Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 15, pp. 297-313, November 2006

ELA
http://math.technion.ac.il/iic/ela

# ZEROS OF UNILATERAL QUATERNIONIC POLYNOMIALS*

STEFANO DE LEO†, GISELE DUCATI‡, AND VINICIUS LEONARDI§

**Abstract.** The purpose of this paper is to show how the problem of finding the zeros of unilateral $n$-order quaternionic polynomials can be solved by determining the eigenvectors of the corresponding companion matrix. This approach, probably superfluous in the case of quadratic equations for which a closed formula can be given, becomes truly useful for (unilateral) $n$-order polynomials. To understand the strength of this method, it is compared with the Niven algorithm and it is shown where this (full) matrix approach improves previous methods based on the use of the Niven algorithm. For convenience of the readers, some examples of second and third order unilateral quaternionic polynomials are explicitly solved. The leading idea of the practical solution method proposed in this work can be summarized in the following three steps: translating the quaternionic polynomial in the eigenvalue problem for its companion matrix, finding its eigenvectors, and, finally, giving the quaternionic solution of the unilateral polynomial in terms of the components of such eigenvectors. A brief discussion on bilateral quaternionic quadratic equations is also presented.

**Key words.** Quaternions, Eigenvalue problem, Matrices over special rings.

**AMS subject classifications.** 15A33, 20G20.

## 1. Introduction.

As far as we know, the problem of solving quaternionic quadratic equations and the study of the fundamental theorem of algebra for quaternions were first approached by Niven and Eilenberg [1, 2]. After these fundamental works, the question of counting the number of zeros of $n$-order quaternionic polynomials and the problem to find the solutions have been investigated. An interesting application of unilateral quadratic equations (where a closed formula for their solutions can be obtained) is found in solving homogeneous quaternionic linear second order differential equations with constant coefficients [3, 4]. The solution of

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\,\Psi(x) - a_1\,\frac{\mathrm{d}}{\mathrm{d}x}\,\Psi(x) - a_0\,\Psi(x) = 0\ , \tag{1.1}$$

where $\Psi : \mathbb{R} \to \mathbb{H}$ , $a_{0,1} \in \mathbb{H}$ (the set of quaternions) and $x \in \mathbb{R}$, can indeed be reduced, by setting $\Psi(x) = \exp[q\,x]$ ($q \in \mathbb{H}$) and using the $\mathbb{H}$-linearity of Eq.(1.1), to the following quadratic equation

$$q^2 - a_1\,q - a_0 = 0\ . \tag{1.2}$$

*Received by the editors 7 December 2005. Accepted for publication 15 October 2006. Handling Editor: Angelika Bunse-Gerstner.

†Department of Applied Mathematics, University of Campinas, SP 13083-970, Campinas, Brazil (deleo@ime.unicamp.br).

‡Department of Mathematics, University of Parana, PR 81531-970, Curitiba, Brazil (ducati@mat.ufpr.br).

§Department of Physics, University of Parana, PR 81531-970, Curitiba, Brazil (vjhcl02@fisica.ufpr.br).

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 15, pp. 297-313, November 2006

ELA
http://math.technion.ac.il/iic/ela



Generalizing this result, we see that the problem of finding the solution of $n$-order ($\mathbb{H}$-linear) differential equations with constant coefficients,

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\,\Psi(x) - \sum_{s=1}^{n-1} a_s\,\frac{\mathrm{d}^s}{\mathrm{d}x^s}\,\Psi(x) - a_0\,\Psi(x) = 0\ , \tag{1.3}$$

where $a_{0,1,\ldots,n-1} \in \mathbb{H}$, can be immediately transformed to the problem of finding the zeros of the corresponding $n$-order unilateral quaternionic polynomial,

$$A_n(q) := q^n - \sum_{s=0}^{n-1} a_s\,q^s\ . \tag{1.4}$$

The important point to be noted here is that such a solution method based on finding the zeros of the corresponding quaternionic polynomial for $\mathbb{H}$-linear differential equations with constant coefficients does not work for $\mathbb{R}$ and $\mathbb{C}$-linear differential equations. For example, in non-relativistic quaternionic quantum mechanics [5], the one-dimensional Schrödinger equation (for quaternionic stationary states) in presence of a constant quaternionic potential, reads

$$i\,\frac{\hbar^2}{2m}\,\frac{\mathrm{d}^2}{\mathrm{d}x^2}\,\Phi(x) - (i\,V_1 + j\,V_2 + k\,V_3)\,\Phi(x) = -\,\Phi(x)\,E\,i\ , \tag{1.5}$$

and, due to the $\mathbb{C}$-linearity, its solution cannot be expressed in terms of a quaternionic exponential function, $\exp[q\,x]$. Recent studies of quaternionic barrier [6] and well [7] potentials confirmed that the solution of Eq.(1.5) has to be expressed as the product of two factors: a quaternionic constant coefficient, $p$, and a complex exponential function, $\exp[z\,x]$ ($z \in \mathbb{C}$). Using $\Phi(x) = p\,\exp[z\,x]$, the Eq.(1.5) reduces to

$$i\,\frac{\hbar^2}{2m}\,p\,z^2 - (i\,V_1 + j\,V_2 + k\,V_3)\,p = -\,p\,E\,i\ , \tag{1.6}$$

which obviously does not represent a quaternionic unilateral polynomial. For a more complete discussion of $\mathbb{R}$ and $\mathbb{C}$ linear quaternionic differential equation theory, we refer the reader to the papers cited in refs. [3, 8].

The problem of finding the zeros of unilateral quaternionic polynomials was solved in 1941 by using a *two-step* algorithm. In his seminal work [9], Niven proposed to divide the unilateral $n$-order quaternion polynomial (1.4) by a quadratic polynomial with real coefficients ($c_{0,1} \in \mathbb{R}$)

$$C_2(q) := q^2 - c_1\,q - c_0\ . \tag{1.7}$$

Then, by using the polynomial equation

$$A_n(q) = B_{n-2}(q)\,C_2(q) - D_1(q)\ , \tag{1.8}$$

where

$$B_{n-2}(q) := q^{n-2} - \sum_{s=0}^{n-3} b_s\,q^s\ ,\quad D_1(q) := d_1\,q + d_0\ ,\quad b_{0,1,\ldots n-3}\,,\ d_{0,1} \in \mathbb{H}\ ,$$

the problem of finding the zeros of $A_n(q)$ was translated in the following *two-step* problem:

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 15, pp. 297-313, November 2006

ELA
http://math.technion.ac.il/iic/ela



- step 1 - determine $d_0$ and $d_1$ in terms of $c_{0,1}$ and $a_{0,1,\ldots,n-1}$;
- step 2 - obtain two coupled real equations which allow to calculate $c_0$ and $c_1$.

