Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 15, pp. 274-284, October 2006

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IRREDUCIBLE TOEPLITZ AND HANKEL MATRICES*

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Abstract. An infinite matrix is called irreducible if its directed graph is strongly connected. It is proved that an infinite Toeplitz matrix is irreducible if and only if almost every finite leading submatrix is irreducible. An infinite Hankel matrix may be irreducible even if all its finite leading submatrices are reducible. Irreducibility results are also obtained in the finite cases.

Key words. Infinite Toeplitz, Hankel matrices, Finite leading submatrices, Irreducibility, Strongly connected digraphs.

AMS subject classifications. 05C50, 05C40, 11A05, 11D04, 15A48, 47B35.

1. Introduction. The concepts of irreducibility of a finite square matrix with nonnegative real entries were treated in a remarkable historical perspective in a paper by H. Schneider [9]. In the paper [6] by I. Marek and K. Zitny concepts of irreducibility for possibly infinite matrices were studied and compared, e.g. the concepts of G. Frobenius [2], D. König [5] and H. Geiringer [3] with those of others. Based on their results on the (essential) equivalence of all these concepts, we accept the following definition due to D. König [5], which is also standard in the field of countable Markov chains (cf., e.g., [8]):

DEFINITION 1.1. Assume that N is a positive integer or the cardinal $\aleph_0$ (which we shall simply denote by ∞). Accordingly, let $D(N)$ be either the finite sequence $\{0, 1, 2, \ldots, N\}$ or the infinite sequence $\{0, 1, 2, \ldots\} \equiv \mathbf{N}_0$. Let

$$M(N) : D(N) \times D(N) \to \mathbf{C}$$

be a matrix with complex entries. It is called *irreducible*, if its directed graph ($\equiv$ digraph) is *(strongly) connected*.

For the corresponding concepts of elementary graph theory we refer the reader to König [5] or Ore [7]. Recall that the graph of M is *strongly connected*, by definition, if each vertex k is *accessible* (or, equivalently, *reachable*) from each vertex $j \neq k$. This means that there is a *path* between them, i.e. a finite sequence of states($\equiv$ vertices) $\{i_1, \ldots, i_r\}$ such that

$$m_{ji_1} m_{i_1 i_2} \ldots m_{i_r k} \neq 0.$$

In this case we say that k *is accessible from* j *in* $r+1$ *steps*. Clearly, these concepts depend only on the zero-nonzero structure of the matrix M, so we can and will always consider the *indicator or Boolean matrix* of M containing the entry 1 at the

*Received by the editors 21 March 2006. Accepted for publication 24 September 2006. Handling Editor: Richard A. Brualdi.

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Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
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place of each nonzero entry of M. It is also clear that in the question of reducibility the convergence problems concerning an infinite matrix (representing perhaps an operator) are irrelevant. Note that we shall use the words *diagonal, main diagonal, perpendicular diagonal* in the classical sense (hence *not in the sense* used frequently in combinatorial matrix theory).

We want to study the reducibility problem of finite and infinite Toeplitz and Hankel matrices. According to the preceding paragraph, this is the same as to study the reducibility of matrices of type T and H, respectively, in the following

DEFINITION 1.2. We shall say that the square matrix is *of type T (of type H)* if its each diagonal parallel (perpendicular) to the main diagonal contains either exclusively 0, or exclusively nonzero entries. Consequently, in their indicator matrices the corresponding diagonals contain either exclusively 0, or exclusively 1 as entries. The corresponding *indicator matrices* will be denoted *by $T(N)$ and $H(N)$*, respectively. If we consider an infinite matrix $T(\infty)$, then $T(N)$ for $N \in \mathbf{N}_0$ will denote its left upper corner $(N+1)\times(N+1)$ (leading) submatrix or, equivalently, finite section (and similarly for type H). We shall consider the 1×1 zero matrix as *reducible.*

A moment's reflection will convince us that for *any* infinite matrix $M(\infty)$ the irreducibility *of an infinite number* of the finite sections $M(N_r)$ $(r = 1, 2, \ldots)$ implies the irreducibility of $M(\infty)$. On the other hand, the converse for general matrices is false.

