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THE STRUCTURE OF LINEAR OPERATORS STRONGLY PRESERVING MAJORIZATIONS OF MATRICES*

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Abstract. A matrix majorization relation $A \prec_r B$ (resp., $A \prec_\ell B$) on the collection M_n of all $n \times n$ real matrices is a relation $A = BR$ (resp., $A = RB$) for some $n \times n$ row stochastic matrix R (depending on A and B). These right and left matrix majorizations have been considered by some authors under the names "matrix majorization" and "weak matrix majorization," respectively. Also, a multivariate majorization $A \prec_{rmul} B$ (resp., $A \prec_{\ell mul} B$) is a relation $A = BD$ (resp., $A = DB$) for some $n \times n$ doubly stochastic matrix D (depending on A and B). The linear operators $T : M_n \to M_n$ which strongly preserve each of the above mentioned majorizations are characterized. Recall that an operator $T : M_n \to M_n$ strongly preserves a relation $\mathcal{R}$ on M_n when $\mathcal{R}(T(X), T(Y))$ if and only if $\mathcal{R}(X, Y)$. The results are the sharpening of well-known representations $TX = CXD$ or $TX = CX^tD$ for linear operators preserving invertible matrices.

Key words. Row stochastic matrix, Doubly stochastic matrix, Matrix majorization, Weak matrix majorization, Left(right) multivariate majorization, Linear preserver.

AMS subject classifications. 15A04, 15A21, 15A51.

1. Introduction. A matrix R with nonnegative entries is called *row stochastic* if the sum of every row of R is 1. A matrix D is called *doubly stochastic* if both A and its transpose A^t are row stochastic. The following notation will be fixed throughout the paper: M_{nm} for the collection of all $n \times m$ real matrices, $M_n = M_{nn}$ for the collection of all $n \times n$ real matrices, $\mathcal{RS}(n)$ for the set of all $n \times n$ row stochastic matrices, $\mathcal{DS}(n)$ for the set of all $n \times n$ doubly stochastic matrices, $\mathcal{P}(n)$ for the set of all $n \times n$ permutation matrices, $\mathbb{R}^n$ for the set of all real $n \times 1$ (column) vectors, and $\mathbb{R}_n$ for the set of all real $1 \times n$ (row) vectors. The letter J stands for the (rank-1) square matrix all of whose entries are 1. (The size of J is understood from the context.)

Let $A, B \in M_{nm}$. We write $A \prec_r B$ (resp. $A \prec_\ell B$) if $A = BR$ (resp. $A = RB$) for some row stochastic matrix R. These relations may be referred to as *matrix majorizations* from the *right* and the *left*, respectively. The right and the left matrix majorizations have been already considered in the references [6] and [11] as "matrix majorization" and "weak matrix majorization," respectively. Also, we write $A \prec_{rmul} B$ (resp. $A \prec_{\ell mul} B$) if $A = BD$ (resp. $A = DB$) for some doubly stochastic matrix D. The latter majorizations are referred to as "multivariate majorizations." Since $A \prec_{\ell mul} B$ if and only if $A^t \prec_{rmul} B^t$, we will restrict our attention to $\prec_{\ell mul}$ and may abbreviate it as $\prec_m$ for convenience.

Let $\mathcal{A}$ be a linear space of matrices, T be a linear operator on $\mathcal{A}$, and $\mathcal{R}$ be a relation on $\mathcal{A}$. We say T *strongly preserves* $\mathcal{R}$ when $\mathcal{R}(T(X), T(Y))$ if and only if $\mathcal{R}(X, Y)$.

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In the present paper, we study the structure of linear strong preservers of the above mentioned matrix majorizations on M_n or M_{nm}. In section 2 we show that every linear mapping which strongly preserves multivariate matrix majorization $\prec_{\ell mul}$ has the form $X \mapsto PXR + JXS$, for all $X \in M_{nm}$, where $P \in \mathcal{P}(n)$ and $R, S \in M_m$ are such that $R(R + nS)$ is invertible. It follows that if $T : M_{nm} \to M_{nm}$ is a linear strong preserver of $\prec_{rmul}$, then $TX = RXP + SXJ$ for all $X \in M_{nm}$, where $P \in \mathcal{P}(m)$ and $R, S \in M_n$ are such that $R(R + mS)$ is invertible.