To facilitate the understanding of this paper, we explicitly discuss the Niven algorithm in the next section. Since its proof is quite detailed and, furthermore, since its use can be *completely* avoided, we merely state the results without proof.

In a recent paper [10], Serôdio, Pereira and Vitoria (SPV) proposed a practical technique to compute the zeros of unilateral quaternion polynomials. The SPV approach is based on the use of the Niven algorithm [9]. Their idea was to improve such an algorithm by calculating the real coefficients $c_0$ and $c_1$ by using, instead of the second part of the Niven algorithm, the (complex) eigenvalues of the companion matrix associated with the quaternionic polynomial $A_n(q)$. The SPV approach, avoiding the solution of the coupled real equations (step 2), surely simplifies the Niven algorithm and gives a more practical method to find the zeros of $n$-order unilateral quaternionic polynomials. However, once the real coefficients $c_0$ and $c_1$ are obtained in terms of the moduli and real parts of the complex eigenvalues of the companion matrix, we have to find the quaternionic coefficients $d_0$ and $d_1$. It seems that we have *no* alternative choice and we have to come back to the Niven algorithm (step 1).

In this paper, we show that the matrix approach (based on the use of the quaternion eigenvalue problem) leads us to find *directly* the solutions of unilateral quaternion polynomials by calculating the eigenvectors of the companion matrix. This allow us to *completely* avoid the use of the Niven algorithm and, consequently, to simplify the method to finding the zeros of unilateral quaternionic polynomials.

We interrupt our discussion at this point to introduce, the Niven algorithm and the vector-matrix notation. These tools would permit a more pleasant reading of the article and render our exposition self-consistent.

**2. The Niven algorithm.** In this section, we aim to concentrate on the Niven algorithm. We shall first discuss the method proposed by Niven for a generic $n$-order quaternionic polynomial, then, to illustrate the potential and problems of the Niven algorithm, we consider an explicit example, i.e. a quadratic polynomial.

In order to determine the quaternionic coefficients $d_0$ and $d_1$ in terms of the real coefficients of the second order polynomial $C_2(q)$ and of the quaternionic coefficients of the $n$-order polynomials $A_n(q)$, we expand the polynomial product $B_{n-2}(q)C_2(q)$,

$$\begin{aligned} B_{n-2}(q)C_2(q) = {} & q^n - (b_{n-3} + c_1)\, q^{n-1} - (b_{n-4} - c_1\, b_{n-3} + c_0)\, q^{n-2} + \\ & - \sum_{s=2}^{n-3} (b_{s-2} - c_1\, b_{s-1} - c_0\, b_s)\, q^s + \\ & + (c_1\, b_0 + c_0\, b_1)\, q + c_0\, b_0 \ . \end{aligned} \tag{2.1}$$

For a particular choice of the quaternionic coefficients of the polynomial $B_{n-2}(q)$, i.e.

$$\begin{aligned} b_{n-3} &= a_{n-1} - c_1 \ , \\ b_{n-4} &= a_{n-2} + c_1\, b_{n-3} - c_0 \ , \\ b_{s-2} &= a_s + c_1\, b_{s-1} + c_0\, b_s \qquad (s = 2, 3, \cdots, n-3) \ , \end{aligned} \tag{2.2}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 15, pp. 297-313, November 2006

ELA
http://math.technion.ac.il/iic/ela



we can rewrite Eq.(2.1) as follows

$$B_{n-2}(q)C_2(q) = A_n(q) + (a_1 + c_1\, b_0 + c_0\, b_1)\, q + a_0 + c_0\, b_0 \ . \tag{2.3}$$

Comparing this equation with the polynomial equation (1.8), we obtain (step 1)

$$\begin{aligned} d_1 &= a_1 + c_1\, b_0(c_{0,1}; a_{2,3,\cdots,n-1}) + c_0\, b_1(c_{0,1}; a_{2,3,\cdots,n-1}) \ , \\ d_0 &= a_0 + c_0\, b_0(c_{0,1}; a_{2,3,\cdots,n-1}) \ . \end{aligned} \tag{2.4}$$

This show that by some simple algebraic manipulations, we can express the quaternionic coefficients $d_{0,1}$ in terms of the real coefficients $c_{0,1}$ and of the quaternionic coefficients $a_{0,1,\ldots,n-1}$. This represents step 1 of the Niven algorithm. As stated in the introduction, to complete the Niven algorithm, we have to determine the real coefficients $c_{0,1}$ (step 2). Once such coefficients are obtained, we can explicitly calculate the coefficients $d_{0,1}$ by using Eq.(2.4) which, after step 2, only contains known quantities. It is natural to ask why the quaternionic coefficients $d_{0,1}$ are important in calculating the quaternionic solution of the polynomial $A_n(q)$. The answer is given by observing that if $q_*$ is a solution of $A_n(q)$ $[\Rightarrow A_n(q_*) = 0]$ and $\{\, c_0\,,\, c_1\,\} = \{-\,|q_*|^2\,,\, 2\,\mathrm{Re}[q_*]\,\}$ $[\Rightarrow C_2(q_*) = 0]$, then from Eq.(1.8), we find

$$D_1(q_*) = 0 \quad \Rightarrow \quad q_* = -\,\bar{d}_1\, d_0 \,/\, |d_1|^2 \ . \tag{2.5}$$

The quaternionic solution can thus be expressed in terms of the coefficients $d_{0,1}$. Turning to the problem of determining the real coefficients $c_{0,1}$, Niven noted that

$$c_0 = -\,|q_*|^2 = -\,|d_0|^2 \,/\, |d_1|^2 \quad \text{ and } \quad c_1 = 2\,\mathrm{Re}[q_*] = -2\,\mathrm{Re}[\bar{d}_1\, d_0] \,/\, |d_1|^2 \ .$$

Consequently, in order to find $c_{0,1}$ we have to solve the following system (step 2)

$$\begin{aligned} c_0\, |d_1(c_{0,1}; a_{1,2,3,\cdots,n-1})|^2 + |d_0(c_{0,1}; a_{0,2,3,\cdots,n-1})|^2 &= 0 \ , \\ c_1\, |d_1(c_{0,1}; a_{1,2,3,\cdots,n-1})|^2 + 2\,\mathrm{Re}[\bar{d}_1(c_{0,1}; a_{1,2,3,\cdots,n-1}) d_0(c_{0,1}; a_{0,2,3,\cdots,n-1})] &= 0 \ . \end{aligned} \tag{2.6}$$

Each *real* coupled solution $\{\, c_0\,,\, c_1\,\}$ gives the modulus and the real part of the quaternionic polynomial solution.