The main aim of this paper is to solve the converse problem for the case of matrices of type T and of type H. We shall prove that $T(\infty)$ is irreducible if and only if there is $N_1 \in \mathbf{N}$ such that for *every* $N \geq N_1$ the submatrix $T(N)$ is irreducible. On the other hand, it can happen that the matrix $H(\infty)$ (of type H) is irreducible, and *all finite sections* $H(N)$ are reducible. Further, we shall obtain results on the irreducibility of *finite* matrices $T(N)$ and $H(N)$ of types T and H, respectively.

Note that it will be convenient to let the subscripts run always from 0. Further, we shall be working with integers, so the *interval* $[a, b]$ for $a, b \in \mathbf{Z}$ will denote the set

$$[a, b] := \{z \in \mathbf{Z} : a \leq z \leq b\}.$$

The greatest common divisor of integers in the set S will be written as $gcd(S)$. Clearly, $gcd(S) > 0$ means that S is nonvoid. The algebraic difference of the sets A and B will be denoted by

$$A - B := \{a - b : a \in A,\ b \in B\},$$

and for $n \in \mathbf{N}$ we set

$$nA := \{na : a \in A\}.$$

2. Matrices of type T. Let N be a positive integer or the symbol ∞. Let $B(N) := [-N, N]$ if N is an integer, let $B(\infty) := \mathbf{Z}$, and let

$$b : B(N) \to [0, 1]$$

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be a finite two-way sequence if N is an integer, and a two-way infinite sequence if $N = \infty$. Define

$$t_{ij} := b(j - i) \qquad (i, j \in B(N) \cap \mathbf{N}_0).$$

The (finite or infinite) square matrix

$$T \equiv T(N) \equiv T(N, b) := \{t_{ij} : i, j \in B(N) \cap \mathbf{N}_0\}$$

of order $(N + 1)$ (or ∞) is the indicator or Boolean matrix of any corresponding matrix of type T.

The following notation will be used in Section 2 throughout. Consider the sequence of exactly those integers $0 < p_1 < p_2 < p_3 < \ldots$ for which $b(p_r) = 1$. In a similar vein, consider the sequence of exactly those integers $0 > -n_1 > -n_2 > -n_3 > \ldots$ for which $b(-n_s) = 1$. The strictly increasing (finite or infinite) sequences of positive integers $\{p_r\}$ and $\{n_s\}$ describe exactly the *nonzero* places of the 1s in the zeroth row (column, respectively) of the matrix T. Define

$$p := gcd(p_1, p_2, \ldots), \quad n := gcd(n_1, n_2, \ldots), \quad g := gcd(p, n).$$

It is clear that these greatest common divisors are the gcd-s *of finite numbers* of terms from the sequences:

$$p = gcd(p_1, p_2, \ldots, p_u),\ n = gcd(n_1, n_2, \ldots, n_v),\ g = gcd(p_1, \ldots, p_u, n_1, \ldots, n_v).$$

THEOREM 2.1. *Using the above notation, the infinite matrix $T(\infty)$ of type T is irreducible if and only if $p > 0, n > 0$ (or, equivalently, there is [apart from the place 0] at least one 1 both in the zeroth row and in the zeroth column), and $g = 1$.*

Proof. By definition, the matrix is irreducible iff its directed graph is strongly connected. This is equivalent to the statement that the state 0 is reachable from every state $m \in \mathbf{N}$, and every state $j \in \mathbf{N}$ is reachable from the state 0. The second condition implies that there exist nonnegative integers $x_i \equiv x_i(j)$ and $y_k \equiv y_k(j)$ such that

$$j = \sum_{i=1}^{u} x_i p_i - \sum_{k=1}^{v} y_k n_k.$$

The reason is that the structure of the indicator matrix $T(\infty)$ shows that the allowed 1-step transitions are exactly either p_i states in the positive direction or n_k states in the negative direction. This implies that $p > 0$, and also that $g|j$ (i.e. g divides j) for every $j \in \mathbf{N}$. It follows that $g = 1$. The first condition implies that there exist nonnegative integers $w_i \equiv w_i(m)$ and $z_k \equiv z_k(m)$ such that

$$0 = m + \sum_{i=1}^{u} w_i p_i - \sum_{k=1}^{v} z_k n_k.$$

Hence we obtain that the set of the n_k's cannot be empty, i.e. $n > 0$, and the *necessity* of the stated conditions is proved.

