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CONTROLLABILITY OF SERIES CONNECTIONS*

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Abstract. In this paper the controllability of series connections of arbitrary many linear systems is studied. As the main result, necessary and sufficient conditions are given, under which the system obtained as a result of series connections of arbitrary many linear systems is controllable.

Key words. Controllability, Linear systems, Completion, Invariant factors.

AMS subject classifications. 93B05, 15A21.

1. Introduction. Let S_i be a time-invariant linear system, with state x_i, input u_i and output y_i, $i = 1, \ldots, m$:

$$\xrightarrow{u_i} \boxed{S_i} \xrightarrow{y_i} \tag{1.1}$$

Suppose that the system S_i is described by the following system of linear differential equations:

$$\dot{x}_i = A_i x_i + B_i u_i, \tag{1.2}$$
$$y_i = C_i x_i, \tag{1.3}$$

where $A_i \in \mathbb{K}^{n_i \times n_i}$, $B_i \in \mathbb{K}^{n_i \times m_i}$, $C_i \in \mathbb{K}^{p_i \times n_i}$, $i = 1, \ldots, m$, $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$; for details see e.g., [3].

Let $j \in \{1, \ldots, m\}$. The algebraic properties of the system S_j depend on the properties of the triple of matrices (A_j, B_j, C_j). Recall that the system S_j is *controllable* if and only if the pair (A_j, B_j) is *controllable*, where the controllability of a pair is defined as follows:

Definition 1.1. Let $\mathbb{F}$ be a field. Let $A_j \in \mathbb{F}^{n_j \times n_j}$, $B_j \in \mathbb{F}^{n_j \times m_j}$. The pair (A_j, B_j) is said to be *controllable* if one of the following (equivalent) conditions is satisfied:

$$\begin{array}{ll} 1) & \min\limits_{\lambda \in \overline{\mathbb{F}}} \operatorname{rank} \begin{bmatrix} \lambda I - A_j & -B_j \end{bmatrix} = n_j \\ 2) & \text{all invariant factors of the matrix pencil } \begin{bmatrix} \lambda I - A_j & -B_j \end{bmatrix} \\ & \text{are trivial} \\ 3) & \operatorname{rank} \begin{bmatrix} B_j & A_j B_j & A_j^2 B_j & \cdots & A_j^{n_j - 1} B_j \end{bmatrix} = n_j. \end{array} \tag{1.4}$$

In this case, we also say that the matrix $\begin{bmatrix} A_j & B_j \end{bmatrix}$ and the corresponding matrix pencil $\begin{bmatrix} \lambda I - A_j & -B_j \end{bmatrix}$ are controllable.

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By series connections of the linear systems $S_1, \ldots, S_m$ we mean connections where the input of the system S_{i+1} is a linear function of the output of S_i, $i = 1, \ldots, m-1$, i.e.,

$$u_{i+1} = X_i y_i, \quad i = 1, \ldots, m-1, \tag{1.5}$$

where $X_i \in \mathbb{F}^{m_{i+1} \times p_i}$. As a result of this connection, we obtain a new linear system S, with input u_1, output y_m and state $\begin{bmatrix} x_1^T & \cdots & x_m^T \end{bmatrix}^T$.

Thus, studying the properties of the system S, arise the following matrix completion control problem:

Problem 1.2. *Let $\mathbb{F}$ be a field. Find necessary and sufficient conditions for the existence of matrices $X_i \in \mathbb{F}^{m_{i+1} \times p_i}$, $i = 1, \ldots, m-1$, such that the matrix*

$$\left[\begin{array}{ccccc|c} A_1 & 0 & 0 & \cdots & 0 & B_1 \\ B_2X_1C_1 & A_2 & 0 & \ddots & 0 & 0 \\ 0 & B_3X_2C_2 & A_3 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \ddots & B_mX_{m-1}C_{m-1} & A_m & 0 \end{array}\right] \tag{1.6}$$

is controllable.

In Section 4 (Theorem 4.1), we give a complete solution to Problem 1.2 when $\mathbb{F}$ is an infinite field. Furthermore, in Section 5, we obtain solutions over arbitrary fields of particular cases of the previous problem.

Similar problems, especially in the case $m = 2$, have been studied previously; see for example the results of I. Baragaña and I. Zaballa [1], and F. C. Silva [8].

2. Notation and Auxiliary results. Let $\mathbb{F}$ be a field. For any polynomial $f \in \mathbb{F}[\lambda]$, $d(f)$ denotes its degree. If $f(\lambda) = \lambda^k - a_{k-1}\lambda^{k-1} - \cdots - a_1\lambda - a_0 \in \mathbb{F}[\lambda]$, where $k > 0$, then the matrix

$$C(f(\lambda)) := \begin{bmatrix} e_2^{(k)} & \cdots & e_k^{(k)} & a \end{bmatrix}^T,$$

where $e_i^{(k)}$ is the ith column of the identity matrix I_k and $a = [a_0 \ \cdots \ a_{k-1}]^T$, is called *the companion matrix* for the polynomial $f(\lambda)$.

If $A(\lambda) \in \mathbb{F}[\lambda]^{n \times m}$, with $r = \operatorname{rank} A(\lambda)$, and $\psi_1 | \cdots | \psi_r$ are the invariant factors of $A(\lambda)$, make a convention that $\psi_i = 1$ for $i \leq 0$ and $\psi_i = 0$ for $i \geq r+1$.

Definition 2.1. Let $A, A' \in \mathbb{F}^{n \times n}$, $B, B' \in \mathbb{F}^{n \times l}$. Two matrices

$$M = \begin{bmatrix} A & B \end{bmatrix}, \qquad M' = \begin{bmatrix} A' & B' \end{bmatrix} \tag{2.1}$$

are *feedback equivalent* if there exists a nonsingular matrix

$$P = \begin{bmatrix} N & 0 \\ V & T \end{bmatrix}$$

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where $N \in \mathbb{F}^{n\times n}$, $V \in \mathbb{F}^{l\times n}$, $T \in \mathbb{F}^{l\times l}$, such that $M' = N^{-1}MP$.

If M and M' are feedback equivalent, then we also say that the corresponding pairs (A, B) and (A', B') are feedback equivalent.

It is easy to verify that two matrices M and M' are feedback equivalent if and only if the corresponding matrix pencils

$$R = \begin{bmatrix} \lambda I - A & -B \end{bmatrix} \quad \text{and} \quad R' = \begin{bmatrix} \lambda I - A' & -B' \end{bmatrix} \tag{2.2}$$

are strictly equivalent, for details see [4].

If $(A, B) \in \mathbb{F}^{n\times n} \times \mathbb{F}^{n\times m}$, is a controllable matrix pair, then it is feedback equivalent to the pair (A_c, B_c) with

$$A_c = \operatorname{diag}(A_1, \ldots, A_s), \qquad B_c = \begin{bmatrix} \operatorname{diag}(B_1, \ldots, B_s) & 0 \end{bmatrix},$$

where

$$A_i = \begin{bmatrix} 0 & I_{k_i-1} \\ 0 & 0 \end{bmatrix} \in \mathbb{F}^{k_i\times k_i}, \quad B_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \in \mathbb{F}^{k_i\times 1}, \quad 1 \le i \le s.$$

The pair (A_c, B_c) is called *the Brunovsky canonical form* of the pair (A, B), and the positive integers $k_1 \ge \cdots \ge k_s$ are called *the nonzero controllability indices* of (A, B). Analogously as in [2], we introduce the following definition.

Definition 2.2. Two polynomial matrices $A(\lambda) \in \mathbb{F}[\lambda]^{p\times q}$ and $B(\lambda) \in \mathbb{F}[\lambda]^{p\times q}$ are *SP-equivalent* if there exist invertible matrices $P \in \mathbb{F}^{p\times p}$ and $Q(\lambda) \in \mathbb{F}[\lambda]^{q\times q}$ such that

$$PA(\lambda)Q(\lambda) = B(\lambda).$$

Lemma 2.3. [2] *Let $\mathbb{F}$ be an infinite field and $f(x)$, $g(x)$, $h(x)$ be nonzero polynomials over $\mathbb{F}$. Then there exists $\alpha \in \mathbb{F}$ such that*

$$\gcd(f(x) + \alpha g(x), h(x)) = \gcd(f(x), g(x), h(x)). \tag{2.3}$$

In fact, in [2] was proved that (2.3) is not valid only for finitely many $\alpha \in \mathbb{F}$. Hence, (2.3) is valid for a generic (almost every) $\alpha \in \mathbb{F}$.