• **Second order quaternionic polynomials**. The main (practical) problem in using the Niven algorithm is in step 2, that is finding the real solutions of the coupled system (2.6). To illustrate the difficulty in using the Niven algorithm, we solve the following quaternionic quadratic equation

$$q^2 + j\, q + (1 - k) = 0 \quad [\, A_2(q) = q^2 - a_1\, q - a_0 \ ,\ \ a_0 = k - 1 \ ,\ \ a_1 = -j\,] \ . \tag{2.7}$$

Second order polynomials represent the more simple situation in which we can test the Niven algorithm. In this case, the quaternionic polynomial $B_{n-2}(q)$ reduces to $B_0(q) = -b_0 = 1$. Consequently,

$$\begin{aligned} B_0(q)C_2(q) &= q^2 - c_1\, q - c_0 \\ &= q^2 - a_1\, q - a_0 + (\, a_1 - c_1\,)\, q + a_0 - c_0 \\ &= A_2(q) + D_1(q) \ . \end{aligned}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 15, pp. 297-313, November 2006

ELA
http://math.technion.ac.il/iic/ela



From the previous equation or equivalently by setting $b_1 = 0$ and $b_0 = -1$ in Eq.(2.4), we obtain

$$\begin{aligned} d_1 &= a_1 - c_1 \ , \\ d_0 &= a_0 - c_0 \ . \end{aligned} \tag{2.8}$$

Now, for second order polynomials, Eq.(2.6) reduces to

$$\begin{aligned} c_0\,|a_1 - c_1|^2 + |a_0 - c_0|^2 &= 0 \ , \\ c_1\,|a_1 - c_1|^2 + 2\,\mathrm{Re}[(\bar{a}_1 - c_1)(a_0 - c_0)] &= 0 \ . \end{aligned} \tag{2.9}$$

The solution of this system is *not* simple. In this special case, the choice of

$$a_1 = -j \quad \text{and} \quad a_0 = k - 1 \ ,$$

reduces the system (2.9) to

$$\begin{aligned} c_0\,(1 + c_1^2) + (1 + c_0)^2 + 1 &= 0 \ , \\ c_1\,(1 + c_1^2) + 2\,c_1\,(1 + c_0) &= 0 \ . \end{aligned} \tag{2.10}$$

The discussion can be then simplified by considering

$$c_1 \neq 0 \quad \text{and} \quad c_1 = 0 \ .$$

In the first case no real coupled solution exists. For $c_1 = 0$, we have

$$(\,c_0\,,\,c_1\,) = \left\{ \begin{array}{l} (-1, 0) \\ (-2, 0) \end{array} \right. \qquad \Rightarrow \qquad (\,d_0\,,\,d_1\,) = \left\{ \begin{array}{l} (k, -j) \\ (k+1, -j) \end{array} \right. \ .$$

Finally, by using Eq.(2.5) we conclude that the solutions of the quadratic equation (2.7) are given by

$$q_* = \left\{ \begin{array}{l} -\,i \\ -\,(i + j) \end{array} \right. \ . \tag{2.11}$$

By direct substitution, we can easily verify that $-i$ and $-(i + j)$ represent the zeros of our quadratic polynomial. Before concluding this brief review of the Niven algorithm and underlining that it surely is an important (old) technique for obtaining the solution of $n$-order unilateral quaternionic polynomial, we wish to emphasize that the system (2.9) and, in general, the system (2.4) are *not* very practical and their solutions could require laborious calculations.

**3. Companion matrix and SPV method.** Taking as our guide the standard complex matrix theory, we associate to the $n$-order unilateral quaternionic polynomial $A_n(q)$ the following companion matrix

$$\begin{pmatrix} a_{n-1} & a_{n-2} & \cdots & a_1 & a_0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \ . \tag{3.1}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 15, pp. 297-313, November 2006

ELA
http://math.technion.ac.il/iic/ela



In standard complex theory, the zeros of $n$-order polynomials can be determined by calculating the eigenvalues of the corresponding companion matrix. Any direct attempt based upon a *straightforward* expansion of the determinant $|M_n[\mathbb{H}] - \lambda\, I|$ is surely destined for failure due to the non-commutativity of the quaternionic algebra [11]. The quaternionic eigenvalue problem has recently been investigated by means of the complex representation of quaternionic matrices. It is clear that the rigorous presentation of the quaternionic eigenvalue problem would require a deep knowledge of quaternionic matrix theory. To facilitate the understanding of this paper, we do not discuss any theoretical aspect of the quaternionic eigenvalue problem here and aim to give a practical matrix solution method based on complex translation rules. We refer the reader to the articles cited in refs. [12, 13, 14] for a through discussion of quaternionic diagonalization and Jordan form. In order to appreciate some of the potentialities of the complex representation of quaternionic matrices, we shall consider in detail the case of second-order polynomials and show how we can immediately obtain the coefficients $c_0$ and $c_1$ by calculating the real part and the modulus of the eigenvalues associated to the complex matrix which represents the complex translation of the quaternionic companion matrix (SPV method [10]).

Before analyzing the companion matrix associated to a second-order unilateral quaternionic polynomial and calculate its eigenvalues, we give the main idea of the complex translation by using a generic $2 \times 2$ quaternionic matrix,

$$M_2[\mathbb{H}] = \begin{pmatrix} q_1 & q_2 \\ q_3 & q_4 \end{pmatrix} , \qquad q_{1,2,3,4} \in \mathbb{H} \, .$$

The complex translation would proceed along the following lines. A first possibility is the use of the symplectic form directly for the quaternionic matrix $M_2[\mathbb{H}]$,

$$M_2[\mathbb{H}] = \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} + j \begin{pmatrix} w_1 & w_2 \\ w_3 & w_4 \end{pmatrix} ,$$

i.e.,

$$M_2[\mathbb{H}] \;\rightarrow\; M_4[\mathbb{C}] = \begin{pmatrix} z_1 & z_2 & -\bar{w}_1 & -\bar{w}_2 \\ z_3 & z_4 & -\bar{w}_3 & -\bar{w}_4 \\ w_1 & w_2 & \bar{z}_1 & \bar{z}_2 \\ w_3 & w_4 & \bar{z}_3 & \bar{z}_4 \end{pmatrix} , \tag{3.2}$$

where $z_{1,2,3,4}$ and $w_{1,2,3,4} \in \mathbb{C}$. In this case the symplectic decomposition directly applies to the complex matrix blocks. Another possibility is represented by the use of the symplectic decomposition for the quaternionic elements of $M_2[\mathbb{H}]$,

$$M_2[\mathbb{H}] = \begin{pmatrix} z_1 + j\, w_1 & z_2 + j\, w_2 \\ z_3 + j\, w_3 & z_4 + j\, w_4 \end{pmatrix} ,$$

i.e.,

$$M_2[\mathbb{H}] \;\rightarrow\; \widetilde{M}_4[\mathbb{C}] = \begin{pmatrix} z_1 & -\bar{w}_1 & z_2 & -\bar{w}_2 \\ w_1 & \bar{z}_1 & w_2 & \bar{z}_2 \\ z_3 & -\bar{w}_3 & z_4 & -\bar{w}_4 \\ w_3 & \bar{z}_3 & w_4 & \bar{z}_4 \end{pmatrix} . \tag{3.3}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 15, pp. 297-313, November 2006