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A publication of the International Linear Algebra Society
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According to the above argument, in order to prove that the stated conditions are *sufficient*, we shall show that for every $m \in \mathbf{N}$ both the equations

$$\sum_{i=1}^{u} x_i p_i - \sum_{k=1}^{v} y_k n_k = +m \quad [\text{or } = -m]$$

have *nonnegative integer solutions* $\{x_i(m), y_k(m)\}$ (and similarly for $-m$). We shall prove it for the first case, the second being completely similar.

It is well known that any infinite set S of positive integers, which is closed under addition, contains all but a finite number of the positive multiples of its greatest common divisor (see, e.g., [8, Lemma A.3, p.183]). Consider the cases

$$S_1 := \{\sum_{i=1}^{u} x_i p_i : x_i \in \mathbf{N}_0,\ \text{there exists } x_j > 0\},$$

$$S_2 := \{\sum_{k=1}^{v} y_k n_k : y_k \in \mathbf{N}_0,\ \text{there exists } y_j > 0\}.$$

It is clear that $gcd(S_1) = p > 0$, and $gcd(S_2) = n > 0$. Hence there is a positive integer M such that $x, y \in \mathbf{N},\ x, y \geq M$ imply $xp \in S_1,\ yn \in S_2$. Further, the assumptions $p > 0,\ n > 0,\ g = 1$ imply that the linear Diophantine equation

$$xp - yn = m$$

has an infinite number of such solutions (x, y) that $x, y \geq M$. By the above argument, the stated solution then exists, and the sufficiency is proved. □

It is remarkable that the irreducibility of a finite leading submatrix $T(N)$ of $T(\infty)$ seems to be harder to decide. If $T(\infty)$ has the determining parameters denoted as above, then it is clearly sensible to study only those submatrices $T(N)$ which already contain *all* the 1s in the zeroth row and in the zeroth column. Hence we shall assume that $N \geq \max[p_u,\ n_v]$, and we have the following

THEOREM 2.2. *If the finite leading submatrix $T(N)$ (of order $(N+1)$) of $T(\infty)$ is irreducible then, using the above notation, we have $p > 0, n > 0, g = 1$, further*

$$N + 1 \geq p_1 + n_1 \equiv \min\{p_k : k = 1, \ldots, u\} + \min\{n_s : s = 1, \ldots, v\}.$$

In the converse direction: if $p > 0, n > 0, g = 1$, further

$$N + 1 \geq p_u + n_v \equiv \max\{p_k : k = 1, \ldots, u\} + \max\{n_s : s = 1, \ldots, v\},$$

then $T(N)$ is irreducible.

Proof. A short inspection of the first half of the preceding proof shows that the irreducibility of $T(N)$ implies the conditions $p > 0, n > 0, g = 1$ in the finite case, too. Further, if $p_1 + n_1 > N + 1$, then the matrix $T(N)$ contains a zero row (and column), which contradicts irreducibility.

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In the converse direction: if $p > 0, n > 0, g = 1$, then the second half of the preceding proof shows that for every $m \in \mathbf{Z}$ satisfying $-N \leq m \leq N$ there exist *nonnegative integers* x_i and y_k such that

$$\sum_{i=1}^{u} x_i p_i - \sum_{k=1}^{v} y_k n_k = m.$$

In the case $N < \infty$ we shall show that, under our additional condition, the above representation of the state m can be written in such an order of the summation that all "partial sums" stay within the interval $[0, N]$. Clearly, if we have achieved this for a given partial sum, and all the remaining terms have the same sign, then the condition that m is also in the interval $[0, N]$ ensures that all the remaining partial sums stay within. On the other hand, if the remaining terms have different signs, then the condition $p_u + n_v \leq N + 1$ ensures that we can choose the next term (p_i or n_k) so that we still remain in the interval $[0, N]$. Indeed, if we are in state q and, a contrario, $q + p_u > N$ and $q - n_v < 0$, then $p_u + n_v > N + 1$, a contradiction. □

The following theorem is *the main result* on the irreducibility of infinite and finite matrices *of type* T, and follows simply from the preceding ones.

