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A NOTE ON NEWTON AND NEWTON–LIKE INEQUALITIES FOR M–MATRICES AND FOR DRAZIN INVERSES OF M–MATRICES*

MICHAEL NEUMANN† AND JIANHONG XU‡

Abstract. In a recent paper Holtz showed that M–matrices satisfy Newton's inequalities and so do the inverses of nonsingular M–matrices. Since nonsingular M–matrices and their inverses display various types of monotonic behavior, monotonicity properties adapted for Newton's inequalities are examined for nonsingular M–matrices and their inverses.

In the second part of the paper the problem of whether Drazin inverses of singular M–matrices satisfy Newton's inequalities is considered. In general the answer is no, but it is shown that they do satisfy a form of Newton–like inequalities.

In the final part of the paper the relationship between the satisfaction of Newton's inequality by a matrix and by its principal submatrices of order one less is examined, which leads to a condition for the failure of Newton's inequalities for the whole matrix.

Key words. M–matrices, Nonnegative matrices, Generalized inverses, Newton's inequalities.

AMS subject classifications. 15A09, 15A24, 15A42.

1. Introduction. In a recent paper, Holtz [5] showed that *Newton's inequalities* hold for the class of matrices consisting of the nonsingular M–matrices and their inverses.

Newton's inequalities involve the (*normalized elementary*) *symmetric functions* that are defined as follows.

DEFINITION 1.1. *For $x_1, \ldots, x_n \in \mathbb{C}$ and for $1 \le j \le n$, define the symmetric functions*

$$E_j(x_1, \ldots, x_n) \ := \ \frac{\displaystyle\sum_{1 \le i_1 < \ldots < i_j \le n} x_{i_1} \cdots x_{i_j}}{\displaystyle\binom{n}{j}} \tag{1.1}$$

and, by convention, $E_0(x_1, \ldots, x_n) = 1$.

The original Newton inequalities, which date back to 1707 [10], concern the case when the n numbers $x_1, \ldots, x_n$ in Definition 1.1 are nonnegative scalars. In this case the inequalities say that:

$$E_j^2(x_1, \ldots, x_n) \ \ge \ E_{j-1}(x_1, \ldots, x_n)E_{j+1}(x_1, \ldots, x_n), \quad j = 1, \ldots, n-1, \tag{1.2}$$

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†Department of Mathematics, University of Connecticut, Storrs, Connecticut 06269–3009, USA (neumann@math.uconn.edu).

‡Department of Mathematics, Southern Illinois University at Carbondale, Carbondale, Illinois 62901–4408, USA (jxu@math.siu.edu).

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again using the convention that $E_0(x_1, \ldots, x_n) = 1$. The inequalities were also obtained by Maclaurin [7]. Actually, as shown in [4], (1.2) continues to hold when $x_1, \ldots, x_n$ are real but not necessarily all nonnegative. More recently, Rosset [12] and Niculescu [11] studied generalized Newton's inequalities with higher order terms of $E_j(x_1, \ldots, x_n)$.

The problem in extending Newton's inequalities to the set of eigenvalues of $n \times n$ matrices is that even if the matrices are real, their eigenvalues may not be real. In the case of real matrices we know, of course, that their nonreal eigenvalues come in conjugate pairs.

Before we proceed, a word about notation. If $A \in \mathbb{R}^{n,n}$ has the eigenvalues $x_1, \ldots, x_n$, then we shall form functions E_j on these eigenvalues and put:

$$E_j(A) \; = \; E_j(x_1, \ldots, x_n), \quad j = 0, 1, \ldots, n, \tag{1.3}$$

and

$$E(A) \; = \; [E_0(A), \ldots, E_n(A)] \, .$$

Let $B = [b_{i,j}] \in \mathbb{R}^{n,n}$ be a nonnegative matrix whose Perron root is $r = \rho(B)$. The matrix $A = sI - B$ is called an M–matrix if $s \geq \rho(B)$. It is well known from the Perron–Frobenius, see, for example, Berman and Plemmons[1] [2], that if $s > \rho(B)$, then A is nonsingular. As reported above, recently Holtz [5] proved that the class of nonsingular M–matrices and their inverses both satisfy Newton's inequalities with respect to their eigenvalues. It was also pointed out by Holtz in [5] that, using a continuity argument, Newton's inequalities continue to hold on the closure of the set of nonsingular M–matrices, i.e. the class consisting of both nonsingular and singular M–matrices.

Nonsingular M–matrices and their inverses possess various types of monotonic behavior. For example, it is known that if A_1 and A_2 are $n \times n$ nonsingular M–matrices and $A_1 \leq A_2$ entrywise, then $A_2^{-1} \leq A_1^{-1}$, again entrywise. A further type of monotonic behavior is that $\det(A_1) \leq \det(A_2)$. Consider now a nonsingular M–matrix dependent on a parameter, namely, $A(s) = sI - B$, as s varies in the interval $(\rho(B), \infty)$. For a matrix $F \in \mathbb{R}^{n,n}$, we shall denote by

$$N_j(F) \; = \; E_j^2(F) - E_{j-1}(F)E_{j+1}(F), \quad j = 1, \ldots, n-1, \tag{1.4}$$

and by

$$N(F) \; = \; [N_1(F), \ldots, N_{n-1}(F)] \, .$$

These N_j shall be called *discriminants* for the matrix F. Returning to the one parameter family of nonsingular M–matrices, $A(s) = sI - B$, we show that for all

[1]Actually, background material for all the results on nonnegative matrices and M–matrices which we shall use in this paper can be found in this reference.