ELA
http://math.technion.ac.il/iic/ela



For the problem of finding the zeros of quaternionic polynomial, the complex representations (3.2) and (3.3) represent equivalent choices. In fact, there exist a permutation matrix

$$P_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

which converts $M_4[\mathbb{C}]$ to $\widetilde{M}_4[\mathbb{C}]$ by using a simple similarity transformation,

$$\widetilde{M}_4[\mathbb{C}] = P_4\, M_4[\mathbb{C}]\, P_4 \ .$$

In Appendix A, we extend this similarity for the matrices $M_{2n}[\mathbb{C}]$ and $\widetilde{M}_{2n}[\mathbb{C}]$ (complex counterparts of the $n$-dimension quaternionic matrix $M_n[\mathbb{H}]$),

$$\widetilde{M}_{2n}[\mathbb{C}] = P_{2n}\, M_{2n}[\mathbb{C}]\, P_{2n} \ ,$$

by explicitly giving the permutation matrix $P_{2n}$. Such a matrix will be obtained by working with the complex representations of the quaternionic column vectors associated to the matrix representations (3.2) and (3.3). Due to this equivalence, in all the explicit examples given in this paper, we shall use, without loss of generality, the complex matrix representation $M_{2n}[\mathbb{C}]$.

• **Second order quaternionic polynomials**. Let us consider the unilateral quaternionic polynomial discussed in the previous section, Eq. (2.7). In this case, the quaternionic companion matrix is

$$M_2[\mathbb{H}] = \begin{pmatrix} -j & k-1 \\ 1 & 0 \end{pmatrix} \ . \tag{3.4}$$

The SPV method allows us to obtain the real coefficients $c_0$ and $c_1$ by calculating the eigenvalues of the complex matrix $M_4[\mathbb{C}]$,

$$\{\, \lambda_1 \,,\, \lambda_2 \,,\, \bar{\lambda}_1 \,,\, \bar{\lambda}_2 \,,\} \ ,$$

which, due to the particular structure of the complex symplectic translation, always appear in conjugate pairs [12]. By using the complex translation rules (3.2), we immediately obtain the following complex counterpart of $M_2[\mathbb{H}]$,

$$M_4[\mathbb{C}] = \begin{pmatrix} 0 & -1 & 1 & -i \\ 1 & 0 & 0 & 0 \\ -1 & -i & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \ , \tag{3.5}$$

whose eigenvalues are easily calculated by using $|M_4[\mathbb{C}] - \lambda\, I_4| = 0$. As observed above, the eigenvalues appear in conjugate pairs,

$$\{\, \lambda_1 = i \,,\, \lambda_2 = i\,\sqrt{2} \ ,\ \bar{\lambda}_1 = -\,i \,,\, \bar{\lambda}_2 = -\,i\,\sqrt{2}\,\} \ .$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 15, pp. 297-313, November 2006

ELA
http://math.technion.ac.il/iic/ela



The real coefficients $c_0$ and $c_1$,

$$(\, c_0\,,\, c_1\,) = \left\{ \begin{array}{lcl} (\, -|\lambda_1|^2\,,\, 2\,\mathrm{Re}[\lambda_1]\,) & = & (-1, 0) \\ (\, -|\lambda_2|^2\,,\, 2\,\mathrm{Re}[\lambda_2]\,) & = & (-2, 0) \end{array} \right. ,$$

can now be obtained without the need of solving the system (2.10). This method proposed by Serôdio, Pereira and Vitoria [10] avoids the solution of the complicated coupled system (2.6) and represents a simpler way to find the real coefficients $c_0$ and $c_1$. Consequently, it improves the Niven algorithm. Note that step 1 in the Niven algorithm has *yet* to be used for finding the quaternionic coefficients $d_0$ and $d_1$. In the next section, we shall prove that by calculating the eigenvectors of the complex representation of the quaternion companion matrix $M_n[\mathbb{H}]$, we can *directly* obtain the zeros of $n$-order unilateral quaternionic polynomials.

**4. Finding the zeros by using the eigenvectors method.** In complex theory, the zeros of the $n$-order polynomial, $A_n(z) = 0$, are determined by calculating the eigenvalues of the corresponding companion matrix, $M_n[\mathbb{C}]$, i.e. $|M_n[\mathbb{C}] - \lambda\, I| = 0$. For quaternions, this method cannot be carried through successfully. In fact, the eigenvalues of the quaternionic companion matrix are *not* unequivocally determined. Consequently, we have to pursue a different track. Before demonstrating two theorems, which will guide us to the solution of our quaternionic mathematical challenge, we briefly recall the main results concerning the (right) eigenvalue problem for quaternionic matrices. Due to the non-commutativity of the quaternionic algebra, the eigenvalues of

$$M_n[\mathbb{H}] = \underbrace{\begin{pmatrix} a_{n-1}^{(1,i)} & a_{n-2}^{(1,i)} & \cdots & a_1^{(1,i)} & a_0^{(1,i)} \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}}_{Z_n[\mathbb{C}]} + j \underbrace{\left[ -j \begin{pmatrix} a_{n-1}^{(j,k)} & a_{n-2}^{(j,k)} & \cdots & a_1^{(j,k)} & a_0^{(j,k)} \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \right]}_{W_n[\mathbb{C}]} \tag{4.1}$$

belong to the following class of equivalence

$$\{\, u_1\lambda_1\bar{u}_1\,,\, u_2\lambda_2\bar{u}_2\,,\, \ldots\,,\, u_n\lambda_n\bar{u}_n\,\}\,, \tag{4.2}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 15, pp. 297-313, November 2006

ELA
http://math.technion.ac.il/iic/ela



where (by using the standard definition adopted in literature) $\lambda_n$ are the eigenvalues of

$$M_{2n}[\mathbb{C}] = \begin{pmatrix} Z_n[\mathbb{C}] & -W_n^*[\mathbb{C}] \\ W_n[\mathbb{C}] & Z_n^*[\mathbb{C}] \end{pmatrix} \tag{4.3}$$

with $\mathrm{Im}[\lambda_n] \geq 0$, and $u_n$ are quaternionic unitary similarity transformations. The starting point in finding the zeros of $n$-order quaternionic unilateral polynomials is the same as in complex theory; i.e., the solution of the eigenvalue problem for the companion matrix,

$$M_n[\mathbb{H}]\,\Phi = \Phi\,\lambda\ , \qquad \lambda \in \mathbb{C}\ ,\ \ \mathrm{Im}[\lambda] \geq 0\ , \tag{4.4}$$

and

$$\Phi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_n \end{pmatrix}\ , \qquad \varphi_{1,2,\ldots,n} \in \mathbb{H}\ .$$