Theorem 2.3. *The infinite matrix $T(\infty)$ of type T is irreducible if and only if for $N + 1 \geq p_u + n_v$ every finite left upper corner submatrix $T(N)$ of $T(\infty)$ is irreducible.* □

The next Proposition gives a method for deciding which finite leading submatrices $T(N)$ (with a given parameter set) are irreducible.

Proposition 2.4. *Consider a matrix $T(\infty)$ of type T with the parameter set (as above)*

$$(p_1, \dots, p_u), \quad (n_1, \dots, n_v).$$

The irreducibility of the leading submatrices $T(N)$ (with these given parameters) can be checked systematically starting by the integer($\equiv$ order $- 1$)

$$N_s := \max[p_u, n_v, p_1 + n_1 - 1],$$

and progressing upwards by order 1 at each step. If for some $N_0 \geq N_s$ the submatrix $T(N_0)$ is irreducible, so is $T(N)$ for every $N \geq N_0$. The smallest value N for which $T(N)$ is irreducible will be denoted by N_1, and it satisfies

$$N_s \leq N_1 \leq p_u + n_v - 1 =: N_f.$$

Proof. The only non-evident statement is: $T(N-1)$ irreducible implies $T(N)$ irreducible. The reason is the structure of a matrix of type T. All the diagonals indexed by p's and n's will be "elongated" in $T(N)$, i.e. they will contain at least one entry 1 in column N and in row N, respectively. Since $T(N-1)$ is irreducible, every vertex in $[0, N-1]$ is reachable from every other such vertex in (the graph of) $T(N-1)$. By the above remark, from the vertex N we can reach some vertex in

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$[0, N-1]$, and from some such vertex we can reach vertex N in $T(N)$. Hence $T(N)$ is irreducible. $\square$

Note that the checking of irreducibility at each step can be done e.g. by the technique proposed by Y. Malgrange and Tomescu and described by Kaufmann in [4], which is easily adaptable to this order-increasing process.

The following example will show that the interval obtained for N_1 in the above Proposition is the *best possible*. It will be suggestive to write the parameter set in the form $(p_1, \ldots, p_u) - (n_1, \ldots, n_v)$ rather than $(p_1, \ldots, p_u), (n_1, \ldots, n_v)$.

Example 2.5. $(6, 15) - (10, 15)$ implies $N_1 = 15$. Here $15 = N_s = N_1 < N_f = 29$.
$(6, 10) - (6, 10, 15)$ implies $N_1 = 16$. Here $15 = N_s < N_1 < N_f = 24$.
$(6) - (6, 13)$ implies $N_1 = 18$. Here $13 = N_s < N_1 = N_f = 18$.
$(2) - (3)$ implies $N_s = 4 = N_f = N_1$.
$(6, 10) - (15)$ implies $N_1 = 20$. Here $20 = N_s = N_1 < N_f = 24$.

3. Matrices of type H. Let N be a positive integer or ∞, and let $D(N)$ be the finite set $[0, N]$ or $D(\infty) := \mathbf{N}_0$, respectively. Let

$$b : D(2N) \to [0, 1]$$

be a finite or infinite sequence, according as N is finite or ∞. Define

$$h_{ij} := b(i + j) \qquad (i, j \in D(N)).$$

The (finite or infinite) square matrix

$$H \equiv H(N) \equiv H(N, b) := \{h_{ij} : i, j \in D(N)\}$$

of order $(N + 1)$ (or ∞) is the indicator matrix of any corresponding *matrix of type H*.

