrm{u}(A)y]x - (\delta_1 + \delta_2)y^2 - 2\delta_1\mathrm{u}(A)y$$

$$= (\delta_1 - \delta_2)\mathrm{u}(A)^2 + \delta_1^2(\delta_1 + \delta_2).$$

Proof. Let $\lambda \neq \delta_1$ be an eigenvalue of A with $\operatorname{Re}\lambda \neq \delta_2$, and $u, E, H(E), S(E)$ be as defined in the proof of Theorem 3.1. Equality in (3.1), namely

$$(\operatorname{Im}\lambda - \mathrm{u}(A))^2 = (\delta_1 - \operatorname{Re}\lambda)\left(\frac{\mathrm{v}(A) - \mathrm{u}(A)^2}{\operatorname{Re}\lambda - \delta_2} + \operatorname{Re}\lambda - \delta_1\right)$$

holds if and only if

$$\operatorname{Re}\lambda - \delta_2 = \frac{(\delta_1 - \operatorname{Re}\lambda)\,u^*u}{(\delta_1 - \operatorname{Re}\lambda)^2 + (\operatorname{Im}\lambda - \mathrm{u}(A))^2}. \tag{4.4}$$

In this case, the eigenvalue $0 \in \sigma(H(E))$ is simple and corresponds to the eigenvector u that appears in (3.2). Moreover, the matrix E is singular and $0 \in \partial F(E)$ because $\operatorname{Re} F(E) = F(H(E))$. Thus 0 must be a normal eigenvalue of E (see [5]) and every corresponding eigenvector belongs to $\operatorname{Nul}(H(E)) \cap \operatorname{Nul}(S(E)) = \operatorname{span}\{u\}$. Hence, u is an eigenvector of $S(E)$ in (3.5) corresponding to the eigenvalue 0 and so

$$\frac{u^*S_1u}{u^*u} = \mathbf{i}\operatorname{Im}\lambda - \frac{\mathbf{i}\,(\operatorname{Im}\lambda - \mathrm{u}(A))}{(\delta_1 - \operatorname{Re}\lambda)^2 + (\operatorname{Im}\lambda - \mathrm{u}(A))^2}\,u^*u. \tag{4.5}$$

The same arguments applied to $\overline{\lambda} \in \sigma(A)$ yields

$$\frac{u^*S_1u}{u^*u} = \mathbf{i}\operatorname{Im}\overline{\lambda} - \frac{\mathbf{i}\,(\operatorname{Im}\overline{\lambda} - \mathrm{u}(A))}{(\delta_1 - \operatorname{Re}\lambda)^2 + (\operatorname{Im}\overline{\lambda} - \mathrm{u}(A))^2}\,u^*u. \tag{4.6}$$

We set $\lambda = x + \mathbf{i}\,y$ and by (4.5), (4.6) we obtain

$$2y = \frac{2y[(x - \delta_1)^2 + y^2 - \mathrm{u}(A)^2]}{[(\delta_1 - x)^2 + (y - \mathrm{u}(A))^2]\,[(\delta_1 - x)^2 + (y + \mathrm{u}(A))^2]}\,u^*u. \tag{4.7}$$

From (4.7) we now have two cases: Either

(a) $y = \operatorname{Im}\lambda = 0$, in which case since $\lambda \in \Gamma(A)$, we have

$$\mathrm{u}(A)^2 = (\delta_1 - \lambda)\left(\frac{\mathrm{v}(A) - \mathrm{u}(A)^2}{\lambda - \delta_2} + \lambda - \delta_1\right),$$