Nevertheless, once the complex eigenvalues have been calculated we also get infinite *similar* quaternionic eigenvalues

$$M_n[\mathbb{H}]\,\Psi = \Psi\,u\lambda\bar{u}\ , \tag{4.5}$$

where $\Psi = \Phi\,\bar{u}$, $u \in \mathbb{H}$, and $u\bar{u} = 1$. Geometrically, the quaternionic eigenvalue $u\lambda\bar{u}$ represents, for the imaginary part of $\lambda$, the following three-dimensional rotation:

$$\begin{array}{c} \lambda \\ \downarrow \\ (\mathrm{Im}[\lambda], 0, 0) \\ \swarrow\ \searrow \\ \theta = 2\,\arctan\left[\dfrac{|(\mathrm{Re}[-iu], \mathrm{Re}[-ju], \mathrm{Re}[-ku])|}{u_0}\right]\ , \quad \text{axis} : (\mathrm{Re}[-iu], \mathrm{Re}[-ju], \mathrm{Re}[-ku]) \\ \searrow\ \swarrow \\ (\mathrm{Re}[-iu\lambda\bar{u}], \mathrm{Re}[-ju\lambda\bar{u}], \mathrm{Re}[-ku\lambda\bar{u}]) \\ \downarrow \\ u\lambda\bar{u}\ . \end{array}$$

We demonstrate later that, for each complex eigenvalue, $\lambda$, it is possible to choose an appropriate similarity transformation $u^{(q)}\lambda\bar{u}^{(q)}$ which corresponds to one of the zeros of the $n$-order quaternionic unilateral polynomial $A_n(q)$. This breaks down the symmetry between the infinitely many equivalent directions in the eigenvalue quaternionic space, $u\lambda\bar{u}$. The complete set of the zeros of the $n$-order quaternionic unilateral polynomial $A_n(q)$ will be then given by

$$\{\, u_1^{(q_1)}\lambda_1\bar{u}_1^{(q_1)}\ ,\ u_2^{(q_2)}\lambda_2\bar{u}_2^{(q_2)}\ ,\ \ldots\ ,\ u_n^{(q_n)}\lambda_n\bar{u}_n^{(q_n)}\,\}\ . \tag{4.6}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 15, pp. 297-313, November 2006

ELA
http://math.technion.ac.il/iic/ela



Now, we proceed with the main objective of this section and prove that the eigenvalue spectrum (4.6) is connected with the zeros of $A_n(q)$. In doing that, the first step will be the proof of two theorems. Theorem 4.1 and Theorem 4.2 concern the (right) complex eigenvalue problem (4.4).

THEOREM 4.1. *The last component of the quaternionic eigenvector $\Phi$ is non null.*

*Proof.* Due to the special form of the companion matrix, from Eq.(4.4) we get the following equalities:

$$\begin{aligned} a_{n-1}\,\varphi_1 + a_{n-2}\,\varphi_2 + \cdots + a_1\,\varphi_{n-1} + a_0\,\varphi_n &= \varphi_1\,\lambda\ , \\ \varphi_1 &= \varphi_2\,\lambda\ , \\ &\vdots \\ \varphi_{n-1} &= \varphi_n\,\lambda\ . \end{aligned} \tag{4.7}$$

Thus, $\varphi_n = 0$ implies $\varphi_{1,2,\ldots,n-1} = 0$. This means $\varphi_n = 0 \ \Rightarrow\ \Phi = 0$ □

THEOREM 4.2. *Let $\Phi$ be an eigenvector of the companion matrix $M_n[\mathbb{H}]$ associated with the polynomial $A_n(q)$. The solution $q$, expressed in terms of the two last components of $\Phi$, is given by $\varphi_{n-1}\,\varphi_n^{-1}$.*

*Proof.* Combining the last $n-1$ equalities in (4.7), we obtain

$$\begin{aligned} \varphi_{n-1} &= \varphi_n\,\lambda\ , \\ \varphi_{n-2} &= \varphi_{n-1}\,\lambda = \varphi_n\,\lambda^2\ , \\ &\vdots \\ \varphi_1 &= \varphi_2\,\lambda = \ldots = \varphi_n\,\lambda^{n-1}\ . \end{aligned}$$

This implies

$$\Phi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_n \end{pmatrix} = \varphi_n \begin{pmatrix} \lambda^{n-1} \\ \lambda^{n-1} \\ \vdots \\ 1 \end{pmatrix}\ .$$

The first equation of system (4.7) can be then re-written as follows

$$a_{n-1}\,\varphi_n\,\lambda^{n-1} + a_{n-2}\,\varphi_n\,\lambda^{n-2} + \cdots + a_1\,\varphi_n\,\lambda + a_0\,\varphi_n = \varphi_n\,\lambda^n\ .$$

Theorem 4.1 guarantees that $\varphi_n \neq 0$. We can thus multiply (from the right) the previous equation by $\varphi_n^{-1}$ obtaining

$$a_{n-1}\,\varphi_n\,\lambda^{n-1}\,\varphi_n^{-1} + a_{n-2}\,\varphi_n\,\lambda^{n-2}\,\varphi_n^{-1} + \cdots + a_1\,\varphi_n\,\lambda\,\varphi_n^{-1} + a_0 = \varphi_n\,\lambda^n\,\varphi_n^{-1}\ ,$$

or equivalently

$$a_{n-1}\,(\,\varphi_n\,\lambda\,\varphi_n^{-1}\,)^{n-1} + a_{n-2}\,(\,\varphi_n\,\lambda\,\varphi_n^{-1}\,)^{n-2} + \cdots + a_1\,\varphi_n\,\lambda\,\varphi_n^{-1} + a_0 = (\,\varphi_n\,\lambda\,\varphi_n^{-1}\,)^n\ .$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 15, pp. 297-313, November 2006

ELA
http://math.technion.ac.il/iic/ela



Consequently, $q = \varphi_n \, \lambda \, \varphi_n^{-1}$ and by using that $\varphi_{n-1} = \varphi_n \lambda$

$$q = \varphi_{n-1} \, \varphi_n^{-1} \ . \tag{4.8}$$

This completes the proof. □

The results of Theorem 4.1 and Theorem 4.2 can be easily extended to the (right) quaternionic eigenvalue problem (4.5). In fact,

$$\text{“}\, \Phi \neq 0 \quad \Rightarrow \quad \varphi_n \neq 0 \,\text{”} \quad \Leftrightarrow \quad \text{“}\, \Phi \, \bar{u} \, (= \Psi) \neq 0 \quad \Rightarrow \quad \varphi_n \, \bar{u} \, (= \psi_n) \neq 0 \,\text{”}$$

and

$$q = \varphi_{n-1} \, \varphi_n^{-1} \quad \Rightarrow \quad q = \varphi_{n-1} \, \bar{u} \, (\varphi_n \, \bar{u})^{-1} = \varphi_{n-1} \, \bar{u} \, u \, \varphi_n^{-1} = \psi_{n-1} \, \psi_n^{-1} \ .$$