Proposition 2.4. [6, 7, 9] *Let $\mathbb{D}$ be a principal ideal domain. Let $A \in \mathbb{D}^{n\times n}$, $B \in \mathbb{D}^{n\times n}$. Let $\alpha_1 | \cdots | \alpha_n$ be the invariant factors of A, and $\beta_1 | \cdots | \beta_n$ be the invariant factors of B. Let $\gamma_1 | \cdots | \gamma_n$ be the invariant factors of AB. Then we have*

$$\operatorname{lcm}(\alpha_{n-k-i+1}\beta_{i+1} : 0 \le i \le n-k) \mid \gamma_{n-k+1} \mid \gcd(\alpha_{n-i+1}\beta_{n-k+i} : 1 \le i \le k), \tag{2.4}$$

$$k = 1, \ldots, n.$$

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3. Preliminary results. The following proposition deals with the almost canonical form for the SP equivalence of arbitrary square polynomial matrix. Proof goes analogously as the proof of Proposition 2 in [2], thus will be omitted.

PROPOSITION 3.1. *Let $\mathbb{F}$ be an infinite field and let $A(\lambda) \in \mathbb{F}[\lambda]^{n\times n}$. Let $r = \operatorname{rank} A(\lambda) \leq n$. Then $A(\lambda)$ is SP-equivalent to a lower triangular matrix $S(\lambda) = (s_{ij}(\lambda))$, $i, j \in \{1, \dots, n\}$, with the following properties:*

1. $s_{ii}(\lambda) = s_i(\lambda),\ i = 1, \dots, r-1,$ *where* $s_1(\lambda)|\cdots|s_{r-1}(\lambda)$ *are the first $r-1$ invariant factors of $A(\lambda)$*
2. $s_{ii}(\lambda)|s_{ji}(\lambda), \quad 1 \leq i \leq r-1, \quad i \leq j \leq n$
3. $s_r(\lambda) = \gcd(s_{rr}(\lambda), \dots, s_{nr}(\lambda))$ *and* $d(s_{rr}(\lambda)) \geq \cdots \geq d(s_{nr}(\lambda))$ *where $s_r(\lambda)$ is the r-th invariant factor of $A(\lambda)$*
4. *if* $i \leq r-1$ *and* $i < j$ *and* $s_{ji}(\lambda) \neq 0$, *then* $s_{ji}(\lambda)$ *is monic and* $d(s_{ii}(\lambda)) < d(s_{ji}(\lambda))$
5. $s_{ij} = 0, \quad j > r.$

Further on, the matrix $S(\lambda)$ will be called *the SP canonical form* of the matrix $A(\lambda)$.

LEMMA 3.2. *Let $\mathbb{F}$ be a field. Let $A \in \mathbb{F}^{n\times n}$, $B \in \mathbb{F}^{n\times m}$. If there exists $X \in \mathbb{F}^{m\times p}$ such that (A, BX) is controllable, then the pair (A, B) is controllable.*

Proof. There exists an invertible matrix $P \in \mathbb{F}^{n\times n}$ such that

$$PBX = \begin{bmatrix} Y \\ 0 \end{bmatrix}, \quad Y \in \mathbb{F}^{\operatorname{rank} B\times p}.$$

Thus, from the controllability of (A, BX) and since

$$\begin{bmatrix} PAP^{-1} & PBX \end{bmatrix} = \begin{bmatrix} A_1 & A_2 & Y \\ A_3 & A_4 & 0 \end{bmatrix}, \quad A_4 \in \mathbb{F}^{(n-\operatorname{rank} B)\times(n-\operatorname{rank} B)},$$

we have that the pair (A_4, A_3) is controllable. Furthermore, there exists an invertible matrix $Q \in \mathbb{F}^{m\times m}$ such that

$$PBQ = \begin{bmatrix} I_{\operatorname{rank} B} & 0 \\ 0 & 0 \end{bmatrix}.$$

Hence, the controllability of (A, B) is equivalent to the controllability of (A_4, A_3), which concludes our proof. □

LEMMA 3.3. *Let $\mathbb{F}$ be an infinite field. Let $A(\lambda) \in \mathbb{F}[\lambda]^{n\times n}$ and $C(\lambda) \in \mathbb{F}[\lambda]^{a\times b}$ be such that $n = \operatorname{rank} A(\lambda)$. Let $\alpha_1|\cdots|\alpha_n$ be the invariant factors of $A(\lambda)$, and let $\beta_1|\cdots|\beta_s$ be the invariant factors of $C(\lambda)$, where $s = \operatorname{rank} C(\lambda)$.*

There exists $X \in \mathbb{F}^{n\times a}$ such that

$$\begin{bmatrix} A(\lambda) & XC(\lambda) \end{bmatrix} \text{ is equivalent to } \begin{bmatrix} I_n & 0 \end{bmatrix}, \tag{3.1}$$

if and only if

$$\gcd(\alpha_i, \beta_{n+1-i}) = 1, \quad i = 1, \dots, n. \tag{3.2}$$

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Proof. If $a \le b$, then there exists an invertible matrix $Q(\lambda) \in \mathbb{F}[\lambda]^{b\times b}$, such that

$$C(\lambda)Q(\lambda) = \begin{bmatrix} D(\lambda) & 0 \end{bmatrix}, \text{ where } D(\lambda) \in \mathbb{F}[\lambda]^{a\times a}.$$

Thus, instead of $C(\lambda)$ we can consider the matrix $D(\lambda)$.

If $a > b$, then instead of the matrix $C(\lambda)$ consider the matrix

$$D(\lambda) = \begin{bmatrix} C(\lambda) & 0 \end{bmatrix}, \text{ where } D(\lambda) \in \mathbb{F}[\lambda]^{a\times a}.$$

Thus, without loss of generality, we can assume that $a = b$.

Necessity:

Suppose that there exists $X \in \mathbb{F}^{n\times a}$, such that $\begin{bmatrix} A(\lambda) & XC(\lambda) \end{bmatrix}$ is equivalent to $\begin{bmatrix} I_n & 0 \end{bmatrix}$. Denote by $A'(\lambda)$ and $C'(\lambda)$ the Smith canonical forms of the matrices $A(\lambda)$ and $C(\lambda)$, respectively. Then $\begin{bmatrix} A(\lambda) & XC(\lambda) \end{bmatrix}$ is equivalent to $\begin{bmatrix} A'(\lambda) & X(\lambda)C'(\lambda) \end{bmatrix}$, for some $X(\lambda) \in \mathbb{F}[\lambda]^{n\times a}$. Thus, we have that for every $x \in \bar{\mathbb{F}}$, rank $\begin{bmatrix} A'(x) & X(x)C'(x) \end{bmatrix} = n$.

If $\gcd(\alpha_n, \beta_s) = 1$, then the condition is obviously satisfied. Otherwise, let $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, s\}$ be such that $\gcd(\alpha_i, \beta_j) \ne 1$. Let $\lambda_0 \in \bar{\mathbb{F}}$ be a common zero of α_i and β_j. Let $t := \min_{k\in\{1,\dots,n\}}\{k|\alpha_k(\lambda_0) = 0\}$ and $p := \min_{l\in\{1,\dots,s\}}\{l|\beta_l(\lambda_0) = 0\}$. The rank of the matrix $\begin{bmatrix} A'(\lambda_0) & X(\lambda_0)C'(\lambda_0) \end{bmatrix}$ (which is equal to n) is less or equal than the number of its nonzero columns. Since the number of nonzero columns of $A'(\lambda_0)$ is $t-1$ and the number of nonzero columns of $C'(\lambda_0)$ is $p-1$, we have

$$n \le t - 1 + p - 1, \text{ and so } \quad i + j \ge n + 2.$$

Thus, for all indices i and j such that $i + j \le n + 1$, the polynomials α_i and β_j are mutually prime, which proves our condition.

Sufficiency:

Suppose that the condition (3.2) is satisfied. Without loss of generality, we shall consider $A(\lambda)$ in its SP canonical form, and $C(\lambda)$ in its SP equivalent form $M(\lambda)$ which we describe below:

First, put the matrix $C(\lambda) \in \mathbb{F}[\lambda]^{a\times a}$, into its SP canonical form:

$$\left[\begin{array}{ccccc|c} \beta_1 & 0 & & 0 & 0 & 0 \\ a_{11}\beta_1 & \beta_2 & & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & & & \vdots \\ a_{s-21}\beta_1 & a_{s-22}\beta_2 & \cdots & \beta_{s-1} & 0 & 0 \\ a_{s-11}\beta_1 & a_{s-12}\beta_2 & \cdots & a_{s-1s-1}\beta_{s-1} & X_0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ a_{a-11}\beta_1 & a_{a-12}\beta_2 & \cdots & a_{a-1s-1}\beta_{s-1} & X_{a-s} & 0 \end{array}\right], \tag{3.3}$$

where $a_{ij} \in \mathbb{F}[\lambda]$, $1 \le i \le a-1$, $1 \le j \le s-1$, and $\gcd(X_0, \dots, X_{a-s}) = \beta_s$. By using the condition and Lemma 2.3, there exist $x_1, \dots, x_{a-s} \in \mathbb{F}$, such that

$$\gcd(\alpha_{n-s+1}, X_0 + x_1X_1 + \cdots + x_{a-s}X_{a-s}) = 1.$$

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Let $\bar{\beta}_s := X_0 + x_1X_1 + \cdots + x_{a-s}X_{a-s}$. By multiplying the row $s+i$ by x_i, for all $i = 1, \ldots, a-s$, and adding it to the sth row, we obtain the matrix $M(\lambda)$, which is SP equivalent to the matrix $C(\lambda)$, and at the position (s,s) has the polynomial $\bar{\beta}_s$. Further on, the matrix $M(\lambda)$ will be called *the SP-quasi canonical form* of the matrix $C(\lambda)$. Note that $\beta_s|\bar{\beta}_s$ and $\gcd(\alpha_{n-s+1}, \bar{\beta}_s) = 1$.