In section 3 we revisit a result of [2]; we study the linear strong preservers $T : M_n \to M_n$ of $\prec_m$ which send matrices with nonnegative entries to matrices of the same kind, and show that if $n \neq 2$ and if $TI = I$, then $TX = PXP^t$ for all $X \in M_n$, where $P \in \mathcal{P}(n)$. We also present counterexamples for the case $n = 2$.

In section 4 we show that a linear mapping $T : M_n \to M_n$ strongly preserves $\prec_r$, if and only if there exist a permutation matrix P and an invertible matrix L such that $TX = LXP$ for all $X \in M_n$. This is an extension of Theorem 2.5 of [4]. Finally, in section 5 we show that a linear mapping $T : M_n \to M_n$ strongly preserves $\prec_\ell$, if and only if there exist a permutation matrix P and an invertible matrix L such that $TX = PXL$ for all $X \in M_n$. Although the proofs of the last two sections have some ideas in common, however, unlike the case of multivariate majorizations $\prec_{\ell mul}$ and $\prec_{rmul}$, there are still essential differences between $\prec_\ell$ and $\prec_r$.

For more information on majorization we refer the reader to the references cited at the end of the paper. We are mostly concentrating on the open questions raised in [2], [3], [4]. Exact references to the related works are given in the appropriate places. The references [5], [6], [7], [11] are included as general references related to the subject.

2. Linear preservers and multivariate majorization. In this section we characterize linear operators $T : M_{nm} \to M_{nm}$ which strongly preserve $\prec_m$. First we need some known facts and lemmas.

LEMMA 2.1. *Let $T : M_{nm} \to M_{nm}$ be a linear operator that strongly preserves one of the majorizations $\prec_m$, $\prec_r$ or $\prec_\ell$. Then T is invertible.*

The case of $\prec_r$ is proved in [4] and the proof of the other cases is similar.

THEOREM 2.2. *(Birkhoff's Theorem [10]) The totality of extreme points of the collection of all doubly stochastic matrices is the set of all permutation matrices. Moreover, the set of doubly stochastic matrices is the convex hull of the permutation matrices.*

THEOREM 2.3. *(Li and Poon [9]) Let $T : M_{nm} \to M_{nm}$ be a linear operator. The following are equivalent.*

(a) $TX \prec_m TY$ whenever $X \prec_m Y$ for $X, Y \in M_{nm}$.

(b) Either

(i) there exist $A_1, \ldots, A_m \in M_{nm}$ such that $T(X) = \sum_{j=1}^{m}(\sum_{i=1}^{n} x_{ij})A_j$ for all $X = [x_{ij}] \in M_{nm}$; or

(ii) there exist $R, S \in M_m$ and $P \in \mathcal{P}(n)$ such that $T(X) = PXR + JXS$ for all $X \in M_{nm}$.

We now prove the main result of this section.

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THEOREM 2.4. *Let $T : M_{nm} \to M_{nm}$ be a linear operator. The following assertions are equivalent.*

(a) T is invertible and preserves multivariate majorization $\prec_m$.

(b) T has the form

$$X \mapsto PXR + JXS, \quad X \in M_{nm},$$

where $P \in \mathcal{P}(n)$ and $R, S \in M_m$ are such that $R(R + nS)$ is invertible.

(c) T strongly preserves multivariate majorization $\prec_m$.

Proof. Assume (a) holds. Then one of the cases (i) or (ii) of part (b) of Theorem 2.3 holds. The case (i) cannot hold for $n \geq 2$ as one can always find a nonzero matrix X the sum of each of whose columns is zero. For such an X we have $TX = 0$; a contradiction. If (i) holds for $n = 1$, then writing $A_j = [a_{j1}, \ldots, a_{jm}]$ yields $TX = XA$ and, thus, (ii) holds with $R = A$ and $S = 0$. (Here, $A = [a_{ij}] \in M_m$.) In general, if (ii) holds for $n = 1$, then $TX = X(R + S)$ for all $X \in M_{1m}$ and hence $(R + S)$ must be invertible. Thus (b) holds with R replaced by $R + S$ and S replaced by 0.