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that is, equivalently, λ coincides with the (triple) real root of

$$\lambda^3 - (2\delta_1 + \delta_2)\lambda^2 + [\delta_1^2 + 2\delta_1\delta_2 + \mathrm{v}(A)]\lambda - \delta_1^2\delta_2 - \delta_2\mathrm{u}(A)^2 - \delta_1(\mathrm{v}(A) - \mathrm{u}(A)^2) = 0;$$

or

(b) $y \neq 0$ and thus $(x-\delta_1)^2 + y^2 \neq \mathrm{u}(A)^2$; in this case (4.7) can be written as

$$\frac{(x-\delta_1)^2 + (y + \mathrm{u}(A))^2}{(x-\delta_1)^2 + y^2 - \mathrm{u}(A)^2} = \frac{u^*u}{(\delta_1 - x)^2 + (y - \mathrm{u}(A))^2}$$

and so by (4.4) we have

$$x - \delta_2 = \frac{(\delta_1 - x)\,[(x-\delta_1)^2 + (y+\mathrm{u}(A))^2]}{(x-\delta_1)^2 + y^2 - \mathrm{u}(A)^2},$$

completing the proof. □

We can apply the results in the previous section to rotations of $A \in \mathcal{M}_n(\mathbb{C})$ in order to obtain localizations of its spectrum that are complementary to the one obtained by $\Gamma(A)$. For example, we can consider three additional curves, $\Gamma(-A)$, $\Gamma(\mathbf{i}\,A)$ and $\Gamma(-\mathbf{i}\,A)$. The spectrum inclusion region resulting from these curves is illustrated in the next example.

Example 4.7. Let $A = \begin{bmatrix} 4 & 0 & -1 & 0 & 0 \\ 0 & 5 & -1 & 2 & 3 \\ 1 & 1 & 2+\mathbf{i} & -1 & 0 \\ 0 & -2 & 1 & -1 & 3-\mathbf{i} \\ 0 & 1 & 0 & -3-\mathbf{i} & 5 \end{bmatrix}$. In Figure 4.4, the numerical range and the curves $\Gamma(A)$, $\Gamma(-A)$, $\Gamma(\mathbf{i}\,A)$ and $\Gamma(-\mathbf{i}\,A)$ are superimposed on the left; on the right the region carved out of the numerical by the four curves is isolated. The eigenvalues of A are marked with $\bullet$'s.

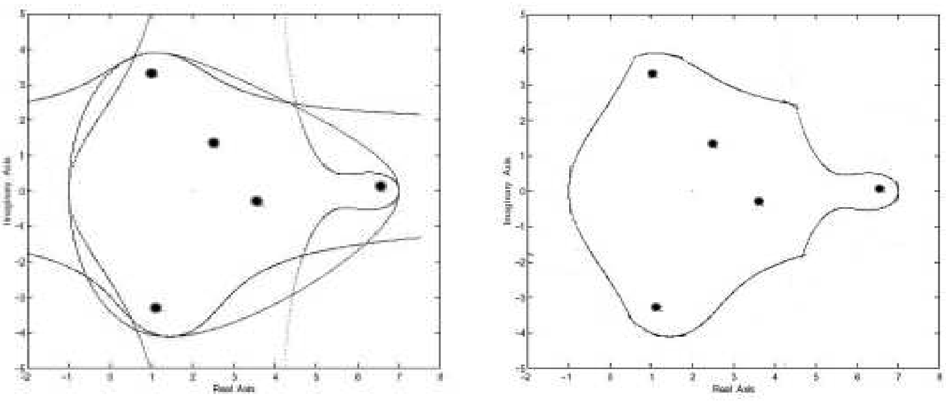

Fig. 4.4. *Left: the numerical range and* $\Gamma(A), \Gamma(-A), \Gamma(\mathbf{i}\,A), \Gamma(-\mathbf{i}\,A)$ *superimposed. Right: spectrum localization provided by* $\Gamma(A), \Gamma(-A), \Gamma(\mathbf{i}\,A), \Gamma(-\mathbf{i}\,A)$ *and the numerical range.*

