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ESSENTIALLY HERMITIAN MATRICES REVISITED[*]

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Abstract. The following case of the Determinantal Conjecture of Marcus and de Oliveira is established. Let A and C be hermitian $n \times n$ matrices with prescribed eigenvalues $a_1, \ldots, a_n$ and $c_1, \ldots, c_n$, respectively. Let κ be a non-real unimodular complex number, $B = \kappa C$, $b_j = \kappa c_j$ for $j = 1, \ldots, n$. Then

$$\det(A - B) \in \operatorname{co} \left\{ \prod_{j=1}^{n} (a_j - b_{\sigma(j)}); \sigma \in S_n \right\},$$

where S_n denotes the group of all permutations of $\{1, \ldots, n\}$ and co the convex hull taken in the complex plane.

1. Introduction. The celebrated Determinantal Conjecture of Marcus [7] and de Oliveira [8] can be stated as follows.

CONJECTURE 1.1. [THE DE OLIVEIRA – MARCUS CONJECTURE (OMC)] *Let A and B be normal $n \times n$ matrices with prescribed complex eigenvalues $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ respectively. Let Δ_0 be the subset of $\mathbb{C}$ given by*

$$\Delta_0 = \operatorname{co} \left\{ \prod_{j=1}^{n} (a_j - b_{\sigma(j)}); \sigma \in S_n \right\}.$$

Then

$$\det(A - B) \in \Delta_0,$$

where S_n denotes the group of all permutations of $\{1, \ldots, n\}$ and co *the convex hull taken in the complex plane.*

The purpose of this paper is to establish the following theorem.

THEOREM 1.2. *Conjecture 1.1 holds in case that A is hermitian and B a non-real scalar multiple of a hermitian matrix.*

Conjecture 1.1 is known in a great many special cases. The case in which A and B are both hermitian was settled by Fiedler[6]. The case in which A is positive definite and B is skew hermitian was settled by da Providência and Bebiano[9]. The case A is positive definite and B a non-real scalar multiple of a hermitian matrix (among others) was settled by N. Bebiano, A. Kovačec, and J. da Providência [5]. The proof of Theorem 1.2 borrows many ideas from these papers. The real content of the present article is to weaken the hypothesis that A is positive definite to A hermitian in [5].

The case in which A and B are both unitary was settled by Bebiano and da Providência[4], the key observation being that Conjecture 1.1 is unchanged under a

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simultaneous application of a fractional linear (Möbius) transformation of the eigen-
values a_j and b_j, allowing a reduction to the case in which A and B are hermitian.

Theorem 1.2 has the following corollary proved using the Möbius transformation
trick and a number of other results.

COROLLARY 1.3. *Let C_A and C_B be circles in the complex plane. Let $a_1, \ldots, a_n \in C_A$ and $b_1, \ldots, b_n \in C_B$, then Conjecture 1.1 holds.*

Proof. The case $C_A = C_B$ can be reduced to the case in which A and B are
both hermitian. The case of non-intersecting circles is established in [1]. The case
of circles that touch can be reduced to the case in which C_B is the real axis and
$C_A = \{z; z \in \mathbb{C}, \Im z = 1\}$ by means of the Möbius transformation trick. This case,
which amounts to showing that

$$(1.1) \qquad \det(iI + C - B) \in \mathrm{co}\left\{ \prod_{j=1}^{n}(i + c_j - b_{\sigma(j)}); \sigma \in S_n \right\}$$

in case b_j, c_j are all real (i.e. B and C hermitian) was handled by N. Bebiano, A.
Kovačec, and J. da Providência in [5]. It can also be deduced from the main result
in an earlier paper Drury[2], where it is shown that (1.1) is valid even if i is treated
as an indeterminate and the convex hull is taken in the ring of polynomials $\mathbb{R}[i]$. The
final case of circles that intersect at two points can be obtained using the Möbius
transformation trick and the result established in this article. ◻

We start by stating some ideas extracted from [9] and [5].

Let Δ be a closed bounded subset of $\mathbb{C}$. Let z be an extreme point of $\mathrm{co}(\Delta)$,
which therefore necessarily lies in Δ. We will say that z is *almost flat* if there is
a smooth curve segment passing through z, lying entirely inside Δ and having zero
curvature at z.

LEMMA 1.4. *The set Δ is contained in the closed convex hull of those extreme
points of $\mathrm{co}(\Delta)$ that are not almost flat.*

LEMMA 1.5. *Let A and B be normal matrices and P skew hermitian. Let $T = A - B$ be invertible and consider a variation $T(t) = \exp(tP)A\exp(-tP) - B$. Then
the expansion*

$$\frac{\det(T(t))}{\det(T)} = 1 + u_1 t + u_2 t^2 + \cdots$$

is valid about $t = 0$ where $u_1 = \mathrm{tr}(T^{-1}[P, A]) = \mathrm{tr}(T^{-1}[P, B])$ and

$$(1.2) \qquad \begin{aligned} u_2 &= \frac{1}{2}\Big(\mathrm{tr}(T^{-1}[P, A])\,\mathrm{tr}(T^{-1}[P, B]) - \mathrm{tr}(T^{-1}[P, A]T^{-1}[P, B]) \Big) \\ &= \mathrm{tr}(T^{-1}[P, A] \wedge T^{-1}[P, B]). \end{aligned}$$