Therefore, to complete the proof of (a) $\Rightarrow$ (b) it remains to prove the invertibility of R and $R + nS$ in case $n \geq 2$. If R is not invertible, there exists a nonzero row vector $X_1 \in \mathbb{R}_m$ such that $X_1R = 0$. Let $X \in M_{nm}$ be a matrix whose rows are all equal to X_1, and let $Y \in M_{nm}$ be a matrix whose first row is nX_1 and the rest are zero. It is clear that $X \neq Y$, $YR = 0 = XR$, and $JY = JX$. Thus

$$\begin{aligned} T(X) &= PXR + JXS = JXS = JYS \\ &= PYR + JYS = T(Y); \end{aligned} \tag{2.1}$$

a contradiction.

Similarly, if $R + nS$ is not invertible, there exists a nonzero row vector $Z_1 \in \mathbb{R}_m$ such that $Z_1(R+nS) = 0$. Let $Z \in M_{nm}$ be a matrix whose rows are all equal to Z_1. Then $Z \neq 0$, $Z(R + nS) = 0$, and $nZ = JZ$. Hence

$$T(Z) = PZR + JZS = PZ(R + nS) = 0; \tag{2.2}$$

a contradiction. Thus (a) $\Rightarrow$ (b).

To prove (b) $\Rightarrow$ (c), we show that T is invertible and T^{-1} satisfies the same condition as T with P replaced by P^t, R replaced by R^{-1} and S replaced by $-(R + nS)^{-1}SR^{-1}$. Define $T' : M_{nm} \to M_{nm}$ by $T'(X) = P^tXR^{-1} - JX(R+nS)^{-1}SR^{-1}$. Then

$$\begin{aligned} (T'T)(X) &= T'(T(X)) = P^tT(X)R^{-1} - JT(X)(R + nS)^{-1}SR^{-1} \\ &= P^t(PXR + JXS)R^{-1} - J(PXR + JXS)(R + nS)^{-1}SR^{-1} \\ &= X + JXSR^{-1} - JX(R + nS)(R + nS)^{-1}SR^{-1} \\ &= X + JXSR^{-1} - JXSR^{-1} = X, \end{aligned} \tag{2.3}$$

which implies that $T' = T^{-1}$. In view Theorem 2.3, (b) $\Rightarrow$ (c) is proven.

The proof of (c) $\Rightarrow$ (a) follows from Lemma 2.1. ☐

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COROLLARY 2.5. *Let $T : M_{nm} \to M_{nm}$ be a linear operator. The following assertions are equivalent.*

(a) T is invertible and preserves multivariate majorization $\prec_{rmul}$.

(b) T has the form

$$X \mapsto RXP + SXJ, \quad X \in M_{nm},$$

where $P \in \mathcal{P}(m)$ and $R, S \in M_n$ are such that $R(R + mS)$ is invertible.

(c) T strongly preserves multivariate majorization $\prec_{rmul}$.

Proof. Define $\tau : M_{mn} \to M_{mn}$ by $\tau(X) = [T(X^t)]^t$ for all $X \in M_{mn}$ and observe that Theorem 2.4 is applicable to τ. □

3. Linear preservers and matrices with nonnegative entries. In this section we obtain the following result of Beasley, Lee, and Lee[2] as a corollary to our Theorem 2.4. We will also construct counterexamples for case $n = 2$.

COROLLARY 3.1. *(Beasley-Lee-Lee[2]) Let $n \neq 2$ and assume $T : M_n \to M_n$ strongly preserves multivariate majorization $\prec_m$. Moreover, assume $T(I) = I$, and T preserves the collection of matrices with nonnegative entries (i.e.; $TM_n(\mathbb{R}^+) \subset M_n(\mathbb{R}^+)$). Then there exists a permutation matrix P such that $T(X) = PXP^t$ for all $X \in M_n$. The conclusion is false for $n = 2$.*