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$j = 1, \ldots, n-1$, $N_j(A(s))$ is monotonically increasing in the interval $(\rho(B), \infty)$, while for $C(s) := (A(s))^{-1}$, we shall show that $N_j(C(s))$ is a monotonically decreasing function in the interval.

It is well known that the nonsingular M–matrices and their inverse are (also) P–matrices, namely, *matrices whose principal submatrices all have a positive determinant.* It is further known, through continuity arguments, that the singular M–matrices are P_0–matrices, that is, matrices whose principal submatrices all have a nonnegative determinant. In [8] it was shown that the Moore–Penrose and Drazin inverses of singular M–matrices are P_0–matrices. This result led us to ask a similar question concerning whether generalized inverses of singular M–matrices (also) satisfy Newton's inequalities.

The answer to the above question turns out to be negative in general, as can be seen from the following example. Let

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{1.5}$$

Then the Drazin inverse [2] A^D of A is given by

$$A^D = \begin{bmatrix} 0.4 & 0.2 & 0 & -0.2 & -0.4 \\ -0.4 & 0.4 & 0.2 & 0 & -0.2 \\ -0.2 & -0.4 & 0.4 & 0.2 & 0 \\ 0 & -0.2 & -0.4 & 0.4 & 0.2 \\ 0.2 & 0 & -0.2 & -0.4 & 0.4 \end{bmatrix}. \tag{1.6}$$

A computation of the elementary symmetric functions of A^D yields that

$$E(A^D) = \begin{bmatrix} 1 & 0.4 & 0.2 & 0.1 & 0.04 & 0 \end{bmatrix}$$

For this example, we see that

$$E_1^2(A^D) - E_0(A^D)E_2(A^D) = 0.4^2 - 0.2 = -0.04 < 0.$$

This observation led us to consider the following question: **Does there exist a constant $0 < c \leq 1$ such that for all $n \geq 3$ and for all singular M–matrices $A \in \mathbb{R}^{n,n}$, the inequalities**

$$E_j^2(A^D) \geq cE_{j-1}(A^D)E_{j+1}(A^D), \quad 1 \leq j \leq n-1, \tag{1.7}$$

hold?

[2] For more background material on Drazin inverses and other generalized inverses of matrices see Ben–Israel and Greville [1] and Campbell and Meyer [3].

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We call inequalities in the form of (1.7) *Newton–like inequalities*. In Section 4 we show that Newton–like inequalities hold on Drazin inverses of M–matrices, regardless of the multiplicity of 0 as their eigenvalue.

Let $A \in \mathbb{R}^{n,n}$. For $i = 1, \ldots, n$, denote by A_i the $(n-1)\times(n-1)$ principal submatrix of A obtained by deleting its i–th row and column. In Section 5 we shall analyze the relation between the symmetric functions of A and the symmetric functions of $A_1, \ldots, A_n$. This will allow us to formulate a condition on $N_j(A_1), \ldots, N_j(A_n)$, for an arbitrary but fixed value of $1 \le j \le n-2$, which leads to the failure of the j–th Newton inequality for A, namely, $N_j(A) < 0$.

2. More Background and Initial Results on Newton–Like Inequalities. First of all, it should be pointed out that Monov [9] recently established Newton–like inequalities in the form

$$E_j^2(x_1, \ldots, x_n) \ge cE_{j-1}(x_1, \ldots, x_n)E_{j+1}(x_1, \ldots, x_n), \quad j = 1, \ldots, n-1, \tag{2.1}$$

for the case when $\Re(x_i) \ge 0$, for all i, and when none of the x_i's are pure imaginary. The main result in [9] states that the constant c can be chosen to be $\cos^2\varphi$, where $0 \le \varphi < \pi/2$ is an upper bound on $|\arg x_i|$ for all i. We shall illustrate here that the Newton–like inequalities arise naturally as we consider the spectra of a nonsingular matrix A and its inverse or as we augment the spectrum of A with zeros. These observations will be used in Section 4 to prove that for Drazin inverses of M–matrices, the constant c in Newton–like inequalities is in $(1/2, 1]$ no matter how close φ is to $\pi/2$.

Let $A \in \mathbb{R}^{n,n}$ and suppose that $\sigma(A) = \{x_1, \ldots, x_n\}$. Denote the *characteristic polynomial* of A by

$$p(x) \;=\; \det(xI - A) \;=\; (x - x_1)\cdots(x - x_n). \tag{2.2}$$

It is a basic fact in matrix theory, see Horn and Johnson[3] [6, p. 42], that the $E_j(A)$'s are real and that

$$p(x) \;=\; \sum_{j=0}^{n}(-1)^j\binom{n}{j}E_j(A)x^{n-j}. \tag{2.3}$$

It is well known that if $A \in \mathbb{R}^{n,n}$ is a P–matrix, particularly a nonsingular M–matrix, then $E_j(A) > 0$, for all $j = 0, \ldots, n$.