Consider the (finite or infinite) set $K \equiv K(2N)$ of exactly those values $k \in D(2N)$ for which $b(k) = 1$. This set describes exactly the places of the 1s in the zeroth row of the matrix $H(\infty)$, and determines exactly the (perpendicular) diagonals of the matrix $H(N)$ in which $h_{ij} = b(i + j) = 1$. Note that all indicator matrices of type H are *symmetric*, thus it is sufficient to consider their *undirected graphs* rather than digraphs when we study the question of reducibility.

We start with the following simple lemma on the matrices $H(N)$.

Lemma 3.1. *The state $j \in D(N)$ is accessible from state 0 in the graph of $H(N)$ in n steps if and only if there is a finite sequence $\{k_1, k_2, \ldots, k_n\}$ of elements of the set $K(2N)$ such that*

$$k_1 \geq 0,\ k_2 - k_1 \geq 0,\ k_3 - k_2 + k_1 \geq 0, \ldots, k_n - k_{n-1} + k_{n-2} - \ldots + (-1)^{n-1} k_1 = j \geq 0$$

and, in the **finite** *case $N \in \mathbf{N}$ in addition, all the left-hand sides above are contained in the interval $[0, N]$.*

Proof. Assume first that the state j is accessible from state 0 in n steps through the states $s_1, s_2, \ldots, s_n = j$ (in that order), i.e.

$$h_{0s_1} h_{s_1 s_2} \ldots h_{s_{n-1} j} = 1.$$

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A publication of the International Linear Algebra Society
Volume 15, pp. 274-284, October 2006

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Define

$$k_1 := s_1,\ k_2 := s_1 + s_2,\ k_3 := s_2 + s_3, \dots, k_n := s_{n-1} + s_n.$$

By the definition of $H(N)$, all the numbers s_r are in the interval $D(N)$, hence all the numbers k_r are contained in the set $K(2N)$. From the equations above we obtain

$$s_1 = k_1, s_2 = k_2 - k_1, s_3 = k_3 - k_2 + k_1, \dots, s_n = k_n - k_{n-1} + \dots + (-1)^{n-1}k_1,$$

and our claims follow. The converse direction is proved by tracing our steps backwards. $\square$

Definition 3.2. If the assertion of Lemma 3.1 holds, we say that *the state j is accessible from the state 0 (in n steps) by using recursion from the interval $D(2N)$*. If $N = \infty$, and the k_q's of Lemma 3.1 are in an interval $[0, d] \subset \mathbf{N}_0$, we say that *the state j is accessible from the state 0 (in n steps) by using recursion from the interval* $[0, d]$. If either statement holds for every $j \in [0, c]$, we say that *the interval* $[0, c]$ is accessible from 0 by using recursion from the interval $D(2N)$ or $[0, d]$, respectively.

We record the simple consequence of Lemma 3.1 as

Theorem 3.3. *The matrix $H(N)$ is irreducible if and only if for every $j \in D(N)$ there is $n \equiv n(j) \in \mathbf{N}$ for which the conditions of Lemma 3.1 are satisfied.* $\square$

The following *necessary* conditions may be useful both in the finite and in the infinite case.

Proposition 3.4. *If the matrix $H(N)$ is irreducible, then $gcd[K(2N)] = 1$. If, in addition, $N < \infty$, then both sets*

$$Q := K(2N) \cap [1, N], \qquad R := K(2N) \cap [N, 2N - 1]$$

are nonvoid, and there are (at least) two distinct elements $q \in Q, r \in R$ satisfying $q + 1 + N \geq r$.

Proof. If $H(N)$ is irreducible, every $j \in [1, N]$ must have a representation of the form stated in Lemma 3.1. This implies that $gcd[K(2N)] = 1$. Assume now that, in addition, $N < \infty$. If the set Q is empty, then it is impossible to reach any state from state 0. If the set R is empty, then it is impossible to reach the state N. Finally, if N is the unique element in $Q \cup R$, then $H(N)$ is clearly reducible. Hence, if $H(N)$ is irreducible, there must exist two distinct elements $q \in Q, r \in R$. If for every such pair we have $q + 1 < r - N$, then there is a zero row in $H(N)$ (e.g. the row $\sup\{q + 1 : q \in Q\}$). $\square$

The following *sufficient* conditions yield useful examples of (finite and infinite) irreducible Hankel matrices.