Consider the submatrix $\bar{M}(\lambda)$ of $M(\lambda)$ formed by the rows $2, \ldots, a-1$, and by the columns $2, \ldots, a-1$. If $s = a$, the invariant factors of $\bar{M}(\lambda)$ are $\beta_2|\cdots|\beta_{s-1}$ and if $s < a$, the invariant factors of $\bar{M}(\lambda)$ are $\beta_2|\cdots|\beta_{s-1}|\beta'_s$, for some polynomial β'_s which satisfies $\beta_s|\beta'_s|\bar{\beta}_s$.

From now on, we shall consider the matrix $M(\lambda)$ instead of the matrix $C(\lambda)$ in (3.1). The proof is further split into three cases:

Case 1. Let $n = a$.

The proof goes by induction on n. The case $n = 1$ is trivial. If $n = 2$, there are two nontrivial possibilities on s : $s = 1$ or $s = 2$.

If $s = 2$, it is enough to prove the existence of $x \in \mathbb{F}$, such that the matrix

$$\left[\begin{array}{cc|cc} \alpha_1 & 0 & \beta_1 & 0 \\ b(\lambda)\alpha_1 & \alpha_2 & (a(\lambda)+x)\beta_1 & \beta_2 \end{array}\right] \tag{3.4}$$

has two invariant factors both equal to 1, where $a(\lambda), b(\lambda) \in \mathbb{F}[\lambda]$.

In fact, we shall prove that there exists $x \in \mathbb{F}$ such that the second determinantal divisor of (3.4), D_2, given by

$$D_2 = \gcd(\beta_1\beta_2, \alpha_1\alpha_2, \beta_1\alpha_2, \alpha_1\beta_2, \alpha_1\beta_1(b(\lambda) - a(\lambda) - x)),$$

is equal to 1.
Since $\mathbb{F}$ is infinite, by applying Lemma 2.3, there exists $x \in \mathbb{F}$, such that

$$D_2 = \gcd(\beta_1\beta_2, \alpha_1\alpha_2, \beta_1\alpha_2, \alpha_1\beta_2, \alpha_1\beta_1).$$

Since $\gcd(\beta_1, \alpha_2) = 1$ and $\gcd(\beta_2, \alpha_1) = 1$, we have $D_2 = 1$, as wanted.

If $s = 1$, we need to prove the existence of $x \in \mathbb{F}$ such that the second determinantal divisor of the matrix

$$\left[\begin{array}{cc|cc} 1 & 0 & p(\lambda) & 0 \\ b(\lambda) & \alpha_2 & q(\lambda)+xp(\lambda) & 0 \end{array}\right]$$

is equal to 1, whenever $\gcd(p(\lambda), \alpha_2) = 1$, $b(\lambda), p(\lambda), q(\lambda) \in \mathbb{F}[\lambda]$. By simple calculation, we have $D_2 = \gcd(p(\lambda)b(\lambda) - q(\lambda) - xp(\lambda), \alpha_2, p(\lambda)\alpha_2, \alpha_2)$. Thus, again by applying Lemma 2.3, we obtain the existence of $x \in \mathbb{F}$ such that $D_2 = 1$.

Now suppose that the claim is true for $n-2$ and prove that it will be valid for n.

Let $\bar{A}(\lambda)$ be a submatrix of $A(\lambda)$ formed by the rows $2, \ldots, n-1$ and the columns $2, \ldots, n-1$. Thus, $\bar{A}(\lambda)$ has $\alpha_2|\cdots|\alpha_{n-1}$ as the invariant factors. In both cases, $s = a$ or $s < a$, the invariant factors of $\bar{M}(\lambda)$ and of $\bar{A}(\lambda)$ satisfy the condition (3.2). Thus, we can apply the induction hypothesis and obtain that there exists $Y \in \mathbb{F}^{(n-2)\times(n-2)}$ such that the matrix $\left[\begin{array}{cc} \bar{A}(\lambda) & Y\bar{M}(\lambda) \end{array}\right]$ is equivalent to $\left[\begin{array}{cc} I_{n-2} & 0 \end{array}\right]$.

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To finish the proof, we shall show that there exists $x \in \mathbb{F}$, such that the matrix $\begin{bmatrix} A(\lambda) & XM(\lambda) \end{bmatrix}$ is equivalent to $\begin{bmatrix} I_n & 0 \end{bmatrix}$, where

$$X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & Y & 0 \\ x & 0 & 1 \end{bmatrix}.$$

Since $\begin{bmatrix} \bar{A}(\lambda) & Y\bar{M}(\lambda) \end{bmatrix}$ is equivalent to $\begin{bmatrix} I_{n-2} & 0 \end{bmatrix}$, and from the forms of matrices $A(\lambda)$ and $M(\lambda)$, the matrix $\begin{bmatrix} A(\lambda) & XM(\lambda) \end{bmatrix}$ is equivalent to the following one

$$\left[\begin{array}{c|c|c|c|c|c} \alpha_1 & 0 & 0 & \beta_1 & 0 & 0 \\ \hline 0 & I_{n-2} & 0 & 0 & 0 & 0 \\ \hline p(\lambda)\alpha_1 & 0 & \alpha_n & (q(\lambda)+x)\beta_1 & * & \beta_n \end{array}\right], \tag{3.5}$$

for some polynomials $p(\lambda)$ and $q(\lambda) \in \mathbb{F}[\lambda]$ ($*$ denotes unimportant entries).

The matrices

$$\begin{bmatrix} \alpha_1 & 0 \\ p(\lambda)\alpha_1 & \alpha_n \end{bmatrix} \text{ and } \begin{bmatrix} \beta_1 & 0 \\ q(\lambda)\beta_1 & \beta_n \end{bmatrix}$$

have $\alpha_1|\alpha_n$ and $\beta_1|\beta_n$ as the invariant factors, respectively, and they are both in SP canonical forms.

Since $\gcd(\alpha_1, \beta_n) = \gcd(\alpha_n, \beta_1) = 1$ by applying the case $n = 2$, there exists $x \in \mathbb{F}$ such that

$$\left[\begin{array}{cc|cc} \alpha_1 & 0 & \beta_1 & 0 \\ p(\lambda)\alpha_1 & \alpha_n & (q(\lambda)+x)\beta_1 & \beta_n \end{array}\right]$$

is equivalent to $\begin{bmatrix} I_2 & 0 \end{bmatrix}$.

Hence, for such $x \in \mathbb{F}$ we have that the matrix (3.5) is equivalent to $\begin{bmatrix} I_n & 0 \end{bmatrix}$, as wanted.

Case 2. Let $n > a$.

Let

$$\tilde{M}(\lambda) = \begin{bmatrix} I_a \\ 0 \end{bmatrix} M(\lambda) \begin{bmatrix} I_a & 0 \end{bmatrix} \in \mathbb{F}[\lambda]^{n\times n}.$$

Then the invariant factors of $\tilde{M}(\lambda)$ are $\beta_1|\cdots|\beta_s$. From the Case 1., there exists $Y \in \mathbb{F}^{n\times n}$ such that

$$\begin{bmatrix} A(\lambda) & Y\tilde{M}(\lambda) \end{bmatrix}$$

is equivalent to $\begin{bmatrix} I_n & 0 \end{bmatrix}$. Now, put $X := Y \begin{bmatrix} I_a \\ 0 \end{bmatrix} \in \mathbb{F}^{n\times a}$.

Case 3. Let $n < a$.