Proof. The case $n = 1$ being clear, we assume without loss of generality that $n \geq 2$. Let P, R and S be as in Theorem 2.4 and define $\tau : M_n \to M_n$ by $\tau(X) = P^tT(X)P$ for all $X \in M_n$. Then $\tau(X) = XRP + JXSP$. Replacing T by τ, we can assume without loss of generality that $P = I$ and that $T(X) = XR + JXS$ for all $X \in M_n$. For a fixed pair $(p, q) \in \{1, \ldots, n\} \times \{1, \ldots, n\}$, let $X = E_{pq}$, where $E_{pq} \in M_n$ is the matrix whose (p, q)-entry is 1, and the rest are 0. Then

$$\begin{aligned} T(E_{pq}) &= E_{pq}R + JE_{pq}S = \sum_{i,j} r_{ij}E_{pq}E_{ij} + \sum_{\alpha,\beta} E_{\alpha\beta} \sum_{i,j} s_{ij}E_{pq}E_{ij} \\ &= \sum_j r_{qj}E_{pj} + \sum_\alpha \sum_j s_{qj}E_{\alpha j} \\ &= \sum_j (r_{qj} + s_{qj})E_{pj} + \sum_{\alpha\neq p} \sum_j s_{qj}E_{\alpha j} \\ &= \sum_j (r_{qj} + s_{qj})E_{pj} + \sum_j \sum_{i\neq p} s_{qj}E_{ij}. \end{aligned} \tag{3.1}$$

Thus T preserves $M_n(\mathbb{R}^+)$ if and only if

$$-r_{qj} \leq s_{qj}, \quad s_{qj} \geq 0 \ \text{ for } \ q, j = 1, \ldots, n. \tag{3.2}$$

Now, assume $T(I) = I$. Then

$$\begin{aligned} I = T(I) &= \textstyle\sum_p T(E_{pp}) = \sum_{p,j}(r_{pj} + s_{pj})E_{pj} + \sum_{p,j}\sum_{i\neq p} s_{pj}E_{ij} \\ &= \textstyle\sum_{i,j}(r_{ij} + s_{ij})E_{ij} + \sum_{i,j}\sum_{p\neq i} s_{pj}E_{ij} \\ &= \textstyle\sum_{i,j}(r_{ij} + s_{ij} + \sum_{p\neq i} s_{pj})E_{ij}. \end{aligned} \tag{3.3}$$

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Let $i, j = 1, 2, \ldots, n$ and $i \neq j$. Then $r_{ij} + s_{ij} + \sum_{p \neq i} s_{pj} = 0$. This, in view of 3.2, implies that

$$r_{ij} = -s_{ij}, \quad \text{and} \quad s_{pj} = 0 \quad \text{for all} \quad p \neq i. \tag{3.4}$$

In particular, $s_{ii} = 0$. Also, $r_{ii} + s_{ii} + \sum_{p \neq i} s_{pi} = 1$, which implies that

$$r_{ii} = 1 - \sum_{p \neq i} s_{pi}. \tag{3.5}$$

If $n \geq 3$, then, for $i, j = 1, 2, \ldots, n$ and $i \neq j$,

$$s_{ij} = 0, \quad r_{ii} = 1 \quad \text{and} \quad r_{ij} = 0. \tag{3.6}$$

Hence $S = 0$, $R = I$, and thus $TX = X$ for all $X \in M_n$.

If $n = 2$, then to have $TM_2(\mathbb{R}^+) \subset M_2(\mathbb{R}^+)$ and $T(I) = I$ for a strong preserver of multivariate majorization, it is necessary and sufficient to have

$$0 \leq s_{21} = 1 - r_{11} = -r_{21} \leq 1, \quad 0 \leq s_{12} = 1 - r_{22} = -r_{12} \leq 1, \tag{3.7}$$

$$s_{11} = s_{22} = 0, \quad \det(R) = \det(R + 2S) = 1 - s_{12} - s_{21} \neq 0. \tag{3.8}$$

In particular, $T(E_{11}) = E_{11}R + JE_{11}S = (1 - s_{21})E_{11} + s_{12}E_{22}$ which is of rank 2 if, moreover, $s_{21} \neq 1$ and $s_{12} \neq 0$. Thus for such cases, T cannot be of the form $X \mapsto CXD$ for any invertible matrices C and D. ◻

The next example shows that, for any $n \geq 2$, there exists an operator $T : M_n \to M_n$ which strongly preserves multivariate majorization as well as the matrices of nonnegative entries but T is not of the form $X \mapsto CXD$ for any invertible matrices $C, D \in M_n$. (cf. Theorem 2.8 of [3].)