Proposition 3.5. *If there are distinct elements $q \in Q$ and $r \in R$ that are neighbours in the order of $K(2N)$, then the finite matrix $H(N)$ is irreducible. Consequently, if $N = \infty$, and there are infinitely many pairs $\{k_r, k_r + 1\} \subset K(N)$, then the matrix $H(\infty)$ is irreducible.*

Proof. Note that the first condition means that either $N - 1, N \in K(2N)$ or $N, N + 1 \in K(2N)$. Assume the first case, the proof for the second being completely

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similar. By assumption, the states $N-1$ and N are accessible in 1 step from 0, hence the following sequence of states is a path in the graph of $H(N)$:

$$0, N-1, 1, N-2, 2, N-3, 3, \dots$$

It is clear from this that $H(N)$ is irreducible.

In the case $N = \infty$ the preceding paragraph shows that each upper left finite section submatrix $H(k_r + 1)$ is irreducible. There are infinitely many of them, hence the matrix $H(\infty)$ is irreducible. $\square$

The following Remark and Examples will show that it may be too optimistic to hope for a simpler characterization of irreducibility than that in Theorem 3.3 (even in the case $N = \infty$).

Remark 3.6. It is clear that if any $H(N)$ is irreducible, and we replace a zero diagonal (perpendicular to the main one) by a diagonal consisting of 1s, then the modified Hankel matrix is also irreducible. In the case $N < \infty$ it might also be tempting to conjecture that if $H(N)$ is irreducible, and we *move* one (perpendicular) diagonal of 1's closer to the main (perpendicular) diagonal (onto the place of a zero diagonal), then the modified Hankel matrix (containing more 1's) is also irreducible. However, this conjecture is false, as is demonstrated by the following examples.

Example 3.7. Let $N := 10, K(2N) := \{8, 10, 12, 15\}$. Then $H(N)$ is irreducible. If we modify $K(2N)$ to become $\{8, 10, 12, 14\}$, then the modified matrix is reducible (the gcd is 2).

In the following example both gcd-s are 1, so the change is perhaps more remarkable.

Example 3.8. Let $N := 10, K(2N) := \{7, 11, 17\}$. Then $H(N)$ is irreducible. If $K(2N) := \{8, 11, 17\}$, then the modified matrix is reducible.

The following examples will demonstrate the applicability of the above results.

Example 3.9. If $K(\infty)$ is the complement of a finite set in $\mathbf{N}_0$, then Proposition 3.5 is applicable, and $H(\infty)$ is irreducible. On the other hand, if $K(\infty)$ is a finite subset of $\mathbf{N}_0$, then all the states accessible from 0 must be not greater than the largest element in $K(\infty)$. Hence $H(\infty)$ is reducible.

Example 3.10. If there is a positive integer $n > 1$ such that $K(\infty) \subset n\mathbf{N}_0$, then $gcd[K(\infty)] \geq n > 1$, hence $H(\infty)$ is reducible. On the other hand, if

$$K(\infty) := [\mathbf{N}_0 \setminus n\mathbf{N}_0] \setminus F,$$

where F is any finite set, then for $n > 2$ Proposition 3.5 applies and yields that $H(\infty)$ is irreducible. If $n = 2$, then denote by f the smallest odd number such that $f + 2\mathbf{N}_0 \subset K(\infty)$. Then each state in the set $f + 2\mathbf{N}_0$ is accessible from 0 in 1 step, and each state $2, 4, 6, \dots$ is accessible from f in 1 step. It easily follows that each state in $\mathbf{N}$ is accessible from 0, hence $H(\infty)$ is irreducible.