Let

$$M'(\lambda) = \begin{bmatrix} I_n & 0 \end{bmatrix} M(\lambda) \begin{bmatrix} I_n \\ 0 \end{bmatrix} \in \mathbb{F}[\lambda]^{n\times n}.$$

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If $n < s$, then the invariant factors of $M'(\lambda)$ are $\beta_1|\cdots|\beta_n$, and if $n \geq s$, then the invariant factors of $M'(\lambda)$ are $\beta_1|\cdots|\beta_{s-1}|\beta_s''$, for some polynomial β_s'' such that $\beta_s|\beta_s''|\bar{\beta}_s$. By applying the Case 1, there exists $Y \in \mathbb{F}^{n\times n}$ such that

$$\begin{bmatrix} A(\lambda) & YM'(\lambda) \end{bmatrix}$$

is equivalent to $\begin{bmatrix} I_n & 0 \end{bmatrix}$. Now, put $X := Y \begin{bmatrix} I_n & 0 \end{bmatrix} \in \mathbb{F}^{n\times a}$. $\square$

REMARK 3.4. Let $A(\lambda)$ be in its SP canonical form and $M(\lambda)$ be the SP-quasi canonical form of the matrix $C(\lambda)$. Let $X_0 \in \mathbb{F}^{n\times a}$ be the matrix defined in the previous lemma, such that

$$\begin{bmatrix} A(\lambda) & X_0M(\lambda) \end{bmatrix} \text{ is equivalent to } \begin{bmatrix} I_n & 0 \end{bmatrix}. \tag{3.6}$$

Let $P \in \mathbb{F}^{n\times n}$ be a lower triangular matrix with units on diagonal. From the proof of Lemma 3.3 (see (3.4)), we have that for a generic matrix P, PX_0 also satisfies (3.6).

Further on in this paper, by S we denote the set of all lower triangular matrices with units on diagonal, P, such that PX_0 satisfies (3.6), and we define

$$G := \{PX_0 | P \in S\}.$$

LEMMA 3.5. *Let $\mathbb{F}$ be an infinite field. Let $A(\lambda) \in \mathbb{F}[\lambda]^{n\times n}$ be such that $n = \operatorname{rank} A(\lambda)$, and let $\alpha_1|\cdots|\alpha_n$ be its invariant factors. Let $D(\lambda) \in \mathbb{F}[\lambda]^{m\times m}$ be such that $m = \operatorname{rank} D(\lambda)$ and let $\beta_1|\cdots|\beta_m$ be its invariant factors. Let $C(\lambda) \in \mathbb{F}[\lambda]^{a\times n}$, $a \leq m$, and let $\gamma_1|\cdots|\gamma_s$ be its invariant factors, $s = \operatorname{rank} C(\lambda)$. Let $\mu_1|\cdots|\mu_n$ be the invariant factors of*

$$\begin{bmatrix} A(\lambda) \\ C(\lambda) \end{bmatrix}.$$

If

$$\gcd(\gamma_i, \beta_{m+1-i}) = 1, \quad i = 1, \ldots, m, \tag{3.7}$$

then there exists $X \in \mathbb{F}^{m\times a}$, such that

$$\begin{bmatrix} D(\lambda) & XC(\lambda) \end{bmatrix}$$

is equivalent to $\begin{bmatrix} I_m & 0 \end{bmatrix}$, and such that every zero of a polynomial ϵ^X_{m+i}, $i = 1, \ldots, n$, is a zero of the polynomial α_i or of the polynomial $\gcd(\beta_j, \mu_{m+i-j+1})$, for some $j = 1, \ldots, m$, where $\epsilon^X_1|\cdots|\epsilon^X_{m+n}$ are the invariant factors of

$$T(\lambda) = \left[\begin{array}{c|c} A(\lambda) & 0 \\ \hline XC(\lambda) & D(\lambda) \end{array}\right]. \tag{3.8}$$

Proof. Without loss of generality, consider the matrix $D(\lambda)$ in its SP canonical form, and the matrix $C(\lambda)$ in its SP-quasi canonical form. By the condition (3.7),

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and by applying Lemma 3.3 there exists $X_0 \in \mathbb{F}^{m\times a}$, such that $\begin{bmatrix} D(\lambda) & X_0C(\lambda) \end{bmatrix}$ is equivalent to $\begin{bmatrix} I_m & 0 \end{bmatrix}$. Even more, by Remark 3.4, for every $X \in G$, the matrix

$$\begin{bmatrix} D(\lambda) & XC(\lambda) \end{bmatrix} \text{ is equivalent to } \begin{bmatrix} I_m & 0 \end{bmatrix}. \tag{3.9}$$

Also, note that for every $X \in G$, the invariant factors of

$$\begin{bmatrix} A(\lambda) \\ XC(\lambda) \end{bmatrix}$$

are exactly $\mu_1|\cdots|\mu_n$. Indeed, the invariant factors of $\begin{bmatrix} A(\lambda) \\ C(\lambda) \end{bmatrix} \in \mathbb{F}[\lambda]^{(n+a)\times n}$ are the same as the invariant factors of $\begin{bmatrix} A(\lambda) \\ C(\lambda) \\ 0 \end{bmatrix} \in \mathbb{F}[\lambda]^{(n+m)\times n}$, since $a \le m$.

If $a = m$, then by the proof of the previous Lemma, every matrix $X \in G$ is invertible, and so the invariant factors of $\begin{bmatrix} A(\lambda) \\ C(\lambda) \end{bmatrix}$ and $\begin{bmatrix} A(\lambda) \\ XC(\lambda) \end{bmatrix}$ coincide.

If $a < m$, then (see case 2. in the previous lemma) we defined $X := Y \begin{bmatrix} I_a \\ 0 \end{bmatrix}$, where Y is an invertible matrix. Thus,

$$\begin{bmatrix} A(\lambda) \\ XC(\lambda) \end{bmatrix} = \begin{bmatrix} A(\lambda) \\ Y\begin{bmatrix} I_a \\ 0 \end{bmatrix} C(\lambda) \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ 0 & Y \end{bmatrix} \begin{bmatrix} A(\lambda) \\ C(\lambda) \\ 0 \end{bmatrix},$$

and so its invariant factors are $\mu_1|\cdots|\mu_n$.

Now, from (3.9), there exist invertible matrices $P(\lambda) \in \mathbb{F}[\lambda]^{m\times m}$, and

$$Q(\lambda) = \begin{bmatrix} Q_1(\lambda) & Q_2(\lambda) \\ Q_3(\lambda) & Q_4(\lambda) \end{bmatrix} \in \mathbb{F}[\lambda]^{(n+m)\times(n+m)}, \text{ where } Q_1(\lambda) \in \mathbb{F}[\lambda]^{n\times n},$$

such that

$$\begin{bmatrix} I & 0 \\ 0 & P(\lambda) \end{bmatrix} \left[\begin{array}{c|c} A(\lambda) & 0 \\ \hline XC(\lambda) & D(\lambda) \end{array}\right] Q(\lambda) = \begin{bmatrix} A(\lambda)Q_1(\lambda) & A(\lambda)Q_2(\lambda) \\ 0 & I \end{bmatrix}.$$

Thus, the invariant factors of $A(\lambda)Q_1(\lambda)$ are exactly $\epsilon^X_{m+1}|\cdots|\epsilon^X_{m+n}$. On the other hand, we have that

$$\begin{bmatrix} I & 0 \\ 0 & P(\lambda) \end{bmatrix} \left[\begin{array}{c|c} I & 0 \\ \hline XC(\lambda) & D(\lambda) \end{array}\right] Q(\lambda) = \begin{bmatrix} Q_1(\lambda) & Q_2(\lambda) \\ 0 & I \end{bmatrix}.$$

Hence, the nontrivial invariant factors of $Q_1(\lambda)$ coincide with the nontrivial invariant factors of $D(\lambda)$. So, the invariant factors of the matrix $Q_1(\lambda)$, denoted by $\beta'_1|\cdots|\beta'_n$, satisfy $\beta'_i = \beta_{i+m-n}$, $i = 1, \ldots, n$.

Now, by applying Proposition 2.4 to the matrix product $A(\lambda)Q_1(\lambda)$, we have that for every $X \in G$

$$\epsilon^X_{i+m} | \gcd(\alpha_i\beta_m, \ldots, \alpha_n\beta_{i+m-n}), \quad i = 1, \ldots, n. \tag{3.10}$$

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Denote by $\phi_i := \gcd(\alpha_i\beta_m, \ldots, \alpha_n\beta_{i+m-n})$, $i = 1, \ldots, n$. Let $\lambda_1^i, \ldots, \lambda_{k_i}^i \in \bar{\mathbb{F}}$ be distinct zeros of ϕ_i, $i = 1, \ldots, n$.

Let $l \in \{1, \ldots, k_i\}$. We shall show the following:

$(*)$ If λ_l^i is not a zero of the polynomial $\alpha_i \prod_{j=1}^m \gcd(\beta_j, \mu_{m+i-j+1})$, then for a generic $X \in G$, λ_l^i is not a zero of the corresponding ϵ_{m+i}^X.

This will obviously prove that for generic $X \in G$, for every $i = 1, \ldots, n$, every zero of the polynomial ϵ_{m+i}^X is a zero of the polynomial α_i or of the polynomial $\gcd(\beta_j, \mu_{m+i-j+1})$ for some $j = 1, \ldots, m$, as wanted.

Thus, we are left with proving $(*)$.