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5. Conclusions. We have provided an inequality satisfied by each eigenvalue λ of $A \in \mathcal{M}_n(\mathbb{C})$. Upon rotation of A by $\pi/2$, π and $3\pi/2$, it generalizes the classical fact generally attributed to Bendixson [1, Theorem II] that

$$\min\{\mu : \mu \in \sigma(H(A))\} \ \leq \ \operatorname{Re}\lambda \ \leq \ \max\{\mu : \mu \in \sigma(H(A))\}$$

and

$$\min\{\nu : \nu \in \sigma(S(A)/\mathbf{i}\,)\} \ \leq \ \operatorname{Im}\lambda \ \leq \ \max\{\nu : \nu \in \sigma(S(A)/\mathbf{i}\,)\}$$

by replacing the lines represented by these lower and upper bounds with cubic curves. We note that there have been other improvements of Bendixson's results by replacing the bounding box with circular and hyperbolic regions that depend on all the eigenvalues of the Hermitian and skew-Hermitian parts; see Wielandt [8] and references therein.

In our results, the curve $\Gamma(A)$ depends only on the quantities $\mathrm{v}(A)$ and $\mathrm{u}(A)$ defined in (2.1), as well as on the two largest eigenvalues of $H(A)$, δ_1 and δ_2. As a consequence, the additional computational effort for $\Gamma(A)$ over Bendixson's results is reasonably small and with the help of a graphing device, $\Gamma(A)$ can provide a new efficient localization for the eigenvalues of A.

We conclude with possible directions for future research:

(1) Consider arbitrary rotations $e^{\mathbf{i}\,\theta}A$ of A in order to obtain a family of localizing curves and thus sharper localization results. Specifically, determine the intersection of all the localization regions arising from Theorem 3.1 applied to $e^{\mathbf{i}\,\theta}A$ as θ ranges in $[0, 2\pi)$. This effort appears to be analogous to the characterization of the numerical range as an intersection of half-planes [5, Theorem 1.5.12]. As the computational effort is likely to be substantial for matrices of large order, it may instead be interesting to determine a minimal number of localizing curves so that the intersection of the corresponding regions is contained entirely in the numerical range of A.

(2) Referring to Observation 4.2 item (4), the notation thereby and the case when $\Gamma(A)$ is not a simple open curve ($\Delta \geq 0$), investigate the number of eigenvalues of A that can lie to the right of the point s_2 on the line $\mathcal{L} = \{z \in \mathbb{C} : \operatorname{Im} z = \mathrm{u}(A)\}$.

(3) Recall that when $\delta_1 = \delta_2$, $\Gamma(A)$ degenerates to a single point. Pursue non-trivial generalizations of $\Gamma(A)$ when the largest eigenvalue of $H(A)$ is not simple. Referring to the proof of Theorem 3.1, this may be possible by orthogonally reducing $H(A)$ relative to the entire eigenspace corresponding to its largest eigenvalue.

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REFERENCES

[1] I. Bendixson. Sur les racines d'une équation fondamentale. *Acta Mathematica*, 25:359-365, 1902.

[2] S. Friedland, Eigenvalues of almost skew-symmetric matrices and tournament matrices. In *Gubernatorial and graph-theoretical problems in linear algebra* (R. A. Brualdi, S. Friedland, and V. Klee, Eds), IMA Vol. Math. Appl. 50, Springer-Verlag, New York, 1993, pp. 189–206.

[3] S. Friedland and M. Katz. On the maximal spectral radius of even tournament matrices and the spectrum of almost skew-symmetric compact operators. *Linear Algebra and its Applications*, 208/209:455–469, 1994.

[4] R. Horn and C.R. Johnson. *Matrix Analysis*, Cambridge University Press, 1990.

[5] R. Horn and C.R. Johnson. *Topics in Matrix Analysis*, Cambridge University Press, 1991.

[6] S. Kirkland, P. Psarrakos, and M. Tsatsomeros. On the location of the spectrum of hypertournament matrices. *Linear Algebra and its Applications*, 323:37–49, 2001.

[7] J.J. McDonald, P.J. Psarrakos, and M.J. Tsatsomeros. Almost skew-symmetric matrices. *Rocky Mountain Journal of Mathematics*, 34:269–288, 2004.

[8] H. Wielandt. On eigenvalues of sums of normal matrices. *Pacific Journal of Mathematics*, 5:633-638, 1955.