Example 3.11. The following example will show that it can happen that *an infinite Hankel matrix $H(\infty)$ is irreducible, and its every finite (left upper) section matrix $H(N)$ is reducible.* It will make essential use of the notion of accessibility (from 0) by using recursion from an interval as defined after Lemma 3.1.

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Consider the following recursively defined sequence of positive integers:

$$j_0 := 1,\ j_1 := 2,\ j_r := 3j_{r-1} + j_{r-2} + 1 \quad (r = 2, 3, \ldots).$$

The sequence j is clearly strictly increasing. Define also the sequences

$$c_r := 2j_r, \quad d_r := 3j_r + j_{r-1} \equiv j_{r+1} - 1 \quad (r = 1, 2, \ldots).$$

It is clear that then

$$j_r < c_r < d_r < j_{r+1} \quad (r = 1, 2, \ldots).$$

Define the (Boolean) Hankel matrix $H(\infty)$ by prescribing that the zeroth row should contain the entries 1 exactly at the values of the sequences c and d, all other entries there should be 0, i.e. let

$$K(\infty) := \{c_r : r \in \mathbf{N}\} \cup \{d_r : r \in \mathbf{N}\}.$$

Then the following assertions on the left upper section matrices $H(N)$ (of order $N+1$) are immediately clear.

The matrices $H(0)$ and $H(1)$ are zero matrices, hence reducible. For every $r = 1, 2, \ldots$ the condition

$$j_r \leq N \leq d_r - 1 - j_r$$

implies that the matrix $H(N)$ in row (and column) j_r has only zeros (apart from the entry $h_{j_r j_r} = b_{c_r} = 1$). Hence these vertices($\equiv$ states) j_r are isolated in the (undirected) graph of $H(N)$, thus $H(N)$ is reducible for these values of N. Further, if

$$d_r - j_r \leq N \leq d_r \equiv j_{r+1} - 1,$$

then the only nonzero entry (not on the main diagonal) in row j_r is $h_{j_r, d_r - j_r} = 1$. Since $N \leq d_r < 2d_r < c_{r+1}$, the only nonzero entry (not on the main diagonal) in row $d_r - j_r$ is the symmetrically lying $h_{d_r - j_r, j_r} = 1$. Hence the vertices $j_r, d_r - j_r$ form a connected component in the graph of $H(N)$. Therefore $H(N)$ *is reducible for every* $N \in \mathbf{N}$.

Now we shall show that $H(\infty)$ *is irreducible.* We want to prove this by showing that each vertex n in (the graph of) $H(\infty)$ is accessible from the vertex 0. Note that from now on all recursions are understood in $H(\infty)$. Further, by definition, exactly the vertices $c_r,\ d_r$ are accessible in 1 step from 0.

We can show directly that *the closed interval* $[0, d_1]$ *is accessible (from 0) by recursion from the interval* $[0, d_2]$. Indeed, we reach in 1 step $c_1 = 4$ and $d_1 = 7$. Hence we reach 7-4=3 and 4-3=1, thus also 7-1=6. Note that $j_2 = 8, c_2 = 16, d_2 = 26$. Starting from 4 we reach $c_2 - 4 = 12$ and $d_2 - 12 = 14$, hence also $c_2 - 14 = 2$, and finally $d_1 - 2 = 5$ as stated.

Next we shall prove the following statement.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 15, pp. 274-284, October 2006

Let r be any positive integer, and let $c_r + j_{r-1}$ be accessible (from 0) by using recursion from the interval $[0, d_{r+1}]$. Then the vertex j_{r+1} is also accessible (from 0) by using recursion from the interval $[0, d_{r+2}]$.

Notice first that

$$c_r + j_{r-1} = 2j_r + j_{r-1} < 3j_r + j_{r-1} = d_r < c_{r+1}.$$

Since $c_r + j_{r-1}$ is accessible by recursion from the interval $[0, d_{r+1}]$, so is

$$c_{r+1} - (c_r + j_{r-1}) = 2(j_{r+1} - j_r) - j_{r-1}.$$

It follows that the following vertices are also accessible (from 0) by recursion from the interval $[0, d_{r+2}]$:

$$c_{r+2} - \textit{preceding} = 2(j_{r+2} - j_{r+1} + j_r) + j_{r-1},$$

$$d_{r+2} - \textit{preceding} = 6j_{r+1} - j_r + 1 - j_{r-1} > 0,$$

$$c_{r+2} - \textit{preceding} = 2j_{r+2} - 6j_{r+1} + j_r - 1 + j_{r-1} = 3j_r + j_{r-1} + 1 = j_{r+1}.$$

This shows the validity of our claim.