We note that for S an operator on an inner product space E, the operator $S \wedge S$ is
defined on the inner product space $E \wedge E$ by extending $(S \wedge S)(e_1 \wedge e_2) = Se_1 \wedge Se_2$ by
linearity. This definition is further extended to $S_1 \wedge S_2$ for possibly different operators
S_1 and S_2 on E either by the polarization identity

$$S_1 \wedge S_2 = \frac{1}{4}\Big((S_1 + S_2) \wedge (S_1 + S_2) - (S_1 - S_2) \wedge (S_1 - S_2) \Big)$$

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or by $(S_1 \wedge S_2)(e_1 \wedge e_2) = \frac{1}{2}(S_1 e_1 \wedge S_2 e_2 + S_2 e_1 \wedge S_1 e_2)$. We remark in particular that if S is a rank one operator, then $S \wedge S$ is zero.

2. A Quadratic Equation. We now specialize to the case of interest in this article by proving the following.

PROPOSITION 2.1. *Let $n \geq 3$ and $\kappa \in \mathbb{C} \setminus \mathbb{R}$ with $|\kappa| = 1$. Let A be an $n \times n$ invertible hermitian matrix with eigenvalues $a_1, \ldots, a_n$ and B an $n \times n$ matrix with $\kappa^{-1}B$ hermitian and with eigenvalues $b_1, \ldots, b_n$. We assume that the a_j and b_j are generic, specifically we suppose that they are non zero, distinct, distinct from their negatives and that the n^2 numbers $b_j a_k^{-1}$ $(1 \leq j, k \leq n)$ are distinct. Suppose also that A and B have no common nontrivial invariant linear subspace and that $\det(A - B)$ is an extreme point of $\mathrm{co}(\Delta)$, where*

$$\Delta = \{\det(U^\star \operatorname{diag}(a_1, \ldots, a_n)U - V^\star \operatorname{diag}(b_1, \ldots, b_n)V);\ U, V\ unitary\}.$$

Let X denote the matrix $\kappa^{-1}BA^{-1}$. Then either $A - B$ is singular, or X satisfies a quadratic equation with real coefficients

$$(2.1) \qquad \overline{\omega}(I - \overline{\kappa}X) + \omega(I - \kappa X) = \lambda(I - \kappa X)(I - \overline{\kappa}X),$$

$$(2.2) \qquad \lambda X^2 - (\lambda\kappa + \lambda\overline{\kappa} - \kappa\omega - \overline{\kappa\omega})X + (\lambda - \omega - \overline{\omega})I = 0,$$

where ω is a unimodular complex number and $\lambda \in \mathbb{R} \setminus \{0\}$ is suitably chosen. Further, we may write X in one of two possible forms:

- $X = \rho_1 E_1 + \rho_2 E_2$ *where E_1 and E_2 are complementary linear projections, ρ_1, ρ_2 being the roots of* (2.2)
- $X = \rho I + \kappa N$ *where ρ is a double root of* (2.2) *and $N^2 = 0$.*

We remark that the approach used in [5] is to take $X = \kappa^{-1}A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$, a normal matrix, with the advantages that the second case above does not occur, that in the first case the projections E_1 and E_2 are orthogonal projections and that the roots of (2.2) are real and distinct.

Proof. We assume that $A - B$ is non singular. Since $\det(A - B)$ is an extreme point of Δ, it possesses a supporting hyperplane. We choose ω to be a complex number of unit modulus such that the direction $\omega \det(A - B)$ is normal to this hyperplane. It is now clear that for every choice of skew hermitian P the function

$$t \mapsto \Re \overline{\omega} \frac{\det(T(t))}{\det(T)}$$

has a critical point at $t = 0$. Consequently $\Re \overline{\omega} u_1 = 0$. Thus, for all P skew hermitian, we have

$$\Re \overline{\omega} \operatorname{tr}(P[T^{-1}, A]) = 0$$

or equivalently $H = \overline{\omega}[T^{-1}, A]$ is hermitian. Let $C = \kappa^{-1}B$, so that C is hermitian. Letting $X = \kappa^{-1}BA^{-1} = CA^{-1}$, $X^\star = A^{-1}C = A^{-1}XA$. Also $B = \kappa XA$, $T = A - B = (I - \kappa X)A$, $T^{-1} = A^{-1}(I - \kappa X)^{-1}$, $H = \overline{\omega}(A^{-1}(I - \kappa X)^{-1}A - (I - \kappa X)^{-1}) = \overline{\omega}((I - \kappa X^\star)^{-1} - (I - \kappa X)^{-1})$. Since H is hermitian, we get

$$\overline{\omega}((I - \kappa X^\star)^{-1} - (I - \kappa X)^{-1}) = \omega((I - \overline{\kappa}X)^{-1} - (I - \overline{\kappa}X^\star)^{-1}),$$

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or

$$\omega(I - \overline{\kappa}X)^{-1} + \overline{\omega}(I - \kappa X)^{-1} = \overline{\omega}(I - \kappa X^\star)^{-1} + \omega(I - \overline{\kappa}X^\star)^{-1}$$
$$= A^{-1}(\overline{\omega}(I - \kappa X)^{-1} + \omega(I - \overline{\kappa}X)^{-1}A)$$

so that,

$$[A, \overline{\omega}(I - \kappa X)^{-1} + \omega(I - \overline{\kappa}X)^{-1}] = 0.$$