Let $i \in \{1, \ldots, n\}$, $l \in \{1, \ldots, k_i\}$, and λ_l^i be a zero of ϕ_i such that $\alpha_i(\lambda_l^i) \neq 0$ and $\gcd(\beta_j, \mu_{m+i-j+1})(\lambda_l^i) \neq 0$, for all $j = 1, \ldots, m$. Let

$$p = \min_{w=i,\ldots,n+1} \{w | \alpha_w(\lambda_l^i) = 0\}$$
$$t = \min_{w=i+m-n,\ldots,m+1} \{w | \beta_w(\lambda_l^i) = 0\}.$$

Since $\phi_i(\lambda_l^i) = 0$, we have $\alpha_{p-1}(\lambda_l^i) \neq 0 \Rightarrow \beta_{i+m-p+1}(\lambda_l^i) = 0$, and $\beta_{t-1}(\lambda_l^i) \neq 0 \Rightarrow \alpha_{i+m-t+1}(\lambda_l^i) = 0$, which gives $p + t \leq i + m + 1$.

Furthermore, since $\alpha_i(\lambda_l^i) \neq 0$ we have $p > i$ and since $\gcd(\beta_{i+m-n}, \mu_{n+1})(\lambda_l^i) = \beta_{i+m-n}(\lambda_l^i) \neq 0$, we have $t > i + m - n$. Also, since $\beta_t(\lambda_l^i) = 0$, we must have $\mu_{m+i-t+1}(\lambda_l^i) \neq 0$.

Consider the following equivalent form of the matrix $T(\lambda_l^i) \in \mathbb{F}^{(n+m)\times(n+m)}$:

$$\left[\begin{array}{cc|cc} \operatorname{diag}(\alpha_1(\lambda_l^i), \ldots, \alpha_{p-1}(\lambda_l^i)) & 0 & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ 0 & 0 & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{$X\bar{C}(\lambda_l^i)$}} & \operatorname{diag}(\beta_1(\lambda_l^i), \ldots, \beta_{t-1}(\lambda_l^i)) & 0 \\ & & 0 & 0 \end{array}\right], \tag{3.11}$$

where $\bar{C}(\lambda_l^i) \in \mathbb{F}^{a\times n}$.

Since $X = PX_0$, $P \in S$ (see Remark 3.4), the matrix (3.11) becomes

$$\left[\begin{array}{cc|cc} \operatorname{diag}(\alpha_1(\lambda_l^i), \ldots, \alpha_{p-1}(\lambda_l^i)) & 0 & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ 0 & 0 & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{$P\begin{bmatrix} A & B \\ C & D \end{bmatrix}$}} & \operatorname{diag}(\beta_1(\lambda_l^i), \ldots, \beta_{t-1}(\lambda_l^i)) & 0 \\ & & 0 & 0 \end{array}\right], \tag{3.12}$$

where

$$X_0\bar{C}(\lambda_l^i) = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathbb{F}^{m\times n}, \quad A \in \mathbb{F}^{(t-1)\times(p-1)}.$$

Let

$$\begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} := P \begin{bmatrix} A & B \\ C & D \end{bmatrix}, D' \in \mathbb{F}^{(m-t+1)\times(n-p+1)}.$$

Since, $\mu_{m+i-t+1}(\lambda_j^i) \neq 0$, we have

$$\operatorname{rank}\begin{bmatrix} B \\ D \end{bmatrix} \geq \operatorname{rank}\begin{bmatrix} B' \\ D' \end{bmatrix} = \operatorname{rank}\begin{bmatrix} A(\lambda_l^i) \\ XC(\lambda_l^i) \end{bmatrix} - p + 1 =$$

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$$= \operatorname{rank} \begin{bmatrix} \mu_1(\lambda_l^i) & & \\ & \ddots & \\ & & \mu_n(\lambda_l^i) \end{bmatrix} - p + 1 \geq m + i - t - p + 2 \quad (\geq 1).$$

On the other hand, $\epsilon^X_{m+i}(\lambda_l^i) \neq 0$ is equivalent to

$$\operatorname{rank} D' \geq m + i - t - p + 2. \tag{3.13}$$

Indeed, this is because the rank of the matrix (3.11) is equal to $p+t-2+\operatorname{rank} D'$. Since $p \geq i+1$ and $t \geq i+m-n+1$, we have

$$\min\{m - t + 1, n - p + 1\} \geq m + i - p - t + 2.$$

Thus, for a generic matrix $X \in G$, we have that (3.13) is valid, which finishes our proof. □

4. Main result. The following theorem gives our main result:

THEOREM 4.1. *Let $\mathbb{F}$ be an infinite field. Let $A_i \in \mathbb{F}^{n_i \times n_i}$, $B_i \in \mathbb{F}^{n_i \times m_i}$, $i = 1, \ldots, m$, $C_i \in \mathbb{F}^{p_i \times n_i}$, $i = 1, \ldots, m-1$. There exist matrices $X_i \in \mathbb{F}^{m_{i+1} \times p_i}$, $i = 1, \ldots, m-1$, such that*

$$M = \left[\begin{array}{ccccc|c} A_1 & 0 & 0 & \ddots & 0 & B_1 \\ B_2X_1C_1 & A_2 & 0 & \ddots & 0 & 0 \\ 0 & B_3X_2C_2 & A_3 & \ddots & 0 & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & B_mX_{m-1}C_{m-1} & A_m & 0 \end{array}\right] \tag{4.1}$$

is controllable if and only if:

$$(A_i, B_i) \text{ are controllable for all } \quad i = 1, \ldots, m, \tag{4.2}$$

and

$$\gcd(\gamma^i_{k_1}, \mu^{i+1}_{k_2}, \ldots, \mu^{j-1}_{k_{j-i}}, \alpha^j_{k_{j-i+1}}) = 1, \quad 1 \leq i < j \leq m, \tag{4.3}$$

for all indices $k_1, \ldots, k_{j-i+1}$ such that

$$k_1 + \cdots + k_{j-i+1} \leq n_i + \cdots + n_j + j - i.$$

Here $\gamma^i_1 | \cdots | \gamma^i_{y_i}$ are the invariant factors of

$$\begin{bmatrix} \lambda I - A_i & -B_i \\ -C_i & 0 \end{bmatrix}, \quad i = 1, \ldots, m-1, \tag{4.4}$$

y_i is its rank, $\alpha^i_1 | \cdots | \alpha^i_{n_i}$ are the invariant factors of $\lambda I - A_i$, $i = 1, \ldots, m$, and $\mu^i_1 | \cdots | \mu^i_{n_i}$ are the invariant factors of

$$\begin{bmatrix} \lambda I - A_i \\ -C_i \end{bmatrix}, \quad i = 1, \ldots, m-1.$$

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Proof.

Necessity:

From the controllability of (4.1), we have that the pair (A_1, B_1) is controllable and $(A_i, B_iX_{i-1}C_{i-1})$ are controllable for all $i = 2, \dots, m$. By applying Lemma 3.2, we obtain the condition (4.2).

Furthermore, there exist invertible matrices $P_i \in \mathbb{F}^{n_i\times n_i}$, such that

$$P_iB_i = \begin{bmatrix} T_i \\ 0 \end{bmatrix}, \quad T_i \in \mathbb{F}^{\operatorname{rank} B_i\times m_i}, \quad i = 1, \dots, m.$$

Let

$$P_iA_iP_i^{-1} = \begin{bmatrix} A_1^i & A_2^i \\ A_3^i & A_4^i \end{bmatrix}, \quad A_1^i \in \mathbb{F}^{\operatorname{rank} B_i\times \operatorname{rank} B_i}, \quad i = 1, \dots, m.$$

Then (A_4^i, A_3^i) is controllable (moreover the controllability of (A_4^i, A_3^i) is equivalent to the controllability of (A_i, B_i)) and there exist invertible matrices $Q_i(\lambda) \in \mathbb{F}[\lambda]^{n_i\times n_i}$ and $S_i(\lambda) \in \mathbb{F}[\lambda]^{n_i\times n_i}$, $i = 1, \dots, m$, such that

$$Q_i(\lambda)(\lambda I - P_iA_iP_i^{-1})S_i(\lambda) = \begin{bmatrix} A_i(\lambda) & 0 \\ 0 & I \end{bmatrix},$$

where $A_i(\lambda) \in \mathbb{F}[\lambda]^{\operatorname{rank} B_i\times \operatorname{rank} B_i}$, $i = 1, \dots, m$. Note that the first $\operatorname{rank} B_i$ columns of the matrices $Q_i(\lambda)$, $i = 1, \dots, m$ are of the form $\begin{bmatrix} I_{\operatorname{rank} B_i} \\ 0 \end{bmatrix}$. Denote the invariant factors of $A_i(\lambda)$ by $\alpha_1'^i | \cdots | \alpha_{\operatorname{rank} B_i}'^i$, then $\alpha_j'^i := \alpha^i_{j+n_i-\operatorname{rank} B_i}$, $j = 1, \dots, \operatorname{rank} B_i$, $i = 1, \dots, m$.