The following identity can be found in [12] and [11], but it is likely to be found in earlier literature:

Lemma 2.1. (Rossett [12, Eq. (4)]) *For the nonzero scalars $x_1, \ldots, x_n \in \mathbb{C}$, let $x_i' = 1/x_i$, $i = 1, \ldots, n$. Then:*

$$E_n(x_1, \ldots, x_n)E_{n-j}(x_1', \ldots, x_n') = E_j(x_1, \ldots, x_n), \quad 0 \le j \le n. \tag{2.4}$$

[3]Note that $E_j(A)$ in [6] is equivalent to $\binom{n}{j}E_j(A)$ here.

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Recall now that if $A \in \mathbb{R}^{n,n}$ is nonsingular, then $E_n(A) \neq 0$. We next develop the following two lemmas:

LEMMA 2.2. *Let $A \in \mathbb{R}^{n,n}$ be a nonsingular matrix and let $\{k_1, \ldots, k_{n-1}\}$ be a set of nonnegative constants. Then*

$$E_j^2(A) \ \geq \ k_j E_{j-1}(A) E_{j+1}(A), \quad 1 \leq j \leq n-1, \tag{2.5}$$

if and only if

$$E_j^2(A^{-1}) \ \geq \ k_{n-j} E_{j-1}(A^{-1}) E_{j+1}(A^{-1}), \quad 1 \leq j \leq n-1. \tag{2.6}$$

Proof. We know that if $\sigma(A) = \{x_1, \ldots, x_n\}$, then $\sigma(A^{-1}) = \{x_1', \ldots, x_n'\}$. To show the necessity, we apply (2.4) to each factor in (2.5) and obtain that

$$\begin{aligned} &E_n^2(A) E_{n-j}^2(A^{-1}) \\ &\qquad \geq \ k_j E_n^2(A) E_{n-j+1}(A^{-1}) E_{n-j-1}(A^{-1}), \end{aligned}$$

and so

$$E_{n-j}^2(A^{-1}) \ \geq \ k_j E_{n-j+1}(A^{-1}) E_{n-j-1}(A^{-1}).$$

This yields (2.6) upon replacing $n-j$ with j.

The proof of the sufficiency part can be done by observing the symmetry between (2.5) and (2.6). ∎

A simple consequence of Lemma 2.2 is that when A is a nonsingular matrix and satisfies Newton's inequalities, **which corresponds to the case when all the k_j's in Lemma 2.2 are equal to** 1, then its inverse A^{-1} also satisfies Newton's inequalities. This provides a further confirmation to a part of Holtz's [5] results that Newton's inequalities hold for nonsingular M–matrices if and only if they hold for inverses of nonsingular M–matrices.

We next present a modification of Lemma 2.2 which will allow us subsequently to examine the fulfillment of Newton's inequalities by certain generalized inverses of M–matrices.

LEMMA 2.3. *Let $B \in \mathbb{R}^{r,r}$ be a nonsingular matrix with spectrum $\sigma(B) = \{x_1, \ldots, x_r\}$ and suppose that there exists a set of constants $\{k_1, \ldots, k_{r-1}\}$ such that*

$$E_j^2(B) \ \geq \ k_j E_{j-1}(B) E_{j+1}(B), \quad 1 \leq j \leq r-1. \tag{2.7}$$

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For any $n > r$, set $x_{r+1} = \ldots = x_n = 0$. Then
(2.8)

$$E_j^2(x_1,\ldots,x_n) \;\geq\; k_j \frac{\binom{r}{j}^2\binom{n}{j-1}\binom{n}{j+1}}{\binom{n}{j}^2\binom{r}{j-1}\binom{r}{j+1}} E_{j-1}(x_1,\ldots,x_n)E_{j+1}(x_1,\ldots,x_n),$$

for $1 \leq j \leq r-1$. In particular, if $E_j(B) > 0$, for $j = 1,\ldots,r$, then

$$E_j^2(x_1,\ldots,x_n) \;>\; k_j E_{j-1}(x_1,\ldots,x_n)E_{j+1}(x_1,\ldots,x_n), \quad 1 \leq j \leq r-1. \tag{2.9}$$

Proof. To show (2.8), we first observe that for any $1 \leq j \leq r$,

$$\begin{aligned} E_j(x_1,\ldots,x_n) &= \frac{\displaystyle\sum_{1\leq i_1<\ldots<i_j\leq n} x_{i_1}\cdots x_{i_j}}{\binom{n}{j}} \\ &= \frac{\displaystyle\sum_{1\leq i_1<\ldots<i_j\leq r} x_{i_1}\cdots x_{i_j}}{\binom{r}{j}} \frac{\binom{r}{j}}{\binom{n}{j}} \qquad (2.10) \\ &= \frac{\binom{r}{j}}{\binom{n}{j}} E_j(B). \end{aligned}$$

Note that the above connection between $E_j(x_1,\ldots,x_n)$ and $E_j(B)$ continues to hold when $j = 0$.