Since the eigenvalues of A are assumed to be all distinct, we see that the matrix $\overline{\omega}(I - \kappa X)^{-1} + \omega(I - \overline{\kappa}X)^{-1}$ diagonalizes with A. Any eigenspace of this operator is necessarily invariant under both A and X and hence also under B. It follows that the operator $\overline{\omega}(I - \kappa X)^{-1} + \omega(I - \overline{\kappa}X)^{-1} = \lambda I$ for some suitable λ. Thus X satisfies the quadratic equation (2.1). Taking adjoints we get

$$\overline{\lambda I} = \omega(I - \overline{\kappa}X^\star)^{-1} + \overline{\omega}(I - \kappa X^\star)^{-1}$$
$$= A^{-1}(\omega(I - \overline{\kappa}X)^{-1} + \overline{\omega}(I - \kappa X)^{-1})A$$
$$= A^{-1}(\lambda I)A = \lambda I$$

so that λ is necessarily real. Note that if $\lambda = 0$ then the equation (2.1) reduces to a linear one. If this equation vanishes identically, then both $\omega + \overline{\omega} = 0$ and $\kappa\omega + \overline{\kappa}\overline{\omega} = 0$, forcing κ to be real, which is not allowed. Thus X is a scalar multiple of the identity and it follows that A and B commute — a contradiction. So we can assert that $\lambda \neq 0$.

The remainder of the assertions follow easily. ◻

3. The Second Order Term — Distinct Roots Cases. We suppose that we are in the first case of Proposition 2.1.

PROPOSITION 3.1. *With the hypotheses and notations of Proposition 2.1, for P an arbitrary skew hermitian matrix we obtain for suitable scalars C_1 and C_2*

$$\Re\overline{\omega}\Big(u_2 - C_1\operatorname{tr}((E_1Z) \wedge (E_1Z)) - C_2\operatorname{tr}((E_2Z) \wedge (E_2Z))\Big) = 0,$$

where $Q = APA^{-1}$ and $Z = P - Q$.

Proof. Let ρ_j, $(j = 1, 2)$ be the roots of (2.1). It will be noted that $X^\star$ also satisfies a similar quadratic equation, the roots of which are $\overline{\rho_j}$ $(j = 1, 2)$. Then we have that $X = \rho_1 E_1 + \rho_2 E_2$ where E_j $(j = 1, 2)$ are (not necessarily orthogonal) linear projections onto linear subspaces V_j $(j = 1, 2)$, according to the direct sum $V = V_1 \oplus V_2$. We can write $X^\star = \overline{\rho_1}E_1^\star + \overline{\rho_2}E_2^\star$ where $E_j^\star$ $(j = 1, 2)$ are linear projections onto the linear subspaces $V_{j'}^\perp$ according to the direct sum $V = V_2^\perp \oplus V_1^\perp$, the notation j' meaning $3 - j$. Since $X^\star = A^{-1}XA$, we find two ways of writing $X^\star$ as a linear combination of two linear projections, namely

$$\overline{\rho_1}E_1^\star + \overline{\rho_2}E_2^\star = X^\star = \rho_1 A^{-1}E_1 A + \rho_2 A^{-1}E_2 A$$

and there are two cases. In case 1 in which (2.1) has two real roots

$$E_j^\star = A^{-1}E_j A, \quad \overline{\rho_j} = \rho_j, \ j = 1, 2$$

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and in case 2 in which (2.2) has a pair of complex conjugate roots

$$E_j^\star = A^{-1} E_{j'} A, \quad \overline{\rho_j} = \rho_{j'}, \; j = 1, 2.$$

We note that in case 2, $\dim(V_1) = \dim(V_2)$ which forces n to be even.
We examine the first part of the second order term which involves

$$\begin{aligned}
&\operatorname{tr}(T^{-1}[P, A]) \operatorname{tr}(T^{-1}[P, B]) \\
&= \kappa \operatorname{tr}(A^{-1}(I - \kappa X)^{-1}[P, A]) \operatorname{tr}(A^{-1}(I - \kappa X)^{-1}(PXA - XAP)) \\
&= \kappa \operatorname{tr}((I - \kappa X)^{-1} Z) \operatorname{tr}(X(I - \kappa X)^{-1} Z) \\
&= \operatorname{tr}((c_1 E_1 + c_2 E_2)Z) \operatorname{tr}(((c_1 - 1)E_1 + (c_2 - 1)E_2)Z) \\
&= \alpha_1 + \beta_1,
\end{aligned}$$

where using the notation $c_j = (1 - \kappa \rho_j)^{-1}$, α_1 consists of the main terms

$$\alpha_1 = c_1(c_1 - 1) \operatorname{tr}(E_1 Z)^2 + c_2(c_2 - 1) \operatorname{tr}(E_2 Z)^2$$

and β_1 consists of the cross terms

$$\beta_1 = (c_1(c_2 - 1) + c_2(c_1 - 1)) \operatorname{tr}(E_1 Z) \operatorname{tr}(E_2 Z).$$