We conclude this subsection by going back to the discussion on the quaternionic similarity transformation which breaks down the symmetry between the infinitely many equivalent directions in the eigenvalue quaternionic space. By using Theorem 4.2, we can write the zero of the quaternionic polynomial $A_n(q)$ in terms of the last component of the eigenvector $\Phi$ and of the complex eigenvalue $\lambda$,

$$q = \varphi_n \, \lambda \, \varphi_n^{-1} \ . \tag{4.9}$$

This allows us to define (for the companion matrix) a *privileged* (right) quaternionic eigenvalue problem, i.e.

$$\begin{aligned} M_n[\mathbb{H}] \, \Psi^{(q)} &= \Psi^{(q)} \, u^{(q)} \, \lambda \, \bar{u}^{(q)} \qquad (u^{(q)} = \varphi_n / |\varphi_n| \,) \\ &= \Psi^{(q)} \, q \ , \end{aligned} \tag{4.10}$$

with

$$\Psi^{(q)} = \Phi \, \bar{u}^{(q)} = \Phi \, \varphi_n^{-1} \, |\varphi_n| = \begin{pmatrix} \varphi_1 \varphi_n^{-1} \\ \varphi_2 \varphi_n^{-1} \\ \vdots \\ \varphi_{n-1} \varphi_n^{-1} \\ 1 \end{pmatrix} |\varphi_n| \ .$$

Recall that

$$q = \varphi_{n-1} \, \varphi_n^{-1} \qquad \text{and} \qquad \lambda = \varphi_n^{-1} \, \varphi_{n-1} \ .$$

For the complex case, commutativity guarantees $q = \lambda$. The problem of finding the zeros of $n$-order unilateral quaternionic polynomials is enormously simplified by using the results obtained in this section. This will be explicitly seen in the next subsections, where the solutions of second and third order quaternionic unilateral equations are calculated using complex matrix translation. For the convenience of the readers, we illustrate the steps of our method in the following sequence:

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 15, pp. 297-313, November 2006

ELA
http://math.technion.ac.il/iic/ela



$$
\begin{array}{cl}
A_n(q) & \text{(unilateral quaternionic polynomial)} , \\
\Downarrow & \\
M_n[\mathbb{H}] & \text{(quaternionic companion matrix)} , \\
\Downarrow & \\
M_{2n}[\mathbb{C}] & \text{(complex translated matrix)} , \\
\Downarrow & \\
(\omega_1,\, \omega_2,\, ...,\, \omega_n,\, \sigma_1,\, \sigma_2,\, ...,\, \sigma_n)_\lambda & \text{(complex eigenvector of } M_{2n}[\mathbb{C}]) , \\
\Downarrow & \\
(\omega_1 + j\,\sigma_1,\, \omega_2 + j\,\sigma_2,\, ...,\, \omega_n + j\,\sigma_n)_\lambda & \text{(quaternionic eigenvector of } M_n[\mathbb{H}])) , \\
\Downarrow & \\
q = \varphi_{n-1}\,\varphi_n^{-1} & \text{(zero corresponding to } \lambda) , \\
\Downarrow & \\
\lambda = \varphi_n^{-1}\,\varphi_{n-1} & \text{(check of the solution)} .
\end{array}
$$

The importance of the *new* method is due, in great extent, to the possibility of using complex matrix analysis.

**4.1. Second order unilateral polynomials.** In section 2 and in section 3 we have solved (by means of the SPV algorithms) the following (unilateral) quaternionic quadratic equation

$$q^2 + j\,q + (1-k) = 0 \ .$$

We now calculate the zeros of $A_2(q)$ by using our method:

$$
\begin{array}{rcl}
A_2(q) & = & q^2 + j\,q + (1-k) \ , \\
\Downarrow & & \\
M_2[\mathbb{H}] & = & \begin{pmatrix} -j & k-1 \\ 1 & 0 \end{pmatrix} \ , \\
\Downarrow & & \\
M_4[\mathbb{C}] & = & \begin{pmatrix} 0 & -1 & 1 & -i \\ 1 & 0 & 0 & 0 \\ -1 & -i & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \ , \\
\Downarrow & & \\
\begin{pmatrix} \omega_1 \\ \omega_2 \\ \sigma_1 \\ \sigma_2 \end{pmatrix} & = & \left\{ \begin{pmatrix} 0 \\ 0 \\ i \\ 1 \end{pmatrix}_i , \begin{pmatrix} \sqrt{2}\,(\sqrt{2}-1) \\ -i\,(\sqrt{2}-1) \\ i\,\sqrt{2} \\ 1 \end{pmatrix}_{i\sqrt{2}} \right\} \ , \\
\Downarrow & & \\
\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} \omega_1 + j\,\sigma_1 \\ \omega_2 + j\,\sigma_2 \end{pmatrix} & = & \left\{ \begin{pmatrix} -k \\ j \end{pmatrix}_i , \begin{pmatrix} \sqrt{2}\,(\sqrt{2}-1-k) \\ j - i\,(\sqrt{2}-1) \end{pmatrix}_{i\sqrt{2}} \right\} \ , \\
\Downarrow & & \\
q = \varphi_1\,\varphi_2^{-1} & = & \{ -i\,,\, -(i+j) \} \quad \left[ \Rightarrow \quad \lambda = \varphi_2^{-1}\,\varphi_1 = \{ i\,,\, i\,\sqrt{2} \} \right] \ .
\end{array}
$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 15, pp. 297-313, November 2006

ELA
http://math.technion.ac.il/iic/ela



**4.2. Third order unilateral polynomials.** Let us now give an example of solution for a third order quaternionic polynomial.