Example 3.2. Assume $n \geq 2$ and define $T : M_n \to M_n$ by $T(X) = X + JX(I + E_{12})$ for all $X \in M_n$. Since $R(R + nS) = (1 + n)I + nE_{12}$ is invertible, T strongly preserves multivariate majorization. It is easy to see that T preserves the nonnegativity of the entries. However, since T sends the rank-1 matrix E_{11} to a rank-2 matrix, it follows that T is not of the form $X \mapsto CXD$ for any invertible matrices C and D.

Corollary 3.3. *Let $T : M_n \to M_n$ be a linear Operator that strongly preserves multivariate majorization. Then the restriction of T to* span $(\mathcal{DS}(n))$ *has the form $X \mapsto PXL$ for some $P \in \mathcal{P}(n)$ and some $L \in M_n$. The matrix L need not be invertible.*

Proof. Let $T(X) = PXR + JXS$ where $P \in \mathcal{P}(n)$ and $R(R + nS)$ is invertible. If $X \in \mathcal{DS}(n)$, then $JX = J = PJ = PXJ$ and, hence, $TX = PXL$, where $L = R + JS$. To show that L may be singular, let $P = R = I$ and $S = -E_{21}$. Note that $R(R + nS)$ is invertible and, hence, T strongly preserves multivariate majorization. However, $T(I)$ is a matrix of rank $n - 1$ and, hence, the restriction of T to the span of $\mathcal{DS}(n)$ cannot be represented as $X \mapsto CXD$ for any invertible matrices C, D. ◻

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4. Linear preservers and right matrix majorization. In this section we generalize the following result.

THEOREM 4.1. *(Beasley-Lee-Lee[4]) If the linear operator $T : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ strongly preserves the matrix majorization $\prec_r$, then there exist a permutation matrix P and an invertible matrix L in M_n such that $T(X) = LXP$ for all $X \in \operatorname{span} \mathcal{RS}(n)$.*

Here, we will show that the representation of T given in Theorem 4.1 remains valid throughout M_n. To see the nontriviality of such extensions, cf. Theorem 2.4 and Corollary 3.3. To prove the main result, we need to state some known facts.

THEOREM 4.2. *([8]) A linear operator $T : M_n \to M_n$ maps the set of invertible matrices into itself if and only if there exist invertible matrices C, D such that $TX = CXD$ for all $X \in M_n$ or $TX = CX^tD$ for all $X \in M_n$.*

In the following, co$U$ denotes the convex hull of a subset U of a real vector space, and ext$V$ denotes the set of all extreme points of a convex set V.

LEMMA 4.3. *Let n be a natural number and assume that $\{e_1, \ldots, e_n\}$ is the standard basis of $\mathbb{R}_n$ and $e = [1, 1, \ldots, 1]$. For each $k = 1, \ldots, n$, define $X(k)$ to be the (unique) $n \times n$ row stochastic matrix whose k^{th} column is e^t. Moreover, let $E \in M_n$ be an invertible matrix. The following assertions are true.*

(a) $|\text{ext } \mathcal{RS}(n)| = n^n$.

(b) If $E^{-1}X(k)^tE \in \mathcal{RS}(n)$ for all $k = 1, \ldots, n$, then $n = 1$.

(c) If $E^{-1}X(k)E$ and $EX(k)E^{-1}$ belong to $\mathcal{RS}(n)$ for all $k = 1, \ldots, n$, then $E = aQ$ for some nonzero real number a and some permutation matrix Q.

Proof.

(a) Each $X \in \mathcal{RS}(n)$ corresponds to the n-tuple $[X_1, \ldots, X_n] \in (\mathbb{R}_n)^n$, where X_i is the i^{th} row of X $(i = 1, \ldots, n)$. Then

$$\begin{aligned} \mathcal{RS}(n) &\cong (\text{co}\{e_1, \ldots, e_n\})^n \\ &= \text{co}(\{e_1, \ldots, e_n\}^n), \end{aligned} \tag{4.1}$$

which proves (a).