We observe that

$$c_1(c_2 - 1) + c_2(c_1 - 1) = \kappa \frac{\rho_1 + \rho_2}{(1 - \kappa \rho_1)(1 - \kappa \rho_2)} = \omega \lambda \frac{\rho_1 + \rho_2}{\overline{\kappa} - \kappa}.$$

since $\rho_1 + \rho_2$ is real and

$$\lambda \kappa^{-2}(1 - \kappa \rho_1)(1 - \kappa \rho_2)$$

is the result of substituting $\rho = \kappa^{-1}$ into the left hand side of (2.2). So β_1 will be tangential if $\operatorname{tr}(E_1 Z) \operatorname{tr}(E_2 Z)$ is real. In case 1

$$\operatorname{tr}(E_j Z) = \operatorname{tr}((E_j - A^{-1} E_j A)P) = \operatorname{tr}((E_j - E_j^\star)P), \; j = 1, 2,$$

which is real, and in case 2

$$\operatorname{tr}(E_j Z) = \operatorname{tr}((E_j - A^{-1} E_j A)P) = \operatorname{tr}((E_j + E_j^\star - I)P), \; j = 1, 2,$$

which is pure imaginary. In either case, the product $\operatorname{tr}(E_1 Z) \operatorname{tr}(E_2 Z)$ is real.
The second part of the second order term involves

$$\begin{aligned}
&\operatorname{tr}(T^{-1}[P, A]T^{-1}[P, B]) \\
&= \kappa \operatorname{tr}(A^{-1}(I - \kappa X)^{-1}(PA - AP)A^{-1}(I - \kappa X)^{-1}(PXA - XAP)) \\
&= \kappa \operatorname{tr}(X(I - \kappa X)^{-1}P(I - \kappa X)^{-1}P) - \kappa \operatorname{tr}(X(I - \kappa X)^{-1}APA^{-1}(I - \kappa X)^{-1}P) \\
&\quad -\kappa \operatorname{tr}((I - \kappa X)^{-1}P(I - \kappa X)^{-1}XAPA^{-1}) \\
&\quad +\kappa \operatorname{tr}((I - \kappa X)^{-1}APA^{-1}(I - \kappa X)^{-1}XAPA^{-1}) \\
&= \kappa \operatorname{tr}(X(I - \kappa X)^{-1}P(I - \kappa X)^{-1}P) + \kappa \operatorname{tr}(X(I - \kappa X)^{-1}Q(I - \kappa X)^{-1}Q) \\
&\quad -2\kappa \operatorname{tr}(X(I - \kappa X)^{-1}Q(I - \kappa X)^{-1}P) \\
&= \alpha_2 + \beta_2,
\end{aligned}$$

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where

$$\alpha_2 = c_1(c_1 - 1)\operatorname{tr}(E_1 P E_1 P + E_1 Q E_1 Q - 2E_1 P E_1 Q)$$
$$+ c_2(c_2 - 1)\operatorname{tr}(E_2 P E_2 P + E_2 Q E_2 Q - 2E_2 P E_2 Q)$$
$$= c_1(c_1 - 1)\operatorname{tr}(E_1 Z E_1 Z) + c_2(c_2 - 1)\operatorname{tr}(E_2 Z E_2 Z)$$

and

$$\beta_2 = (c_1(c_2 - 1) + c_2(c_1 - 1))\operatorname{tr}(E_1 P E_2 P + E_1 Q E_2 Q)$$
$$- 2c_1(c_2 - 1)\operatorname{tr}(E_2 Q E_1 P) - 2c_2(c_1 - 1)\operatorname{tr}(E_1 Q E_2 P)$$
$$= (c_1(c_2 - 1) + c_2(c_1 - 1))\operatorname{tr}(E_1 Z E_2 Z)$$
$$+ (c_1(c_2 - 1) - c_2(c_1 - 1))\operatorname{tr}(E_1 Q E_2 P - E_1 P E_2 Q).$$

We have $Z^\star = (P - APA^{-1})^\star = -P + A^{-1}PA = A^{-1}ZA$. In case 1 we find

$$(E_1 Z E_2 Z)^\star = A^{-1}ZAA^{-1}E_2 AA^{-1}ZAA^{-1}E_1 A = A^{-1}Z E_2 Z E_1 A$$

so that $\operatorname{tr}(E_1 Z E_2 Z)$ is real. This is also correct in case 2 since then

$$(E_1 Z E_2 Z)^\star = A^{-1}ZA(I - A^{-1}E_2 A)A^{-1}ZA(I - A^{-1}E_1 A)$$

and therefore (using $E_1 + E_2 = I$)

$$\overline{\operatorname{tr}(E_1 Z E_2 Z)} = \operatorname{tr}(Z(I - E_2)Z(I - E_1)) = \operatorname{tr}(Z E_1 Z E_2) = \operatorname{tr}(E_1 Z E_2 Z).$$

Now

$$(3.1) \qquad c_1(c_2 - 1) - c_2(c_1 - 1) = \kappa \frac{\rho_2 - \rho_1}{(1 - \kappa\rho_1)(1 - \kappa\rho_2)} = \omega\lambda \frac{\rho_2 - \rho_1}{\overline{\kappa} - \kappa}$$

again since

$$(3.2) \qquad\qquad \lambda\kappa^{-2}(1 - \kappa\rho_1)(1 - \kappa\rho_2) = (\kappa^{-2} - 1)\overline{\omega}.$$

The quantity (3.1) is a pure imaginary multiple of ω in case 1 and real multiple of ω in case 2.