$$
\begin{array}{rcl}
A_3(q) & = & q^3 + k\,q^2 + i\,q - j\ , \\
& \Downarrow & \\
M_3[\mathbb{H}] & = & \begin{pmatrix} -k & -i & j \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\ , \\
& \Downarrow & \\
M_6[\mathbb{C}] & = & \begin{pmatrix} 0 & -i & 0 & i & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ i & 0 & 1 & 0 & i & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}\ , \\
& \Downarrow & \\
\begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix} & = & \left\{ \begin{pmatrix} -1 \\ i \\ 1 \\ -1 \\ i \\ 1 \end{pmatrix}_{i}\ , \begin{pmatrix} i-1 \\ i\sqrt{2} \\ i+1 \\ i\sqrt{2} \\ i-1 \\ \sqrt{2} \end{pmatrix}_{\frac{1+i}{\sqrt{2}}}\ , \begin{pmatrix} 1-i \\ -\sqrt{2} \\ i+1 \\ -i\sqrt{2} \\ i-1 \\ \sqrt{2} \end{pmatrix}_{\frac{i-1}{\sqrt{2}}} \right\}\ , \\
& \Downarrow & \\
\begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix} & = & \left\{ \begin{pmatrix} -1-j \\ i-k \\ 1+j \end{pmatrix}_{i}\ , \begin{pmatrix} i-1-k\sqrt{2} \\ i\sqrt{2}+j-k \\ 1+i+j\sqrt{2} \end{pmatrix}_{\frac{1+i}{\sqrt{2}}}\ , \begin{pmatrix} 1-i+k\sqrt{2} \\ -\sqrt{2}-j-k \\ 1+i+j\sqrt{2} \end{pmatrix}_{\frac{i-1}{\sqrt{2}}} \right\}\ , \\
& \Downarrow & \\
q = \varphi_2\,\varphi_3^{-1} & = & \left\{ -k\,,\ \frac{j-k+\sqrt{2}}{2}\,,\ \frac{j-k-\sqrt{2}}{2} \right\} \quad \left[ \Rightarrow \quad \lambda = \varphi_3^{-1}\varphi_2 = \left\{ i\,,\ \frac{1+i}{\sqrt{2}}\,,\ \frac{i-1}{\sqrt{2}} \right\} \right]\ .
\end{array}
$$

**5. Bilateral quadratic equations.** So far, we have not analyzed the possibility to have left and right acting coefficients in the quaternionic polynomials. This topic exceeds the scope of this paper. At the moment it represents an open question and satisfactory resolution methods have to be further investigated. Nevertheless, an interesting generalization of the matrix approach introduced in the previous section can be found in discussing bilateral equations [15], i.e.,

$$p^2 - \alpha_1\,p + p\,\beta_1 - \alpha_0 = 0\ . \tag{5.1}$$

For second order bilateral quaternionic polynomial, we introduced the following *generalized* companion matrix

$$\mathcal{M}_2[\mathbb{H}] = \begin{pmatrix} \alpha_1 & \alpha_0 \\ 1 & \beta_1 \end{pmatrix}\ , \tag{5.2}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 15, pp. 297-313, November 2006

ELA
http://math.technion.ac.il/iic/ela



which reduces to the standard one for $\beta_1 = 0$. The (right) complex eigenvalue problem for this matrix,

$$\mathcal{M}_2[\mathbb{H}]\,\Phi = \Phi\,\lambda\ , \tag{5.3}$$

is equivalent to

$$\begin{aligned} \alpha_1\,\varphi_1 + \alpha_0\,\varphi_2 &= \varphi_1\,\lambda\ , \\ \varphi_1 + \beta_1\,\varphi_2 &= \varphi_2\,\lambda\ , \end{aligned}$$

where $\varphi_{1,2}$ are the components of the eigenvector $\Phi$. It is immediate to show that $\varphi_2 = 0$ implies $\Phi = 0$. Consequently, the second component of the (non trivial) eigenvector $\Phi$ is always different from zero. Multiplying the first of the previous equations by $\varphi_2^{-1}$ (from the right), we obtain

$$\alpha_1\,\varphi_1\,\varphi_2^{-1} + \alpha_0 = \varphi_1\,\varphi_2^{-1}\,\varphi_2\,\lambda\,\varphi_2^{-1}\ .$$

From the second equation, we get

$$\varphi_2\,\lambda\,\varphi_2^{-1} = \varphi_1\,\varphi_2^{-1} + \beta_1\ .$$

Finally,

$$\alpha_1\,\varphi_1\,\varphi_2^{-1} + \alpha_0 = \left(\varphi_1\,\varphi_2^{-1}\right)^2 + \varphi_1\,\varphi_2^{-1}\,\beta_1$$

which, once compared with Eq.(5.1), implies

$$p = \varphi_1\,\varphi_2^{-1}\ . \tag{5.4}$$

The solution for the bilateral quadratic equation (5.1) is then *formally* equal to the solution given for the unilateral one. The complex eigenvalue $\lambda$ can be expressed in terms of the $\Phi$ components and of the right acting coefficient $\beta_1$ by

$$\lambda = \varphi_2^{-1}\,\varphi_1 + \varphi_2^{-1}\,\beta_1\,\varphi_2\ . \tag{5.5}$$

It is worth pointing that the bilateral equation (5.1) can be reduced to the following *equivalent* unilateral equation

$$q^2 - a_1\,q - a_0 = 0\ , \tag{5.6}$$

where $a_1 = \alpha_1 + \beta_1$ and $a_0 = \alpha_0 - \alpha_1\,\beta_1$ by using

$$p = q - \beta_1\ . \tag{5.7}$$

The next examples should emphasize one more time the advantage of using of the matrix approach proposed in this paper to solve unilateral (and some particular bilateral) quaternionic equations. It would be desirable to extend this approach to more general polynomials (with left and right acting quaternionic coefficients) but we have not been able to do this and from our point of view this should deserve more attention and further investigation. The question of solving general quaternionic polynomials is at present far from being solved.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 15, pp. 297-313, November 2006

ELA
http://math.technion.ac.il/iic/ela



**5.1. Second order bilateral polynomials.** Let us consider the bilateral quadratic equation

$$p^2 - i\,p + p\,j - k = 0\ . \tag{5.8}$$

To solve this equation, we shall follow the same steps of the resolution method presented in the previous section:

$$\begin{array}{rcl}
\mathcal{M}_2[\mathbb{H}] & = & \begin{pmatrix} i & k \\ 1 & j \end{pmatrix}\ , \\
& \Downarrow & \\
\mathcal{M}_4[\mathbb{C}] & = & \begin{pmatrix} i & 0 & 0 & -i \\ 1 & 0 & 0 & -1 \\ 0 & -i & -i & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}\ , \\
& \Downarrow & \\
\begin{pmatrix} \omega_1 \\ \omega_2 \\ \sigma_1 \\ \sigma_2 \end{pmatrix} & = & \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}_0 , \begin{pmatrix} -1-\sqrt{2} \\ i+i\sqrt{2} \\ -i \\ 1 \end{pmatrix}_{i\sqrt{2}} \right\}\ , \\
& \Downarrow & \\
\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} & = & \left\{ \begin{pmatrix} 1 \\ j \end{pmatrix}_0 , \begin{pmatrix} k-1-\sqrt{2} \\ i+i\sqrt{2}+j \end{pmatrix}_{i\sqrt{2}} \right\}\ , \\
& \Downarrow & \\
p = \varphi_1\,\varphi_2^{-1} & = & \{-j\,,\,i\}\ \ \left[\Rightarrow\ \ \lambda = \varphi_2^{-1}\,\varphi_1 + \varphi_2^{-1}\,\beta_1\,\varphi_2 = \{\,0\,,\,i\,\sqrt{2}\,\}\right]\ .
\end{array}$$