(b) We assume $n \geq 2$ and reach a contradiction. Let $E^{-1} = [f_{ij}]$. Fix $j = 1, \ldots, n$. Then the i^{th} row of $E^{-1}X(j)^tE$ is $f_{ij}eE$ $(i = 1, \ldots, n)$. Since $E^{-1}X(j)^tE \in \mathcal{RS}(n)$, it follows that $f_{1j} = \ldots = f_{nj}$. Thus, rank $E^{-1} = 1 < n$; a contradiction.

(c) Let $E = [e_{ij}]$ and $E^{-1} = F$. For a fixed $i = 1, \ldots, n$, the matrix $E^{-1}X(i)E$ is a rank-one row stochastic matrix $Y(i)$ whose rows are all equal to a single (nonnegative row) vector $(y_{i1}, \ldots, y_{in})$ with $\sum_j y_{ij} = 1$. Writing down the entries of the equation $X(i)E = EY(i)$ yields $e_{ij} = y_{ij} \sum_q e_{pq}$ for all $j, p = 1, \ldots, n$. Hence, $E = cR$, where R is the row stochastic matrix $[y_{ij}]$ and $c = \sum_j e_{1j} = \ldots = \sum_j e_{nj}$. By the symmetry of the assumption, $F = dS$ for some row stochastic matrix S and some constant d. Since $e = EFe = cdRSe = cde$, it follows that, $d = c^{-1}$ and, hence, $RS = I$. Thus R and R^{-1} are both row stochastic and, hence, they are permutation matrices [4]. ☐

We are now ready to prove the main result of the section.

THEOREM 4.4. *A linear operator $T : M_n \to M_n$ Strongly preserves the matrix majorization $\prec_r$ if and only if there exist a permutation matrix P and an invertible matrix L in M_n such that $TX = LXP$ for all $X \in M_n$.*

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Proof. The fact that the condition $TX = LXP$ for all $X \in M_n$ is sufficient for T to be a strong preserver of $\prec_r$ is easy to prove. So we prove the necessity of the condition.

We first show that a strong preserver of the right matrix majorization preserves invertible matrices. Let $A \in M_n$ be invertible. We assume TA is singular and reach a contradiction. It follows that

$$\{(TA)R : R \in \mathcal{RS}(n)\} = T\{AS : S \in \mathcal{RS}(n)\} = (TM_A)(\mathcal{RS}(n)),$$

Where $M_A : M_n \to M_n$ is the mapping defined by $M_A X = AX$ for all $X \in M_n$. Since TM_A is bijective,

$$|\text{ext } (TA)(\mathcal{RS}(n)| = |\text{ext } \mathcal{RS}(n)| = n^n.$$

(See part (a) of Lemma 4.3.) Also, since TA is not invertible, there exists a nonzero (column) vector $a = (a_1, \dots, a_n)^t \in \mathbb{R}^n$ such that $-1 \le a_i \le +1$, and $(TA)a = 0$. Define

$$R = \begin{bmatrix} c_1 & (1-c_1) & 0 & \dots & 0 \\ c_2 & (1-c_2) & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_n & (1-c_n) & 0 & \dots & 0 \end{bmatrix}, \quad S = \begin{bmatrix} d_1 & (1-d_1) & 0 & \dots & 0 \\ d_2 & (1-d_2) & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ d_n & (1-d_n) & 0 & \dots & 0 \end{bmatrix},$$

where $c_i = a_i$ and $d_i = 0$ if $a_i > 0$, and $c_i = 1 + a_i$ and $d_i = 1$ if $a_i \le 0$ $(i = 1, \dots, n)$. Then $R, S \in \mathcal{RS}(n)$, $R \ne S$, but $(TA)R = (TA)S$. Since $S \in \text{ext } \mathcal{RS}(n)$ and since $\text{ext } (TA)(\mathcal{RS}(n)) \subset (TA)(\text{ext } \mathcal{RS}(n))$, it follows that $|\text{ext } (TA)(\mathcal{RS}(n))| < n^n$; a contradiction.

Thus T sends invertible matrices to invertible matrices and, hence, there exist invertible matrices $C, D \in M_n$ such that either $n \ge 1$ and $TX = CXD$ for all $X \in M_n$, or $n \ge 2$ and $TX = CX^tD$ for all $X \in M_n$. In the latter case, let $X = X(i)$ be as in the statement of Lemma 4.3. Since $X(i) \prec_r I$, it follows that $D^{-1}X(i)^tD \in \mathcal{RS}(n)$ for all $i = 1, \dots, n$. In view of part (b) of Lemma 4.3, $n = 1$; a contradiction.