It remains to show that $\operatorname{tr}(E_1 Q E_2 P - E_1 P E_2 Q)$ is real in case 1 and pure imaginary in case 2. In case 1

$$(E_1 Q E_2 P - E_1 P E_2 Q)^\star$$
$$= PA^{-1}E_2 AA^{-1}PAA^{-1}E_1 A - A^{-1}PAA^{-1}E_2 APA^{-1}E_1 A,$$

which gives

$$\overline{\operatorname{tr}(E_1 Q E_2 P - E_1 P E_2 Q)} = \operatorname{tr}(Q E_2 P E_1 - P E_2 Q E_1)$$

and in case 2,

$$\overline{\operatorname{tr}(E_1 Q E_2 P - E_1 P E_2 Q)} = \operatorname{tr}(Q E_1 P E_2 - P E_1 Q E_2).$$

Consequently the second order term is

$$c_1(c_1 - 1)\operatorname{tr}(E_1 Z \wedge E_1 Z) + c_2(c_2 - 1)\operatorname{tr}(E_2 Z \wedge E_2 Z) + \frac{1}{2}(\beta_1 + \beta_2).$$

The term $\frac{1}{2}(\beta_1 + \beta_2)$ is always tangentially aligned. □

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4. The Second Order Term — Equal Roots Case. We suppose that we are in the second case of Proposition 2.1 to be designated case 3 in the sequel.

PROPOSITION 4.1. *With the hypotheses and notations of Proposition 2.1, for P an arbitrary skew hermitian matrix we obtain for suitable scalars C_1 and C_2*

$$\Re\overline{\omega}\Big(u_2 - C_1\operatorname{tr}((NZ)\wedge(NZ)) - C_2\operatorname{tr}(NZ^2)\Big) = 0,$$

where $Q = APA^{-1}$ and $Z = P - Q$.

Proof. We have that (2.1) has a double root ρ and hence ρ is a real. In particular, $\rho \neq \kappa^{-1}$. We have

$$X = \rho I + N, \qquad N^2 = 0.$$

We find that $(I - \kappa X)^{-1} = (1 - \kappa\rho)^{-1}I + \kappa(1 - \kappa\rho)^{-2}N$ and $\kappa X(I - \kappa X)^{-1} = (I - \kappa X)^{-1} - I = \kappa\rho(1 - \kappa\rho)^{-1}I + \kappa(1 - \kappa\rho)^{-2}N$. So the first part of the second order term involves

$$\operatorname{tr}((I - \kappa X)^{-1}Z)\operatorname{tr}(\kappa X(I - \kappa X)^{-1}Z)$$
$$= \operatorname{tr}(((1 - \kappa\rho)^{-1}I + \kappa(1 - \kappa\rho)^{-2}N)Z)\operatorname{tr}((\kappa\rho(1 - \kappa\rho)^{-1}I + \kappa(1 - \kappa\rho)^{-2}N)Z)$$
$$= \kappa^2(1 - \kappa\rho)^{-4}\Big(\operatorname{tr}(NZ)\Big)^2$$

since $\operatorname{tr}(Z) = 0$. The second part of the second order term involves

$$\kappa\operatorname{tr}((I - \kappa X)^{-1}ZX(I - \kappa X)^{-1}Z)$$
$$= \kappa\operatorname{tr}(((1 - \kappa\rho)^{-1}I + \kappa(1 - \kappa\rho)^{-2}N)Z(\rho(1 - \kappa\rho)^{-1}I + (1 - \kappa\rho)^{-2}N)Z)$$
$$= \kappa\rho(1 - \kappa\rho)^{-2}\operatorname{tr}(Z^2)$$
$$\quad + \kappa(1 + \kappa\rho)(1 - \kappa\rho)^{-3}\operatorname{tr}(NZ^2) + \kappa^2(1 - \rho)^{-4}\operatorname{tr}(NZNZ).$$

Now $Z^2 = (P - APA^{-1})^2 = P^2 - PAPA^{-1} - APA^{-1}P + AP^2A^{-1}$ clearly has a real trace. On the other hand, we have analogous to (3.2)

$$\lambda\kappa^{-2}(1 - \kappa\rho)^2 = (\kappa^{-2} - 1)\overline{\omega}$$

so that

$$\kappa\rho(1 - \kappa\rho)^{-2} = \frac{\lambda\omega}{\overline{\kappa} - \kappa}$$

and it follows that $\kappa\rho(1 - \kappa\rho)^{-2}$ and $\kappa\rho(1 - \kappa\rho)^{-2}\operatorname{tr}(Z^2)$ are tangentially aligned. □

5. The Second Order Term — Conclusion. Surprisingly, case 2 is the easiest of the three cases to settle.

PROPOSITION 5.1. *Assume that we are in case 2 arising in the proof of Proposition 3.1. It is possible to choose a skew hermitian matrix P such that E_1ZE_1 and E_2ZE_2 are simultaneously rank one and $\operatorname{tr}(E_1Z) = -\operatorname{tr}(E_2Z) \neq 0$. Consequently, the entire second order term is tangential and the underlying extreme point is flat.*