Using (2.10) in (2.7) yields that

$$\frac{\binom{n}{j}^2}{\binom{r}{j}^2} E_j^2(x_1,\ldots,x_n) \;\geq\; k_j \frac{\binom{n}{j-1}}{\binom{r}{j-1}} E_{j-1}(x_1,\ldots,x_n) \cdot \frac{\binom{n}{j+1}}{\binom{r}{j+1}} E_{j+1}(x_1,\ldots,x_n),$$

from which we obtain that:

$$E_j^2(x_1,\ldots,x_n) \;\geq\; k_j \frac{\binom{r}{j}^2\binom{n}{j-1}\binom{n}{j+1}}{\binom{n}{j}^2\binom{r}{j-1}\binom{r}{j+1}} \cdot E_{j-1}(x_1,\ldots,x_n)E_{j+1}(x_1,\ldots,x_n).$$

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This completes the proof of (2.8).

It can be easily verified that

$$\frac{\binom{r}{j}^2\binom{n}{j-1}\binom{n}{j+1}}{\binom{n}{j}^2\binom{r}{j-1}\binom{r}{j+1}} \;=\; \frac{(n-j)(r-j+1)}{(r-j)(n-j+1)} \;>\; 1.$$

This, together with the fact that $E_j(x_1,\ldots,x_n) > 0$, for $1 \le j \le r$, implies (2.9) holds. ∎

3. Monotonicity of Newton's Inequalities for M–matrices and Their Inverses. In this section we consider the following questions: **Let** $A(s) = sI - B$**, with** $B \ge 0$ **and** $s \in (\rho(B), \infty)$**. Then how do** $N_j(A(s))$ **and** $N_j(((A(s))^{-1})$ **change when** s **changes?**

Consider $E_j(A(s))$, for $j = 0,\ldots,n$. Then it can be easily verified that

$$\frac{d^i}{ds^i}E_j(A(s)) \;=\; j(j-1)\cdots(j-i+1)E_{j-i}(A(s)) \;\ge\; 0, \quad \text{for all } i \le j, \tag{3.1}$$

and for all $0 \le j \le n$. We can now prove the first theorem of this section:

THEOREM 3.1. *Let $B \in \mathbb{R}^{n,n}$ be a nonnegative matrix and consider the one parameter family of nonsingular M–matrices $A(s) = sI - B$, with $s \in (\rho(B), \infty)$. For $1 \le j \le n-1$, let $N_j(A(s))$ be given by the substitution of $F = A(s)$ in (1.4). Then, for all $j = 1,\ldots,n-1$, and for all $s \in (\rho(B), \infty)$, $N_j'(A(s)) \ge 0$, that is, $N_j(A(s))$ is monotonically increasing in the interval.*

Proof. With $i = 1$, on applying (3.1) to the difference given in (1.4), we obtain that for each $1 \le j \le n-1$,

$$\begin{aligned} N_j'(A(s)) \;&=\; 2jE_j(A(s))E_{j-1}(A(s)) - (j-1)E_{j-2}(A(s))E_{j+1}(A(s)) \\ &\quad - (j+1)E_{j-1}(A(s))E_j(A(s)) \\ &=\; (j-1)\left(E_{j-1}(A(s))E_j(A(s)) - E_{j-2}(A(s))E_{j+1}(A(s))\right). \end{aligned}$$

Since

$$E_{j-2}(A(s))E_j(A(s)) \;\le\; E_{j-1}^2(A(s)) \;\text{ and }\; E_{j-1}(A(s))E_{j+1}(A(s)) \;\le\; E_j^2(A(s)),$$

with all the quantities in the display being nonnegative due to the $A(s)$'s being nonsingular M–matrices and so, in particular, P–matrices, we have that

$$E_{j-2}(A(s))E_{j+1}(A(s)) \;\le\; E_{j-1}(A(s))E_j(A(s)). \tag{3.2}$$

Thus $N_j'(A(s)) \ge 0$, for all $s \in (\rho(B), \infty)$, concluding the proof. ∎

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To illustrate the results of the theorem let

$$A = \begin{bmatrix} 2 & -1 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 & 0 \\ 0 & 0 & 3 & -1 & -1 \\ 0 & -1 & 0 & 5 & -1 \\ 0 & -1 & -1 & -1 & 3 \end{bmatrix}$$

and let $B = A + I$. Then on computing the discriminants for A and B we find that:

$$N(B) - N(A) = \begin{bmatrix} 0 & 4.96 & 154.52 & 3497.6 \end{bmatrix}.$$

However it should be pointed out that the conclusions of Theorem 3.1 fail if we merely have two nonsingular M–matrices in which one majorizes the other. As an example let

$$C = \begin{bmatrix} 1.9 & -1.1 & -1.1 & -1.1 & -.1 \\ -1.1 & 2.9 & -1.1 & -1.1 & -.1 \\ -.1 & -.1 & 2.9 & -1.1 & -1.1 \\ -.1 & -1.1 & -.1 & 4.9 & -1.1 \\ -.1 & -1.1 & -1.1 & -1.1 & 2.9 \end{bmatrix} \leq A.$$

Then, again after computing, we find that

$$N(C) - N(A) = \begin{bmatrix} 0.14000 & 1.9029 & 25.521 & 342.1824 \end{bmatrix}.$$

In contrast to the monotonically increasing property of the discriminants for the nonsingular M–matrices $A(s) = sI - B$, the discriminants for $(A(s))^{-1}$ display a monotonic decreasing behavior as shown in the following theorem:

THEOREM 3.2. *Let $B \in \mathbb{R}^{n,n}$ be a nonnegative matrix and consider the one parameter family of nonsingular M–matrices $A(s) = sI - B$, with $s \in (\rho(B), \infty)$. Denote by $A^{-1}(s)$ the inverse of $A(s)$. Then, for all $j = 1, \dots, n-1$, and for all $s \in (\rho(B), \infty)$, $N'_j(A^{-1}(s)) \leq 0$, that is, $N_j(A^{-1}(s))$ is monotonically decreasing in the interval.*

Proof. Due to the length of some of the expressions in the proof, for $j = 0, \dots, n$, we shall set $\widetilde{E}_j := E_j(A^{-1}(s))$ while reserving E_j for $E_j(A(s))$.

By the identity given in (2.4), we can write that for $s > \rho(B)$,

$$\frac{d}{ds}\left[\widetilde{E}_j^2 - \widetilde{E}_{j-1}\widetilde{E}_{j+1}\right]$$
$$= \frac{d}{ds}\left[\frac{E_{n-j}^2 - E_{n-j-1}E_{n-j+1}}{E_n^2}\right].$$

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Now put $k = n - j$. Then, by (3.1), we know that $\frac{d}{ds}E_k = kE_{k-1}$ and so

$$
\begin{aligned}
&\frac{d}{ds}\left[\frac{E_k^2 - E_{k-1}E_{k+1}}{E_n^2}\right] \\
&\quad = \frac{(k-1)E_n\left(E_{k-1}E_k - E_{k-2}E_{k+1}\right) - 2nE_{n-1}\left(E_k^2 - E_{k-1}E_{k+1}\right)}{E_n^3} \\
&\quad = \frac{(k-1)\dfrac{E_n}{E_{n-1}}\left(E_{k-1}E_k - E_{k-2}E_{k+1}\right) - 2n\left(E_k^2 - E_{k-1}E_{k+1}\right)}{\dfrac{E_n^3}{E_{n-1}}}.
\end{aligned}
$$

Since $E_{k-1}E_k - E_{k-2}E_{k+1} \geq 0$ as shown in (3.2) and since directly from Newton's inequalities we can write that

$$
0 \;<\; E_n/E_{n-1} \;\leq\; E_{n-1}/E_{n-2} \;\leq \ldots \leq\; E_{k-1}/E_{k-2} \;\leq\; E_k/E_{k-1},
$$

we have that

$$
\begin{aligned}
&(k-1)\frac{E_n}{E_{n-1}}\left(E_{k-1}E_k - E_{k-2}E_{k+1}\right) - 2n\left(E_k^2 - E_{k-1}E_{k+1}\right) \\
&\leq\; (k-1)\frac{E_k}{E_{k-1}}\left(E_{k-1}E_k - E_{k-2}E_{k+1}\right) - 2n\left(E_k^2 - E_{k-1}E_{k+1}\right) \\
&=\; (k-1-2n)E_k^2 - (k-1)\frac{E_kE_{k-2}E_{k+1}}{E_{k-1}} + 2nE_{k-1}E_{k+1} \\
&\leq\; (k-1-2n)E_k^2 - (k-1)\frac{E_{k-1}E_{k-2}E_{k+1}}{E_{k-2}} + 2nE_{k-1}E_{k+1} \\
&\leq\; (k-1-2n)\left(E_k^2 - E_{k-1}E_{k+1}\right) \;\leq\; 0.
\end{aligned}
$$

Thus

$$
\frac{d}{ds}\left[\widetilde{E}_j^2 - \widetilde{E}_{j-1}\widetilde{E}_{j+1}\right] \;\leq\; 0
$$

which shows that $N_j(A^{-1}(s))$ decreases in the interval $(\rho(B), \infty)$. $\square$

4. Newton–Like Inequalities for Drazin Inverses of M–matrices. In this section we provide certain extended, Newton–like inequalities for the Drazin inverse of a singular M–matrix. We begin with the following result:

THEOREM 4.1. *Let $A \in \mathbb{R}^{n,n}$ be a singular M–matrix and let A^D be its Drazin inverse. Suppose that A^D has spectrum $\sigma(A^D) = \{x_1, \ldots, x_r, x_{r+1}, \ldots, x_n\}$, where $x_1, \ldots, x_r \neq 0$ and where $x_{r+1} = \ldots = x_n = 0$. Then*

$$
E_j^2(A^D) \;\geq\; c_jE_{j-1}(A^D)E_{j+1}(A^D), \quad 1 \leq j \leq n-1, \tag{4.1}
$$

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where the c_j's are constants given by

$$c_j = \frac{(r-j+1)(n-r+j+1)j(n-j)}{(r-j)(n-r+j)(j+1)(n-j+1)}, \quad \text{for} \quad 1 \le j \le r-1, \tag{4.2}$$

and $c_j = 0$, for $r \le j \le n-1$.