**5.2. Equivalent second order unilateral polynomials.** In this subsection, we solve the unilateral quadratic equation obtained from Eq.(5.8) by using $p = q - j$:

$$\begin{array}{rcl}
M_2[\mathbb{H}] & = & \begin{pmatrix} i+j & 0 \\ 1 & 0 \end{pmatrix}\ , \\
& \Downarrow & \\
M_4[\mathbb{C}] & = & \begin{pmatrix} i & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & -i & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}\ , \\
& \Downarrow & \\
\begin{pmatrix} \omega_1 \\ \omega_2 \\ \sigma_1 \\ \sigma_2 \end{pmatrix} & = & \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}_0 , \begin{pmatrix} -\sqrt{2}\,(1+\sqrt{2}) \\ i\,(1+\sqrt{2}) \\ i\,\sqrt{2} \\ 1 \end{pmatrix}_{i\sqrt{2}} \right\}\ , \\
& \Downarrow & \\
\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} \omega_1 + j\,\sigma_1 \\ \omega_2 + j\,\sigma_2 \end{pmatrix} & = & \left\{ \begin{pmatrix} 0 \\ 1+j \end{pmatrix}_i , \begin{pmatrix} -\sqrt{2}\,(1+\sqrt{2}+k) \\ i\,(1+\sqrt{2})+j \end{pmatrix}_{i\sqrt{2}} \right\}\ , \\
& \Downarrow & \\
q = \varphi_1\,\varphi_2^{-1} & = & \{\,0\,,\,i+j\,\}\ \ \left[\Rightarrow\ \ \lambda = \varphi_2^{-1}\,\varphi_1 = \{\,0\,,\,i\,\sqrt{2}\,\}\right]\ .
\end{array}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 15, pp. 297-313, November 2006

ELA
http://math.technion.ac.il/iic/ela



As expected, by recalling that

$$p = q - \beta_1 = q - j = \{ -j\, ,\, i \}$$

we recover the solution of Eq.(5.8) obtained in the previous subsection.

**Appendix. Equivalence between translation rules.** Let

$$\begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_n \end{pmatrix} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_n \end{pmatrix} + j \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \vdots \\ \sigma_n \end{pmatrix} \rightarrow \begin{pmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_n \\ \sigma_1 \\ \sigma_2 \\ \vdots \\ \sigma_n \end{pmatrix}$$

and

$$\begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_n \end{pmatrix} = \begin{pmatrix} \omega_1 + j\,\sigma_1 \\ \omega_2 + j\,\sigma_2 \\ \vdots \\ \omega_n + j\,\sigma_n \end{pmatrix} \rightarrow \begin{pmatrix} \omega_1 \\ \sigma_1 \\ \omega_2 \\ \sigma_2 \\ \vdots \\ \vdots \\ \omega_n \\ \sigma_n \end{pmatrix}$$

be the complex vectors obtained by using the complex translation rules given in Eqs.(3.2) and (3.3). A simple algebraic calculation shows that

$$\begin{pmatrix} \omega_1 \\ \sigma_1 \\ \omega_2 \\ \sigma_2 \\ \vdots \\ \vdots \\ \omega_n \\ \sigma_n \end{pmatrix} = P_{2n} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_n \\ \sigma_1 \\ \sigma_2 \\ \vdots \\ \sigma_n \end{pmatrix} ,$$

where

$$P_{2n\,,\,rs} = \begin{cases} 1 & (r,s) = (1,1)\,,\ (2,n+1)\,,\ \ldots\,,\ (2n-1,n)\,,\ (2n,2n)\ , \\ 0 & \text{otherwise}\ . \end{cases}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 15, pp. 297-313, November 2006

ELA
http://math.technion.ac.il/iic/ela



The foregoing result can be then used to prove the equivalence between the complex matrices $M_{2n}[\mathbb{C}]$ and $\widetilde{M}_{2n}[\mathbb{C}]$,

$$\widetilde{M}_{2n}[\mathbb{C}] = P_{2n}\, M_{2n}[\mathbb{C}]\, P_{2n} \ .$$

**Acknowledgments.** The authors wish to express their gratitude to Prof. Nir Cohen for several helpful comments and for many stimulating conversations. One of the author (S.D.L) greatly thanks the Department of Mathematics (Parana University), where the paper was prepared, for the invitation and hospitality. This is part of the research project financed by the Araucaria Foundation (V.L and G.D).

REFERENCES

[1] I. Niven. The roots of a quaternion. *American Mathematical Monthly*, 49:386–388, 1942.
[2] S. Eilenberg and I. Niven. The fundamental theorem of algebra for quaternions. *Bulletin of American Mathematical Society*, 50:246–248, 1944.
[3] S. De Leo and G. Ducati. Quaternionic differential operators. *Journal of Mathematical Physics*, 42:2236–2265, 2001.
[4] S. De Leo and G. Ducati. Solving simple quaternionic differential equation. *Journal of Mathematical Physics*, 43:2224–2233, 2003.
[5] S. L. Adler. *Quaternionic quantum mechanics and quantum fields*. Oxford University Press, New York, 1995.
[6] S. De Leo, G. Ducati and C. C. Nishi. Quaternionic potentials in non-elativistic quantum mechanics. *Journal of Physics A*, 35:5411–5426, 2002.
[7] S. De Leo and G. Ducati. Quaternionic bound states. *Journal of Physics A*, 38:3443–3454, 2005.
[8] S. De Leo and G. Ducati. Real linear quaternionic differential equations. *Computers and Mathematics with Application*, 48:1893–1903, 2004.
[9] I. Niven. Equations in quaternions. *American Mathematical Monthly*, 48:654–661, 1941.
[10] R. Serôdio, E. Pereira and J. Vitória. Computing the zeros of quaternion polynomials. *Computer and Mathematics with Applications*, 42:1229–1237, 2001.
[11] N. Cohen and S. De Leo. The quaternionic determinant. *Electronic Journal of Linear Algebra* 7:100-111, 2000.
[12] S. De Leo and G. Scolarici. Right eigenvalue equation in quaternionic quantum mechanics. *Journal of Physics A*, 33:2971–2995, 2000.
[13] S. De Leo, G. Scolarici, and L. Solombrino. Quaternionic eigenvalue problem. *Journal of Mathematical Physics*, 43:5812–2995, 2002.
[14] T. Jiang. Algebraic methods for diagonalization of a quaternion matrix in quaternion quantum theory. *Journal of Mathematical Physics*, 46:052106–8, 2005.
[15] R. Carvalho and S. De Leo. *Quaternions in Algebra and Analysis*. Technical report (Unicamp), Campinas, 2002.