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Proof. We have $E_j^\star = A^{-1}E_{j'}A$ and we simply set $P = i\xi \otimes \xi^\star$ where say $\xi \in E_1$ is a nonzero vector. We note that $E_j Z E_j = E_j[P,A]E_{j'}^\star A^{-1}$, so it will suffice to show that $E_j[P,A]E_{j'}^\star$ is rank one and that $\mathrm{tr}(E_j[P,A]E_{j'}^\star A^{-1}) \neq 0$. We obtain

$$[P,A] = i\Big(\xi \otimes (A\xi)^\star - (A\xi) \otimes \xi^\star\Big)$$

so that

$$E_1[P,A]E_2^\star = i\xi \otimes (E_2 A\xi)^\star \text{ and } E_2[P,A]E_1^\star = -i(E_2 A\xi) \otimes \xi^\star$$

both of which are rank one. Furthermore,

$$\mathrm{tr}(E_1[P,A]E_2^\star A^{-1}) = i\xi^\star A E_2^\star A^{-1}\xi = i\xi^\star E_1\xi = i\|\xi\|^2 \neq 0$$
$$\mathrm{tr}(E_2[P,A]E_1^\star A^{-1}) = -i\xi^\star A^{-1}E_2 A\xi = -i\xi^\star E_1^\star\xi = -i\|\xi\|^2 \neq 0.$$

These terms are easily seen to be equal and opposite in any case. We have

$$\mathrm{tr}\Big((E_j Z) \wedge (E_j Z)\Big) = \mathrm{tr}\Big((E_j \wedge E_j)(Z \wedge Z)\Big) = \mathrm{tr}\Big((E_j \wedge E_j)(Z \wedge Z)(E_j \wedge E_j)\Big)$$
$$= \mathrm{tr}\Big((E_j Z E_j) \wedge (E_j Z E_j)\Big) = 0,$$

since $(E_j \wedge E_j)^2 = (E_j \wedge E_j)$. It follows from Proposition 3.1 that $\Re\overline{\omega}u_2 = 0$. ☐

LEMMA 5.2. *Let $\alpha_1,\ldots,\alpha_n$ be distinct non-zero complex numbers such that $\alpha_j + \alpha_k \neq 0$ for all j,k. Let $\beta_1,\ldots,\beta_n$ be complex numbers. Suppose that*

$$\sum_{j=1}^{n} \alpha_j^m = \sum_{j=1}^{n} \beta_j^m \text{ for } m = 1,3,5,\ldots$$

Then there exists $\sigma \in S_n$ such that $\beta_j = \alpha_{\sigma(j)}$ for $j = 1,\ldots,n$.

Sketch of proof.

The hypotheses can be used to show that

(5.1) $$\prod_{j=1}^{n} \frac{1 + \alpha_j z}{1 - \alpha_j z} = \prod_{j=1}^{n} \frac{1 + \beta_j z}{1 - \beta_j z}$$

for all complex z. To prove this, take logarithms for $|z|$ small and expand as a power series in z. Since

$$\ln\left(\frac{1+z}{1-z}\right) = 2 \sum_{\substack{m=1 \\ m \text{ odd}}}^{\infty} \frac{z^m}{m}$$

we have

$$\sum_{j=1}^{n} \ln\left(\frac{1 + \alpha_j z}{1 - \alpha_j z}\right) = 2 \sum_{\substack{m=1 \\ m \text{ odd}}}^{\infty} \frac{\sum_{j=1}^{n} \alpha_j^m}{m} z^m$$
$$= 2 \sum_{\substack{m=1 \\ m \text{ odd}}}^{\infty} \frac{\sum_{j=1}^{n} \beta_j^m}{m} z^m = \sum_{j=1}^{n} \ln\left(\frac{1 + \beta_j z}{1 - \beta_j z}\right).$$

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Analytic continuation then shows that the two functions in (5.1) agree as rational functions. Since the hypotheses imply that cancelation on the left in (5.1) is impossible and since the left hand side has a full complement of n distinct poles, matching poles on the right gives the result. $\square$

PROPOSITION 5.3. *Assume that we are in case 1 arising in the proof of Proposition 3.1. It is possible to choose a skew hermitian matrix P such that $E_1 Z E_1$ and $E_2 Z E_2$ are simultaneously both of rank one and such that $\mathrm{tr}(E_1 Z) = -\mathrm{tr}(E_2 Z) \neq 0$. Consequently, the entire second order term is tangential and the underlying extreme point is flat.*

Proof. We again take $P = i\xi \otimes \xi^\star$ and obtain

$$E_j Z E_j = iE_j \xi \otimes \xi^\star E_j - iE_j A\xi \otimes \xi^\star A^{-1} E_j.$$