Let

$$Y_i := T_{i+1}X_i \in \mathbb{F}^{\operatorname{rank} B_{i+1}\times p_i}, \quad i = 1, \dots, m - 1.$$

Let $\bar{P}_i \in \mathbb{F}^{m_i\times m_i}$ be the invertible matrices such that $-T_i\bar{P}_i = \begin{bmatrix} I_{\operatorname{rank} B_i} & 0 \end{bmatrix}$, $i = 1, \dots, m$. Denote by $P = \operatorname{diag}(P_1, \dots, P_m)$, $Q(\lambda) = \operatorname{diag}(Q_1(\lambda), \dots, Q_m(\lambda))$, $\bar{P} = \operatorname{diag}(P_1^{-1}, \dots, P_m^{-1}, \bar{P}_1)$ and $S(\lambda) = \operatorname{diag}(S_1(\lambda), \dots, S_m(\lambda), I)$.

Furthermore, let $-C_iP_i^{-1}S_i(\lambda) = \begin{bmatrix} C_i(\lambda) & C_i'(\lambda) \end{bmatrix}$, $C_i(\lambda) \in \mathbb{F}[\lambda]^{p_i\times \operatorname{rank} B_i}$, $i = 1, \dots, m - 1$.

Now, consider the matrix

$$M(\lambda) = Q(\lambda)P(\lambda \begin{bmatrix} I & 0 \end{bmatrix} - M)\bar{P}S(\lambda).$$

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The matrix $M(\lambda)$ has the following form

$$
\left[\begin{array}{cc|cc|cc|cc|cc|cc}
A_1(\lambda) & 0 & & & & & & & & & I & 0\\
0 & I & & & & & & & & & 0 & 0\\ \hline
Y_1C_1(\lambda) & Y_1C_1'(\lambda) & A_2(\lambda) & 0 & & & & & & & & \\
0 & 0 & 0 & I & & & & & & & & \\ \hline
 & & Y_2C_2(\lambda) & Y_2C_2'(\lambda) & A_3(\lambda) & 0 & & & & & & \\
 & & 0 & 0 & 0 & I & & & & & & \\ \hline
 & & & & & \ddots & \ddots & & & & & \\
 & & & & & & \ddots & \ddots & & & & \\ \hline
 & & & & & & Y_{m-1}C_{m-1}(\lambda) & Y_{m-1}C_{m-1}'(\lambda) & A_m(\lambda) & & & \\
 & & & & & & 0 & 0 & 0 & I & &
\end{array}\right].
$$

Thus, the matrix $\lambda\left[\begin{array}{cc} I & 0 \end{array}\right] - M$ is equivalent to

$$
\left[\begin{array}{ccccc|cc|c}
Y_1C_1(\lambda) & A_2(\lambda) & & & & & & \\
 & Y_2C_2(\lambda) & A_3(\lambda) & & & & & \\
 & & Y_3C_3(\lambda) & A_4(\lambda) & & & & \\
 & & & \ddots & \ddots & & & \\
 & & & & Y_{m-2}C_{m-2}(\lambda) & A_{m-1}(\lambda) & & \\
 & & & & & Y_{m-1}C_{m-1}(\lambda) & A_m(\lambda) & \\ \hline
 & & & & & & & I
\end{array}\right], \tag{4.5}
$$

where nonmarked entries are equal to zero.

Since the matrix

$$
\left[\begin{array}{cc} Q_i(\lambda)P_i & 0 \\ 0 & I \end{array}\right]
\left[\begin{array}{cc} \lambda I - A_i & -B_i \\ -C_i & 0 \end{array}\right]
\left[\begin{array}{cc} P_i^{-1}S_i(\lambda) & 0 \\ 0 & \bar{P}_i \end{array}\right]
$$

is equal to the following one

$$
\left[\begin{array}{cc|cc} A_i(\lambda) & 0 & I & 0 \\ 0 & I & 0 & 0 \\ \hline C_i(\lambda) & C_i'(\lambda) & 0 & 0 \end{array}\right], \quad i = 1, \dots, m-1,
$$

the invariant factors of the matrix $C_i(\lambda)$, denoted by $\gamma_1'^i | \cdots | \gamma_{y_i - n_i}'^i$, satisfy $\gamma_j'^i := \gamma_{j+n_i}^i$, $j = 1, \dots, y_i - n_i$, $i = 1, \dots, m-1$.

Moreover, if denote by $\mu_1'^i | \cdots | \mu_{\operatorname{rank} B_i}'^i$ the invariant factors of

$$
\left[\begin{array}{c} A_i(\lambda) \\ C_i(\lambda) \end{array}\right] \in \mathbb{F}[\lambda]^{(\operatorname{rank} B_i + p_i) \times \operatorname{rank} B_i}, \quad i = 2, \dots, m-1,
$$

from

$$
\left[\begin{array}{cc} Q_i(\lambda)P_i & 0 \\ 0 & I \end{array}\right]
\left[\begin{array}{c} \lambda I - A_i \\ -C_i \end{array}\right] S_i(\lambda) =
\left[\begin{array}{cc} A_i(\lambda) & 0 \\ 0 & I \\ \hline C_i(\lambda) & C_i'(\lambda) \end{array}\right],
$$

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they satisfy $\mu_j'^i := \mu^i_{j+n_i-\operatorname{rank} B_i}$, $j = 1, \dots, \operatorname{rank} B_i$, $i = 2, \dots, m-1$.

Now the condition (4.3) becomes

$$\gcd(\gamma'^i_{k_1}, \mu'^{i+1}_{k_2}, \dots, \mu'^{j-1}_{k_{j-i}}, \alpha'^j_{k_{j-i+1}}) = 1, \quad 1 \le i < j \le m, \tag{4.6}$$

for all $k_1, \dots, k_{j-i+1}$ *such that*

$$k_1 + \cdots + k_{j-i+1} \le \operatorname{rank} B_{i+1} + \cdots + \operatorname{rank} B_j + j - i.$$

Since the matrix (4.5) is equivalent to $\begin{bmatrix} I & 0 \end{bmatrix}$, every submatrix formed by some of its rows is also equivalent to $\begin{bmatrix} I & 0 \end{bmatrix}$. Let i and j be such that $1 \le i < j \le m$. Consider the submatrix

$$R(\lambda) = \begin{bmatrix} Y_iC_i(\lambda) & A_{i+1}(\lambda) & & & & \\ & Y_{i+1}C_{i+1}(\lambda) & A_{i+2}(\lambda) & & & \\ & & Y_{i+2}C_{i+2}(\lambda) & A_{i+3}(\lambda) & & \\ & & & \ddots & \ddots & \\ & & & & Y_{j-1}C_{j-1}(\lambda) & A_j(\lambda) \end{bmatrix}. \tag{4.7}$$

Let $k_1, \dots, k_{j-i+1}$ be arbitrary indices such that the polynomials $\gamma'^i_{k_1}, \mu'^{i+1}_{k_2}, \dots,$ $\dots, \mu'^{j-1}_{k_{j-i}}, \alpha'^j_{k_{j-i+1}}$ have a common zero $\lambda_0 \in \bar{\mathbb{F}}$. Then, since $R(\lambda)$ is equivalent to $\begin{bmatrix} I & 0 \end{bmatrix}$, we have

$$\operatorname{rank} R(\lambda_0) = \operatorname{rank} B_{i+1} + \cdots + \operatorname{rank} B_j.$$

On the other hand, from the form of $R(\lambda)$, we have

$$\operatorname{rank} R(\lambda_0) \le \operatorname{rank} C_i(\lambda_0) + \sum_{l=i+1}^{j-1} \operatorname{rank} \begin{bmatrix} A_l(\lambda_0) \\ C_l(\lambda_0) \end{bmatrix} + \operatorname{rank} A_j(\lambda_0) \le$$

$$\le k_1 - 1 + \sum_{l=2}^{j-i} (k_l - 1) + k_{j-i+1} - 1 = k_1 + \cdots + k_{j-i+1} - (j - i + 1),$$

as wanted.

Sufficiency:

Since (A_i, B_i) is controllable for every $i = 1, \dots, m$, as in the necessity part of the proof, the matrix (4.1) is equivalent to the matrix (4.5). Thus, it is enough to define $Y_1, \dots, Y_{m-1}$ over $\mathbb{F}$, such that the matrix (4.5) is equivalent to $\begin{bmatrix} I & 0 \end{bmatrix}$, when the condition (4.6) is satisfied.

Further proof goes by induction on m. For $m = 2$, the condition (4.6) becomes

$$\gcd(\gamma'^1_i, \alpha'^2_{\operatorname{rank} B_2+1-i}) = 1, \quad \text{for all } i = 1, \dots, \operatorname{rank} B_2,$$

and so by Lemma 3.3, there exists a matrix $Y_{m-1} \in \mathbb{F}^{\operatorname{rank} B_m \times p_{m-1}}$, such that

$$\begin{bmatrix} A_m(\lambda) & Y_{m-1}C_{m-1}(\lambda) \end{bmatrix}$$

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is equivalent to $\begin{bmatrix} I_{\operatorname{rank} B_m} & 0 \end{bmatrix}$.