Proof. It is known that A is similar to the block diagonal matrix

$$\begin{bmatrix} C & 0 \\ 0 & N \end{bmatrix},$$

where C is nonsingular[4] and where N is nilpotent, while A^D is similar to the block diagonal matrix

$$\begin{bmatrix} C^{-1} & 0 \\ 0 & 0 \end{bmatrix}.$$

Clearly, the spectrum of A is given by $\sigma(A) = \{x'_1, \ldots, x'_r, x'_{r+1}, \ldots, x'_n\}$, where $x'_i = 1/x_i$, for $i = 1, \ldots, r$, and $x'_i = 0$, for $i = r+1, \ldots, n$. Since Newton's inequalities hold for singular M–matrices, we know that

$$E_j^2(A) \ \ge \ E_{j-1}(A)E_{j+1}(A), \quad 1 \le j \le n-1.$$

Next, following similar steps to (2.10), we obtain that for $1 \le j \le r-1$,

$$E_j^2(x'_1, \ldots, x'_r) \ \ge \ k_j E_{j-1}(x'_1, \ldots, x'_r) E_{j+1}(x'_1, \ldots, x'_r), \tag{4.3}$$

where $k_j = \dfrac{\binom{n}{j}^2 \binom{r}{j-1}\binom{r}{j+1}}{\binom{r}{j}^2 \binom{n}{j-1}\binom{n}{j+1}}$. By Lemma 2.2, the inequalities in (4.3) hold if and only if the inequalities

$$E_j^2(x_1, \ldots, x_r) \ \ge \ k_{r-j} E_{j-1}(x_1, \ldots, x_r) E_{j+1}(x_1, \ldots, x_r), \quad 1 \le j \le r-1 \tag{4.4}$$

hold. We can now use Lemma 2.3 to conclude that for $1 \le j \le r-1$,

$$\begin{aligned} E_j^2(A^D) \quad &\ge \quad k_{r-j} \frac{\binom{r}{j}^2 \binom{n}{j-1}\binom{n}{j+1}}{\binom{n}{j}^2 \binom{r}{j-1}\binom{r}{j+1}} E_{j-1}(A^D)E_{j+1}(A^D) \\ &= \quad c_j E_{j-1}(A^D)E_{j+1}(A^D). \end{aligned}$$

Finally we observe that (4.1) also holds for $r \le j \le n-1$, because for this case both sides in (4.1) are zero. ∎

[4]We remark that this matrix C is not necessarily a nonsingular M–matrix.

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Corollary 4.2. *Under the assumption of Theorem 4.1, there exists a constant c, $1/2 < c \leq 1$, and independent of n, such that*

$$E_j^2(A^D) \;\geq\; cE_{j-1}(A^D)E_{j+1}(A^D), \quad 1 \leq j \leq n-1.$$

Proof. It suffices to consider the case when $1 \leq j \leq r-1$. Let

$$\delta \;=\; (r-j+1)(n-r+j+1)j(n-j) - (r-j)(n-r+j)(j+1)(n-j+1).$$

It can be verified that

$$\begin{aligned} \delta &= (n+1)\left[j(n-j) - (r-j)(n-r+j)\right] \\ &= (n+1)\left[-(j-n/2)^2 + (r-j-n/2)^2\right] \\ &= (n+1)(r-n)(r-2j). \end{aligned}$$

We see that when $1 \leq j \leq r/2$, $\delta \leq 0$, showing that $c_j \leq 1$, while when $r/2 \leq j \leq r-1$, $\delta \geq 0$, showing that $c_j \geq 1$. Moreover,

$$\min_{1 \leq j \leq r-1} c_j \;=\; c_1 \;=\; \frac{r(n-r+2)(n-1)}{2(r-1)(n-r+1)n}.$$

To see that $c_1 > 1/2$, note that

$$c_1 \;=\; \frac{1}{2} \cdot \frac{rn-r}{rn-n} \cdot \frac{n-r+2}{n-r+1}. \quad \square \tag{4.5}$$

To illustrate the result of Corollary 4.2 let us return to the example of the M–matrix A given in (1.5) whose Drazin inverse is given in (1.6). Here A has precisely one zero eigenvalue so that $n = 5$ and $r = 4$. Substituting these values in (4.5) yields that in this example $c = 0.8$.

We finally remark that if $A = sI - B$ is a singular and irreducible M–matrix, then A has a $\{1\}$–inverse which satisfies Newton's inequalities. To see this partition A into

$$A \;=\; \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix}.$$

where $A_{1,1}$ is $(n-1)$ by $(n-1)$. It is known that A has rank $n-1$, that $A_{1,1}$ is a nonsingular M–matrix, and that

$$A^- \;=\; \begin{bmatrix} A_{1,1}^{-1} & 0 \\ 0 & 0 \end{bmatrix}$$

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is a $\{1\}$–inverse of A. Next, as Newton's inequalities hold on $A_{1,1}^{-1}$, by Lemma 2.3 we conclude, with $x_1, \ldots, x_{n-1}$, being the nonzero eigenvalues of A^- and with x_n being its only zero eigenvalue, that

$$\begin{aligned} E_j^2(A^-) \;&\geq\; \frac{\binom{r}{j}^2 \binom{n}{j-1}\binom{n}{j+1}}{\binom{n}{j}^2 \binom{r}{j-1}\binom{r}{j+1}} \cdot E_{j-1}(A^-)E_{j+1}(A^-) \\ &\geq\; E_{j-1}(A^-)E_{j+1}(A^-), \end{aligned}$$

for $1 \leq j \leq n-2$. These inequalities continue to hold in the case when $j = n-1$ as $E_n(A^-) = 0$.