Thus $TX = CXD$ for all $X \in M_n$ and, hence, $D^{-1}X(i)D \in \mathcal{RS}(n)$ for all $i = 1, \dots, n$. Hence, in view of part (c) of Lemma 4.3, D is a multiple of a permutation matrix Q. $\square$

The following corollary is proved in [4] under the additional condition $TM_n(\mathbb{R}^+) \subset M_n(\mathbb{R}^+)$.

Corollary 4.5. *If $T : M_n \to M_n$ strongly preserves $\prec_r$ and if $TI = I$, then there exists $P \in \mathcal{P}(n)$ such that $TX = P^tXP$ for all $X \in M_n$.*

Proof. With the notation of the theorem, $I = TI = LP$ and, hence, $L = P^t$. $\square$

5. Linear preservers and left matrix majorization. Finally, we prove a result similar to Theorem 4.4 for $\prec_\ell$. As we mentioned earlier, there is no duality between the two cases and the proofs are basically different. For the proof of the main result of this section we further need the following known results. Proofs are given for reader's convenience.

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LEMMA 5.1. *For every natural number* n *the following assertions are true.*
(a) $\dim \operatorname{span} \mathcal{RS}(n) = n^2 - n + 1$.
(b) $\dim \operatorname{span} \{X \in M_n : X \prec_\ell A\} \leq \mathrm{n}(\operatorname{rank} A)$ *for all* $A \in M_n$.

Proof. (a) The span of $\mathcal{RS}(n)$ consists of all matrices with the property that all the row sums are equal. The dimension of this subspace is clearly $n^2 - n + 1$.

(b) If $X \prec_\ell A$, then every row of X is a linear combination of the rows of A. □

We are now ready to prove the main result of this section.

THEOREM 5.2. *A linear operator* $T : M_n \to M_n$ *strongly preserves the matrix majorization* $\prec_\ell$ *if and only if there exist a permutation matrix* P *and an invertible matrix* L *in* M_n *such that* $TX = PXL$ *for all* $X \in M_n$. *Moreover, if* $TI = I$, *then* $L = P^t$.

Proof. The fact that the condition $TX = PXL$ for all $X \in M_n$ is sufficient for T to be a strong preserver of $\prec_\ell$ is easy to prove. So we prove only the necessity of the condition.

We first show that a strong preserver of a left matrix majorization preserves invertible matrices. Let $A \in M_n$ be invertible. We assume TA is singular and reach a contradiction. It follows, in view of parts (a,b) of Lemma 5.1, that

$$\begin{aligned} n^2 - n &\geq \dim \operatorname{span} \{Y \in M_n : Y \prec_\ell TA\} = \dim \operatorname{span} \{X \in M_n : X \prec_\ell A\} \\ &= \dim(\operatorname{span} \{X \in M_n : X \prec_\ell A\})A^{-1} = \dim \operatorname{span} \{X \in M_n : X \prec_\ell I\} \\ &= \dim \operatorname{span} \mathcal{RS}(n) = n^2 - n + 1; \end{aligned} \tag{5.1}$$

a contradiction. Thus T sends invertible matrices to invertible matrices and, hence, there exist invertible matrices $C, D \in M_n$ such that either $n \geq 1$ and $TX = CXD$ for all $X \in M_n$ or $n \geq 2$ and $TX = CX^tD$ for all $X \in M_n$. In the latter case, let $X = X(i)$ be as in the statement of Lemma 4.3. Since $X(i) \prec_\ell I$, it follows that $CX(i)^tC^{-1} \in \mathcal{RS}(n)$ for all $i = 1, \ldots, n$. It now follows from part (b) of Lemma 4.3 that $n = 1$; a contradiction.

Thus $TX = CXD$ for all $X \in M_n$ and, hence, $CX(i)C^{-1} \in \mathcal{RS}(n)$ for all $i = 1, \ldots, n$. Hence, in view of part (c) of Lemma 4.3, the matrix C is a multiple of a permutation matrix Q. □

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