Now suppose that the condition is sufficient for $m-1$ and we shall prove that it is sufficient for m. Consider the matrix

$$\begin{bmatrix} A_{m-1}(\lambda) & 0 \\ Y_{m-1}C_{m-1}(\lambda) & A_m(\lambda) \end{bmatrix}. \tag{4.8}$$

If $p_{m-1} \leq \operatorname{rank} B_m$, by Lemma 3.5, there exists a matrix Y_{m-1} such that the matrix

$$\begin{bmatrix} A_m(\lambda) & Y_{m-1}C_{m-1}(\lambda) \end{bmatrix}$$

is equivalent to $\begin{bmatrix} I_{\operatorname{rank} B_m} & 0 \end{bmatrix}$, and every zero of the polynomial $\epsilon_{i+\operatorname{rank} B_m}$ is the zero of the polynomial $\alpha_i'^{m-1}$ or of the polynomial $\gcd(\alpha_j'^m, \mu'^{m-1}_{\operatorname{rank} B_m+i-j+1})$, for some $j = 1, \ldots, \operatorname{rank} B_m$, where $\epsilon_1|\cdots|\epsilon_{\operatorname{rank} B_m+\operatorname{rank} B_{m-1}}$ are the invariant factors of (4.8).

If $p_{m-1} > \operatorname{rank} B_m$, instead of $A_m(\lambda)$ consider the matrix $\bar{A}_m(\lambda) := A_m(\lambda) \oplus I_{p_{m-1}-\operatorname{rank} B_m}$. Now, again by Lemma 3.5, there exists a matrix $Y'_{m-1} \in \mathbb{F}^{p_{m-1}\times p_{m-1}}$ such that

$$\begin{bmatrix} \bar{A}_m(\lambda) & Y'_{m-1}C_{m-1}(\lambda) \end{bmatrix}$$

is equivalent to $\begin{bmatrix} I_{p_{m-1}} & 0 \end{bmatrix}$. Then define $Y_{m-1} := \begin{bmatrix} I_{\operatorname{rank} B_m} & 0 \end{bmatrix} Y'_{m-1}$.

In both cases, the matrix (4.8) is equivalent to the matrix

$$\begin{bmatrix} A'_{m-1}(\lambda) & 0 \\ 0 & I_{\operatorname{rank} B_m} \end{bmatrix}, \text{ for some } A'_{m-1}(\lambda) \in \mathbb{F}[\lambda]^{\operatorname{rank} B_{m-1}\times \operatorname{rank} B_{m-1}}.$$

Note that the invariant factors of $A'_{m-1}(\lambda)$, denoted by $\epsilon'_1|\cdots|\epsilon'_{\operatorname{rank} B_{m-1}}$, satisfy $\epsilon'_i = \epsilon_{i+\operatorname{rank} B_m}$, $i = 1, \ldots, \operatorname{rank} B_{m-1}$.

Denote the submatrix of (4.5) formed by the rows $1, \ldots, \sum_{i=2}^{m-1} \operatorname{rank} B_i$ and by the columns $1, \ldots, \sum_{i=1}^{m-2} \operatorname{rank} B_i$, by E. Now, our problem reduces to defining the matrices $Y_1, \ldots, Y_{m-2}$ such that the matrix

$$\left[\begin{array}{c|c} E & \begin{matrix} 0 \\ A'_{m-1}(\lambda) \end{matrix} \end{array}\right]$$

is equivalent to $\begin{bmatrix} I & 0 \end{bmatrix}$.

In order to apply the induction hypothesis, and thus to finish the proof, we need to prove the validity of the following condition

$$\gcd(\gamma'^i_{k_1}, \mu'^{i+1}_{k_2}, \ldots, \mu'^{m-2}_{k_{m-i-1}}, \epsilon'_{k_{m-i}}) = 1, \tag{4.9}$$

for every $i = 1, \ldots, m-2$ and for all indices $k_1, \ldots, k_{m-i}$ such that

$$k_1 + \cdots + k_{m-i} \leq \operatorname{rank} B_{i+1} + \cdots + \operatorname{rank} B_{m-1} + m - i - 1.$$

Suppose that the condition (4.9) is not valid. Then there exists $\lambda_0 \in \bar{\mathbb{F}}$, a common zero of the polynomials $\gamma'^i_{k_1}, \mu'^i_{k_2}, \ldots, \mu'^{m-2}_{k_{m-i-1}}$ and $\epsilon'_{k_{m-i}}$ for some indices $k_1, \ldots, k_{m-i}$ satisfying $k_1 + \cdots + k_{m-i} \leq \operatorname{rank} B_{i+1} + \cdots + \operatorname{rank} B_{m-1} + m - i - 1$.

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Hence, λ_0 is a zero of the polynomial $\epsilon_{k_{m-i}+\operatorname{rank} B_m}$. Now, since Y_{m-1} is defined by Lemma 3.5, λ_0 is a zero of the polynomial $\alpha'^{m-1}_{k_{m-i}}$ or of the polynomial $\gcd(\mu'^{m-1}_{\operatorname{rank} B_m+k_{m-i}-l+1}, \alpha'^{m}_{l})$, for some index $l \in \{1, \dots, \operatorname{rank} B_m\}$.

If $\alpha'^{m-1}_{k_{m-i}}(\lambda_0) = 0$, then

$$\gcd(\gamma'^{i}_{k_1}, \mu'^{i+1}_{k_2}, \dots, \mu'^{m-2}_{k_{m-i-1}}, \alpha'^{m-1}_{k_{m-i}}) \neq 1$$

which is a contradiction by (4.6).

If $\gcd(\mu'^{m-1}_{\operatorname{rank} B_m+k_{m-i}-l+1}, \alpha'^{m}_{l})(\lambda_0) = 0$, then

$$\gcd(\gamma'^{i}_{k_1}, \mu'^{i+1}_{k_2}, \dots, \mu'^{m-2}_{k_{m-i-1}}, \mu'^{m-1}_{\operatorname{rank} B_m+k_{m-i}-l+1}, \alpha'^{m}_{l}) \neq 1,$$

which is again a contradiction. Thus, (4.9) is valid, as wanted. ∎

5. Special cases. In this section we study some special cases of the Problem 1.2 over arbitrary fields.

THEOREM 5.1. *Let $\mathbb{F}$ be a field. Let $A_i \in \mathbb{F}^{n_i \times n_i}$, $B_i \in \mathbb{F}^{n_i \times m_i}$, $i = 1, \dots, m$, $C_i \in \mathbb{F}^{p_i \times n_i}$, $i = 1, \dots, m-1$. Let $\operatorname{rank} B_i = 1$, $i = 1, \dots, m$, and $\operatorname{rank} C_i = 1$, $i = 1, \dots, m-1$. There exist matrices $X_i \in \mathbb{F}^{m_{i+1} \times p_i}$, $i = 1, \dots, m-1$, such that*

$$\left[\begin{array}{ccccc|c} A_1 & 0 & 0 & \ddots & 0 & B_1 \\ B_2X_1C_1 & A_2 & 0 & \ddots & 0 & 0 \\ 0 & B_3X_2C_2 & A_3 & \ddots & 0 & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & B_mX_{m-1}C_{m-1} & A_m & 0 \end{array}\right] \tag{5.1}$$

is controllable, if and only if

$$\begin{array}{ll} (i) & (A_i, B_i) \text{ is controllable}, \quad i = 1, \dots, m, \\ (ii) & \gcd(\gamma^i_{n_i+1}, \alpha^j_{n_j}) = 1, \quad 1 \leq i < j \leq m, \end{array}$$

where $\gamma^i_1 | \cdots | \gamma^i_{n_i+1}$ are the invariant factors of the matrix

$$\begin{bmatrix} \lambda I - A_i & -B_i \\ -C_i & 0 \end{bmatrix}, \tag{5.2}$$

$i = 1, \dots, m-1$. Also, $\alpha^j_{n_j}$ is the only nontrivial invariant factor of $\lambda I - A_j$, $j = 1, \dots, m$.