5. Conditions for the Failure of Newton's Inequalities. In this section we explore conditions under which for a matrix $A \in \mathbb{R}^{n,n}$, Newton's inequalities will fail. For that purpose let $\sigma(A) = \{x_1, \ldots, x_n\}$ and, for $i = 1, \ldots, n$, denote by A_i the *principal submatrix of A obtained from A by deleting its i–th row and column.*

We begin by obtaining a relation between $E_j(A)$ and $E_j(A_i)$, which can be regarded as the special case of a formula in [5, p. 712]. As earlier in the paper, let $p(x) = \det(xI - A)$ be the characteristic polynomial of A. It is well known, see for example Horn and Johnson [6, Exercise 4, p. 93], that

$$\frac{d}{dx}p(x) \;=\; \sum_{i=1}^{n} \det(xI - A_i).$$

Thus, from the representation of the characteristic polynomial as given in (2.3) when it is applied to matrices of order $n-1$ we obtain that

$$\frac{1}{n} \cdot \frac{d}{dx}p(x) \;=\; \sum_{j=0}^{n-1} (-1)^j \binom{n-1}{j} \left[\frac{1}{n}\sum_{i=1}^{n} E_j(A_i)\right] x^{n-j-1}. \tag{5.1}$$

Next, let $y_1, \ldots, y_{n-1}$ be the roots of $\frac{d}{dx}p(x)$. Then

$$\frac{1}{n} \cdot \frac{d}{dx}p(x) \;=\; \sum_{j=0}^{n-1} (-1)^j \binom{n-1}{j} E_j(y_1, \ldots, y_{n-1}) x^{n-j-1}. \tag{5.2}$$

Upon equating the coefficients of likewise powers of x in (5.1) and (5.2), we obtain that for $0 \leq j \leq n-1$,

$$E_j(y_1, \ldots, y_{n-1}) \;=\; \frac{1}{n}\sum_{i=1}^{n} E_j(A_i).$$

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Now, according to Niculescu [11, Section 2], $E_j(A) = E_j(y_1, \ldots, y_{n-1})$ for $0 \le j \le n-1$. Hence,

$$E_j(A) \;=\; \frac{1}{n}\sum_{i=1}^{n} E_j(A_i), \quad 0 \le j \le n-1.$$

We are now ready to prove the following result. For simplicity of notation, we write $E_{k,i} := E_k(A_i)$ for $i = 1, \ldots, n$ and $k = 0, \ldots, n-1$.

Proposition 5.1. *Let $A \in \mathbb{R}^{n,n}$. Suppose that for some $k = 1, \ldots, n-2$, and for all $i = 1, \ldots, n$,*

$$N_k(A_i) \;\le\; 0, \quad E_{k-1,i} \;\ge\; 0 \;\text{ and }\; E_{k+1,i} \;\ge\; 0. \tag{5.3}$$

Then

$$N_k(A) \;\le\; 0, \tag{5.4}$$

that is, the k–th Newton inequality fails to hold.

Proof. Again, for $j = 0, \ldots, n$, set $E_j := E_j(A)$. Now from the definition of $N_k(A)$ in (1.4) we can write that:

$$\begin{aligned}
n^2 N_k(A) \;&=\; n^2\left(E_k^2 - E_{k-1}E_{k+1}\right)\\
&=\; \left(\sum_{i=1}^{n} E_{k,i}\right)^2 - \left(\sum_{i=1}^{n} E_{k-1,i}\right)\left(\sum_{i=1}^{n} E_{k+1,i}\right)\\
&=\; \sum_{i=1}^{n}\left(E_{k,i}^2 - E_{k-1,i}E_{k+1,i}\right)\\
&\quad + \sum_{1\le i<m\le n}\left(2E_{k,i}E_{k,m} - E_{k-1,i}E_{k+1,m} - E_{k+1,i}E_{k-1,m}\right)
\end{aligned}$$

$$\begin{aligned}
&\le\; \sum_{i=1}^{n}\left(E_{k,i}^2 - E_{k-1,i}E_{k+1,i}\right)\\
&\quad + \sum_{1\le i<m\le n}\left(2\sqrt{E_{k-1,i}E_{k+1,i}E_{k-1,m}E_{k+1,m}} - E_{k-1,i}E_{k+1,m} - E_{k+1,i}E_{k-1,m}\right)\\
&=\; \sum_{i=1}^{n}\left(E_{k,i}^2 - E_{k-1,i}E_{k+1,i}\right) - \sum_{1\le i<m\le n}\left(\sqrt{E_{k-1,i}E_{k+1,m}} - \sqrt{E_{k+1,i}E_{k-1,m}}\right)^2\\
&\le\; 0.
\end{aligned}$$

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This shows that if the k–th Newton's inequalities fail **for all** the $(n-1)\times(n-1)$ principal submatrices of A, then the k–th Newton's inequality fails on the entire matrix and our proof is done. □