To make this of rank one, we choose ξ to satisfy $(A - \alpha_1 E_1 - \alpha_2 E_2)\xi = 0$ on the assumption that α_j $(j = 1, 2)$ are nonzero complex numbers such that $A - \alpha_1 E_1 - \alpha_2 E_2$ is singular. It follows that $E_j A\xi = \alpha_j E_j \xi$ forcing $E_j Z E_j$ to be rank one for both $j = 1, 2$. Our result will follow unless $\mathrm{tr}(E_j Z) = 0$ for all such ξ. Now we have

$$
\begin{aligned}
\mathrm{tr}(E_j Z) &= \mathrm{tr}(E_j Z E_j) \\
&= i\left(\xi^\star E_j \xi - \xi^\star A^{-1} E_j A\xi\right) \\
&= i\left(\xi^\star E_j \xi - \alpha_j \xi^\star A^{-1} E_j \xi\right) \\
&= i\left(\xi^\star E_j \xi - \alpha_j \xi^\star E_j^\star A^{-1}\xi\right) \\
&= i\left(\xi^\star E_j \xi - \frac{\alpha_j}{\overline{\alpha_j}} \xi^\star A E_j^\star A^{-1}\xi\right) \\
&= i\left(1 - \frac{\alpha_j}{\overline{\alpha_j}}\right) \xi^\star E_j \xi.
\end{aligned}
$$

This last quantity must vanish for both $j = 1$ and $j = 2$ since $\mathrm{tr}(E_1 Z) + \mathrm{tr}(E_2 Z) = \mathrm{tr}(Z) = 0$. But $\xi^\star E_1 \xi + \xi^\star E_2 \xi = \|\xi\|^2 \neq 0$. It follows that either $1 - \frac{\alpha_1}{\overline{\alpha_1}} = 0$ or $1 - \frac{\alpha_2}{\overline{\alpha_2}} = 0$. In other words, either α_1 or α_2 is real. To finish the proof, let $m \geq 3$ be an odd integer, $t > 0$ and consider the characteristic roots ν of $(E_1 + tzE_2)A = (E_1 + t^{-1}z^{-1}E_2)^{-1}A$ where z is a primitive m^{th} root of unity. For each such ν, $A - \nu E_1 - t^{-1}z^{-1}\nu E_2$ is singular. It follows that each such characteristic root ν is either real or a real multiple of z. Consequently $((E_1 + tzE_2)A)^m$ has only real characteristic roots and indeed $\mathrm{tr}\left(((E_1 + tzE_2)A)^m\right)$ is real. Let k be an integer with $0 < k < m$. Since $\mathrm{tr}\left(((E_1 + tzE_2)A)^m\right)$ is a polynomial in the positive variable t, the coefficient of t^k in this expression will also be real. This means that $z^k \mathrm{tr}(W_k)$ is real where W_k is the coefficient of t^k in $((E_1 + tE_2)A)^m$. By working with two different primitive m^{th} roots z_1 and z_2 such that $z_1^k z_2^{-k}$ is not real (possible since $m \geq 3$ is odd) we see that $\mathrm{tr}(W_k) = 0$ for $k = 1, 2, \ldots, m-1$. Therefore we deduce

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that $\mathrm{tr}\left(((z_1 E_1 + z_2 E_2)A)^m\right) = z_1^m \,\mathrm{tr}((E_1 A)^m) + z_2^m \,\mathrm{tr}((E_2 A)^m)$ for m an odd integer $m \geq 1$.

Observe that if $n_j = \dim(V_j) = \mathrm{rank}(E_j)$, then $n = n_1 + n_2$ and $E_j A$ has exactly n_j non-zero characteristic roots (counted according to multiplicity). It cannot have fewer since $n = \mathrm{rank}(A) \leq \mathrm{rank}(E_1 A) + \mathrm{rank}(E_2 A)$. Taking $z_1 = z_2 = 1$ and applying Lemma 5.2, we see that the non-zero characteristic roots of $E_1 A$ and of $E_2 A$ make up the eigenvalues of A. But, recalling that $B = (\kappa\rho_1 E_1 + \kappa\rho_2 E_2)A$, we see also that $\mathrm{tr}(B^m) = (\kappa\rho_1)^m \,\mathrm{tr}((E_1 A)^m) + \kappa(\rho_2)^m \,\mathrm{tr}((E_2 A)^m)$ for m odd $m \geq 1$. Applying Lemma 5.2 again, allows the b_j to be identified. It follows that the eigenvalues b_j of B have the form $\kappa\rho_{\tau(j)} a_{\sigma(j)}$ where $\tau : \{1,2,\ldots,n\} \longrightarrow \{1,2\}$ takes the value 1 n_1 times and the value 2 n_2 times and $\sigma \in S_n$. But for $n \geq 3$ this contradicts our hypothesis that the quantities $b_j a_k^{-1}$ are all distinct. $\square$

PROPOSITION 5.4. *Assume that we are in case 3. It is possible to choose a skew hermitian matrix P such that NZ has rank one, $\mathrm{tr}(NZ^2) = 0$ and such that $\mathrm{tr}(NZ) \neq 0$. Consequently, the entire second order term is tangential and the underlying extreme point is flat.*