Assume now that $r \geq 1$, and that the closed interval $[0, d_r]$ is accessible (from 0) by using recursion from the interval $[0, d_{r+1}]$. Then the interval $[0, d_{r+1}]$ is accessible (from 0) by using recursion from the interval $[0, d_{r+2}]$.

Indeed, using the assumption, by recursion from the interval $[0, d_{r+1}]$ we can also reach the states in the interval

$$c_{r+1} - [0, d_r] = 2j_{r+1} - [0, j_{r+1} - 1] = [j_{r+1} + 1, 2j_{r+1}].$$

From what has been proved above, we can also reach the state j_{r+1} by recursion from the interval $[0, d_{r+2}]$. Hence we can do the same for the interval $[1, j_{r+1}]$. Since $c_{r+1} = 2j_{r+1}$, we can reach the whole interval $[0, c_{r+1}]$ by recursion from the interval $[0, d_{r+2}]$. Further, we have

$$d_{r+1} - c_{r+1} = j_{r+2} - 1 - 2j_{r+1} = 3j_{r+1} + j_r + 1 - 1 - 2j_{r+1} = j_{r+1} + j_r < c_{r+1}.$$

It follows that

$$d_{r+1} - [0, c_{r+1}] = [j_{r+1} + j_r, d_{r+1}] \supset [c_{r+1}, d_{r+1}].$$

We have obtained that we can reach the whole interval $[0, d_{r+1}]$ by recursion from the interval $[0, d_{r+2}]$ as we had claimed.

The proof of the irreducibility of the matrix $H(\infty)$ proceeds by induction. We have seen that the interval $[0, d_1]$ is accessible from 0 by recursion from the interval $[0, d_2]$. We have also proved that if the interval $[0, d_r]$ is accessible from $[0, d_{r+1}]$, then the interval $[0, d_{r+1}]$ is accessible by using recursion from $[0, d_{r+2}]$. By induction, we

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 15, pp. 274-284, October 2006

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obtain that every interval of the form $[0, d_r]$ is accessible from the state 0. Hence *the graph of* $H(\infty)$ *is (strongly) connected*, i.e. the matrix is irreducible as claimed. $\square$

REMARK 3.12. The unconventional agreement that the 1×1 *zero matrix is reducible* was used only to make possible the elegant formulation of the statement of Example 6, and nowhere else collided with established usage.

REMARK 3.13. Assume that $n \in \mathbf{N}$, T is a Toeplitz matrix, and P is the permutation matrix containing 1's exactly on the main perpendicular diagonal (both of order n). It is well known that the matrix $H := PT$ is then a Hankel matrix, and this remains valid if we say "of type T and H" instead of "Toeplitz and Hankel", respectively. We shall call $H = PT$ the matrix (of type H) corresponding to the matrix T. It is also well known that the matrices T and $H = PT$ are *fully indecomposable* at the same time (for this notion see, e.g., [1, pp. 110-118]). Further, if T is assumed to be (only) irreducible, then [1, Theorem 4.2.3] shows that the indicator of the matrix $T + I$ (where I is the identity of order n) is fully indecomposable (and of type T). Hence $H := P(T + I)$ is also fully indecomposable (and of type H). *A fortiori*, H is irreducible.

On the other hand, the matrix

$$H := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

is irreducible, whereas $T := P^{-1}H = PH = I$ is reducible.

This shows that there is a certain (simple) connection between the irreducibility of *some* matrices of type T and the corresponding matrices of type H (both of *finite order* n). However, our results demonstrate that this connection is by far not sufficient to deal with the problems treated here.

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