In view of the above proposition it is tempting to conjecture that if one modifies the conditions of the proposition to that of that for some $k = 1,\ldots,n-2$, and for all $i = 1,\ldots,n$, $N_k(A_i) \geq 0$, $E_{k-1,i} \geq 0$, and $E_{k+1,i} \geq 0$, then $N_k(A) \geq 0$. That this is **not** the case can be seen by taking

$$A = \begin{bmatrix} 0.5779 & 0.1202 & -0.6326 & -0.4212 \\ 0.2568 & 0.2905 & 0.6116 & 0.01260 \\ 0.5483 & -0.1244 & 0.8705 & 0.1633 \\ 0.1306 & -0.05360 & 0.4026 & 0.8043 \end{bmatrix}.$$

A computation now shows that

$$E(A) = \begin{bmatrix} 1.0 & 0.6358 & 0.4507 & 0.3210 & 0.2251 \end{bmatrix}$$

and that

$$N(A) = \begin{bmatrix} -0.04650 & -0.0009556 & 0.001627 \end{bmatrix}.$$

A further computation yields that for $k = 0,\ldots,3$, and for $i = 1,\ldots,4$,

$$[E_k(A_i)] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0.6551 & 0.7509 & 0.5576 & 0.5796 \\ 0.3992 & 0.6680 & 0.2971 & 0.4386 \\ 0.2401 & 0.5870 & 0.1326 & 0.3245 \end{bmatrix}$$

and that $k = 1, 2$, and $k = 1,\ldots,4$,

$$[N_k(A_i)] = \begin{bmatrix} 0.02993 & -0.1042 & 0.01383 & -0.1027 \\ 0.002101 & 0.005496 & 0.01433 & 0.004297 \end{bmatrix}.$$

We observe that in this example, for $k = 2$, $N_2(A) \leq 0$, while $E_{1,1},\ldots,E_{4,1} \geq 0$ and $E_{1,3},\ldots,E_{4,3} \geq 0$, and, in contrast to the leftmost condition in (5.3), $N_2(A_i) \geq 0$, for $i = 1,\ldots,4$.

When applied in an inductive manner, Proposition 5.1 allows us to easily check whether certain Newton's inequalities must fail. Consider, for example, the case when Newton's inequalities fail on all 2×2 leading principal submatrices of an $n\times n$ matrix $A = [a_{i,j}]$. We observe that for any such leading principal submatrix

$$B = \begin{bmatrix} a_{i,i} & a_{i,j} \\ a_{j,i} & a_{j,j} \end{bmatrix}, \quad i < j,$$

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$E_0(B) = 1$, $E_1(B) = \mathrm{trace}(B)/2 = (a_{i,i} + a_{j,j})/2$, and $E_2(B) = \det(B) = a_{i,i}a_{j,j} - a_{i,j}a_{j,i}$. Thus Netwon's inequalities fail on B if and only if

$$(a_{i,i} - a_{j,j})^2 < -4a_{i,j}a_{j,i}. \tag{5.5}$$

This observation, together with Proposition 5.1, imply that an $n \times n$ matrix A fails to satisfy the first Newton's inequality $E_1^2(A) \leq E_0(A)E_2(A)$ when (5.5) holds for all $i \neq j$. In particular, when A has a constant diagonal, the first Newton's inequality fails to hold on A if $a_{i,j}a_{j,i} < 0$, for all $i \neq j$, i.e., if the off–diagonal entries of A are all nonzero and exhibit a skew–symmetric sign pattern.

REFERENCES

[1] A. Ben-Israel and T. N. E. Greville. *Generalized Inverses: Theory and Applications*. 2nd ed., Springer, New York, 2003.
[2] A. Berman and R. J. Plemmons. *Nonnegative Matrices in the Mathematical Sciences*. SIAM, Philadelphia, 1994.
[3] S. L. Campbell and C. D. Meyer, Jr. *Generalized Inverses of Linear Transformations*. Dover Publications, Yew York, 1991.
[4] G. Hardy, J. E. Littlewood, and G. Pólya. *Inequalities*. Cambridge Mathematical Library, 2nd ed., 1952.
[5] O. Holtz. M–matrices satisfy Newton's inequalities. *Proc. Amer. Math. Soc.*, 133:711–717, 2005.
[6] R. A. Horn and C. R. Johnson. *Matrix Analysis*. Cambridge Univ. Press, New York, 1985.
[7] C. Maclaurin. A second letter to Martin Folkes, Esq.; concerning the roots of equations, with demonstration of other rules of algebra. *Philos. Trans. Roy. Soc. London Ser. A*, 36:59–96, 1729.
[8] S. R. Mohan, M. Neumann, and K. G. Ramamurthy. Nonnegativity of principal minors of generalized inverses of M–matrices. *Linear Algebra Appl.*, 58:247–259, 1984.
[9] V. Monov. Newton's inequalities for families of complex numbers. *JIPAM. J. Inequal. Pure Appl. Math.*, Vol. 6, No. 3, 2005.
[10] I. Newton. *Arithmetica universalis:sive de comppositione et resolutione arithmetica liber*. 1707.
[11] C. P. Niculescu. A new look at Newton's inequalities. *JIPAM. J. Inequal. Pure Appl. Math.*, Vol. 1, No. 2, 2000.
[12] S. Rosset. Normalized symmetric functions, Newton's inequalities, and a new set of stronger inequalities. *Amer. Math. Monthly*, 96:815–820, 1989.