Proof. First we need to find $N^\star$. Since $X = \rho I + N$ and ρ is real, we have that $N^\star = A^{-1} N A$. Now suppose that α and β are such that $A - \alpha I - \beta N$ is a singular matrix and suppose that ξ is a vector such that $A\xi = \alpha\xi + \beta N\xi$. We take $P = i\xi \otimes \xi^\star$. We obtain,

$$
\begin{aligned}
Z &= i(\xi \otimes \xi^\star - A\xi \otimes \xi^\star A^{-1}) \\
&= i(\xi \otimes \xi^\star - \alpha\xi \otimes \xi^\star A^{-1} - \beta N\xi \otimes \xi^\star A^{-1}) \\
NZ &= iN\xi \otimes (\xi^\star - \alpha\xi^\star A^{-1})
\end{aligned}
$$

since $N^2 = 0$. So NZ has rank one. Furthermore

$$
\begin{aligned}
NZ^2 &= -N\xi \otimes \xi^\star(I - \alpha A^{-1})(\xi \otimes \xi^\star - A\xi \otimes \xi^\star A^{-1}) \\
\mathrm{tr}(NZ^2) &= -(\xi^\star N\xi)(\xi^\star(I - \alpha A^{-1})\xi) + (\xi^\star A^{-1} N\xi)(\xi^\star(I - \alpha A^{-1})A\xi) \\
&= -(\xi^\star N\xi)(\xi^\star(I - \alpha A^{-1})\xi) + (\xi^\star A^{-1} N\xi)(\xi^\star(A - \alpha I)\xi) \\
&= -(\xi^\star N\xi)(\xi^\star(I - \alpha A^{-1})\xi) + \beta(\xi^\star A^{-1} N\xi)(\xi^\star N\xi) \\
&= (\xi^\star N\xi)(\xi^\star(\beta A^{-1} N - I + \alpha A^{-1})\xi) \\
&= (\xi^\star N\xi)(\xi^\star(A^{-1}(A - \alpha I) - I + \alpha A^{-1})\xi) = 0.
\end{aligned}
$$

So our result will follow unless $\mathrm{tr}(NZ) = 0$ in all these situations. We note that

$$
\begin{aligned}
\mathrm{tr}(NZ) &= i\xi^\star(I - \alpha A^{-1})N\xi \\
&= i\xi^\star(A - \alpha I)A^{-1} N\xi \\
&= i\xi^\star(\overline{\alpha} - \alpha)A^{-1} N\xi + i\xi^\star\overline{\beta}N^\star A^{-1} N\xi \\
&= i(\overline{\alpha} - \alpha)\xi^\star A^{-1} N\xi
\end{aligned}
$$

since $N^\star A^{-1} N = A^{-1} N A A^{-1} N = A^{-1} N^2 = 0$. We claim that α is real. If not then $\xi^\star A^{-1} N\xi = 0$. So $0 = \xi^\star A^{-1}\beta N\xi = \xi^\star A^{-1}(A - \alpha I)\xi$ or $\|\xi\|^2 = \alpha\xi^\star A^{-1}\xi$ and it follows that α is real after all.

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Now consider the characteristic roots of $(I+zN)A$ where z is an arbitrary complex number. If ν is a characteristic root, then there exists a vector ξ such that $(I + zN)A\xi = \nu\xi$ or $A\xi = \nu(I - zN)\xi$. It follows that ν is real. Therefore, for every integer m, and every complex z, $\operatorname{tr}((I + zN)A)^m$ is real. It follows that $\operatorname{tr}((I + zN)A)^m = \operatorname{tr}(A^m)$ for $m = 0, 1, 2, \ldots$. Applying Lemma 5.2 we deduce that $(I + zN)A$ has precisely the same characteristic roots as A. But $BA^{-1} = \kappa X = \kappa\rho I + \kappa N$ and $\rho \neq 0$ since B is non-singular. Choosing $z = \rho^{-1}$, we find that the eigenvalues of B are proportional to those of A. But this eventuality is not allowed by our hypotheses. $\square$

6. Final Steps. *Proof of Theorem 1.2.* We prove the result by strong induction on n. For $n = 1$ and $n = 2$ the result is easy to verify by direct calculation.

Let $n \geq 3$. We suppose that $a_1, \ldots, a_n$ and $c_1, \ldots, c_n$ are real and that $b_j = \kappa a_j$ for some fixed $\kappa \in \mathbb{C} \setminus \mathbb{R}$ with $|\kappa| = 1$. We consider Conjecture 1.1 in this case. It is clear that OMC is stable under perturbations. Explicitly, this means that if OMC holds for $a_1^{(k)}, a_2^{(k)}, \ldots, b_n^{(k)}$ for every $k = 1, 2, \ldots$ and if $\lim_{k\to\infty} a_j^{(k)} = a_j$ and $\lim_{k\to\infty} b_j^{(k)} = b_j$ for $j = 1, 2, \ldots, n$ then it also holds for $a_1, a_2, \ldots, b_n$. It therefore suffices to establish our conjectures for generic sets of eigenvalues as described in Proposition 2.1.

Now suppose that $\Delta \not\subseteq \Delta_0$. Then it follows from Lemma 1.4 that there is an extreme point z of $\operatorname{co}(\Delta)$ which is not almost flat and such that $z \notin \Delta_0$. We can assert that $z \neq 0$ since OMC is known in this case [3]. Let A and B be the corresponding matrices. Applying Propositions 2.1, 3.1 and 4.1, we can conclude that A and B possess a common nontrivial invariant linear subspace. The orthogonal complement is also simultaneously invariant. The fact that the eigenvalue sets consist of distinct elements allows the matrices (or rather the corresponding operators) to be decomposed simultaneously on spaces of lower dimension. This allows a contradiction to be established from the strong induction hypothesis. Hence $\Delta \subseteq \Delta_0$ as required. $\square$

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