Proof. First, since $\operatorname{rank} C_i = 1$, $i = 1, \dots, m-1$, there exist invertible matrices $P_i \in \mathbb{F}^{p_i \times p_i}$ such that

$$\bar{C}_i = P_iC_i = \begin{bmatrix} c_i \\ 0 \end{bmatrix}, \text{ where } c_i \in \mathbb{F}^{1 \times n_i}, \quad i = 1, \dots, m-1.$$

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Let

$$\bar{X}_i = X_i P_i^{-1}, \quad i = 1, \ldots, m-1.$$

Further on, instead of matrix (5.1), we shall consider the matrix

$$M = \left[\begin{array}{ccccc|c} A_1 & 0 & 0 & \ddots & 0 & B_1 \\ B_2\bar{X}_1\bar{C}_1 & A_2 & 0 & \ddots & 0 & 0 \\ 0 & B_3\bar{X}_2\bar{C}_2 & A_3 & \ddots & 0 & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & B_m\bar{X}_{m-1}\bar{C}_{m-1} & A_m & 0 \end{array}\right]. \tag{5.3}$$

Necessity:

Suppose that there exist matrices $\bar{X}_1, \ldots, \bar{X}_{m-1}$ such that the matrix (5.3) is controllable. Like in Theorem 4.1, we obtain the condition (i). Also, note that the fact that rank of the matrix (5.2) is equal to $n_i + 1$ follows from the controllability of the pair (A_i, B_i), $i = 1, \ldots, m$.

Furthermore, we shall consider the matrix $\lambda \begin{bmatrix} I & 0 \end{bmatrix} - M$. As in Theorem 4.1, matrix $\lambda \begin{bmatrix} I & 0 \end{bmatrix} - M$ is equivalent to the matrix (4.5), and since in this case $\operatorname{rank} B_i = 1$, $i = 1, \ldots, m$, the matrix (4.5) is of the following form

$$\left[\begin{array}{cccccc|c} q_1 & \alpha_{n_2}^2 & & & & & \\ & q_2 & \alpha_{n_3}^3 & & & & \\ & & q_3 & \ddots & & & \\ & & & \ddots & \ddots & & \\ & & & & q_{m-1} & \alpha_{n_m}^m & \\ \hline & & & & & & I \end{array}\right], \tag{5.4}$$

where the polynomials $\alpha_{n_i}^i$ are the only nontrivial invariant factors of the matrices $\lambda I - A_i$, $i = 2, \ldots, m$, and the polynomials q_i, $i = 1, \ldots, m-1$ (see the proof of Theorem 4.1) are the last nonzero invariant factors of the matrices

$$\begin{bmatrix} \lambda I - A_i & -B_i \\ -\bar{C}_i & 0 \end{bmatrix}, \tag{5.5}$$

i.e.

$$q_i = \gamma_{n_i+1}^i, \quad i = 1, \ldots, m-1.$$

Hence, we have that the matrix

$$N = \begin{bmatrix} \gamma_{n_1+1}^1 & \alpha_{n_2}^2 & & & & \\ & \gamma_{n_2+1}^2 & \alpha_{n_3}^3 & & & \\ & & \gamma_{n_3+1}^3 & \ddots & & \\ & & & \ddots & \ddots & \\ & & & & \gamma_{n_{m-1}+1}^{m-1} & \alpha_{n_m}^m \end{bmatrix} \tag{5.6}$$

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has only trivial invariant factors. Thus, $D_{m-1} = 1$ (D_{m-1} is the $(m-1)$th determinantal divisor of (5.6), i.e. the greatest common divisor of minors of order $m-1$ of (5.6)). We shall prove that this is equivalent to the condition (ii).

Indeed, since

$$D_{m-1} = \gcd(\Pi_{m-1}, \Pi_{m-2}, \ldots, \Pi_0),$$

where

$$\Pi_i := \gamma^1_{n_1+1} \ldots \gamma^i_{n_i+1} \alpha^{i+2}_{n_{i+2}} \ldots \alpha^m_{n_m}, \quad i = 0, \ldots, m-1,$$

we shall prove that:

$$D_{m-1} = 1 \Leftrightarrow \gcd(\gamma^i_{n_i+1}, \alpha^j_{n_j}) = 1,\ 1 \leq i < j \leq m, \text{ i.e.,}$$
$$D_{m-1} \neq 1 \Leftrightarrow \exists i, j : 1 \leq i < j \leq m, \text{ such that } \gcd(\gamma^i_{n_i+1}, \alpha^j_{n_j}) \neq 1.$$

Let $D_{m-1} \neq 1$. Let $\lambda_0 \in \bar{\mathbb{F}}$ be a common zero of $\Pi_{m-1}, \ldots, \Pi_0$. Since Π_{m-1} is the product only of $\gamma^i_{n_i+1}$'s, at least one of $\gamma^i_{n_i+1}$, $i = 1, \ldots, m-1$, must have λ_0 as its zero. Let $k := \min\{i | \gamma^i_{n_i+1}(\lambda_0) = 0\}$, $1 \leq k \leq m-1$. Then $\Pi_{k-1} = \gamma^1_{n_1+1} \cdots \gamma^{k-1}_{n_{k-1}+1} \alpha^{k+1}_{n_{k+1}} \cdots \alpha^m_{n_m}$. Thus, there exists $j > k$ such that $\alpha^j_{n_j}(\lambda_0) = 0$. Then, obviously, $\gcd(\gamma^k_{n_k+1}, \alpha^j_{n_j}) \neq 1$, as wanted.

Conversely, suppose that there exist indices i and j such that $1 \leq i < j \leq m$ and $\gcd(\gamma^i_{n_i+1}, \alpha^j_{n_j}) \neq 1$, i.e., $\exists \lambda_0 \in \bar{\mathbb{F}}$ such that $\gamma^i_{n_i+1}(\lambda_0) = 0$ and $\alpha^j_{n_j}(\lambda_0) = 0$. Then every Π_l, $l = 0, \ldots, m-1$, has λ_0 as its zero. Indeed, if $i \leq l$, then since $\gamma^i_{n_i+1}(\lambda_0) = 0$, we have $\Pi_l(\lambda_0) = 0$. If $i > l$, then $j > i \geq l+1$ and since $\alpha^j_{n_j}(\lambda_0) = 0$, we have $\Pi_l(\lambda_0) = 0$.

Sufficiency:

Let the conditions (i) and (ii) be valid. Then as in the proof of Theorem 4.1, we can consider the matrix $\lambda \begin{bmatrix} I & 0 \end{bmatrix} - M$ in the equivalent form (4.5). Define

$$Y_i = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{F}^{1 \times p_i}, \quad i = 1, \ldots, m-1.$$

Then (4.5) becomes

$$\begin{bmatrix} N & 0 \\ 0 & I \end{bmatrix}. \tag{5.7}$$

In the necessity part of the proof, we have proved that the condition (ii) is equivalent to the fact that the matrix N has all invariant factors equal to 1. Thus, the matrix (5.7) has all invariant factors equal to 1, as wanted. ∎

In the following theorem, we consider the series connection of the linear systems S_i, $i = 1, \ldots, m$, in the case when $\operatorname{rank} B_i = n_i$, $i = 2, \ldots, m$, and $\operatorname{rank} C_i = n_i$, $i = 1, \ldots, m-1$.

THEOREM 5.2. *Let $\mathbb{F}$ be a field. Let $A_i \in \mathbb{F}^{n_i \times n_i}$, $i = 1, \ldots, m$, $B_1 \in \mathbb{F}^{n_1 \times m_1}$, $\operatorname{rank} B_1 = s$. Let $l_i \geq 0$, $i = 1, \ldots, m-1$. There exist matrices $X_i \in \mathbb{F}^{n_{i+1} \times n_i}$,*

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$i = 1, \ldots, m-1$, *such that the matrix*

$$\left[\begin{array}{ccccc|c} A_1 & 0 & 0 & \ddots & 0 & B_1 \\ X_1 & A_2 & 0 & \ddots & 0 & 0 \\ 0 & X_2 & A_3 & \ddots & 0 & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & X_{m-1} & A_m & 0 \end{array}\right] \tag{5.8}$$

is controllable and such that $\operatorname{rank} X_i \leq l_i$, $i = 1, \ldots, m-1$, *if and only if*

(i) (A_1, B_1) *is controllable*

(ii) $\min\{s, l_i, n_i\} \geq \max_{j=i+1,\ldots,m} \{r_j\}, \quad i = 1, \ldots, m-1.$

Here, by r_i *we have denoted the number of nontrivial invariant factors of* $\lambda I - A_i$, $i = 1, \ldots, m$.

Proof.

Necessity:

Since the matrix (5.8) is controllable, we directly obtain the condition (i). Also, considering the submatrices of (5.8) formed by its last $\sum_{i=1}^{j} n_{m-i+1}$, $j = 1, \ldots, m$, rows, we can apply the result from Theorem 1 in [8] and thus obtain that $s \geq \max_{j=2,\ldots,m}\{r_j\}$, and $\operatorname{rank} X_i \geq \max_{j=i+1,\ldots,m}\{r_j\}$, $i = 1, \ldots, m-1$. Thus, we obtain the condition (ii).

Sufficiency:

Let $c_1 \geq \cdots \geq c_s > 0$ be the nonzero controllability indices of the pair (A_1, B_1). Let $\alpha_1^i | \cdots | \alpha_{n_i}^i$ be the invariant factors of $\lambda I - A_i$, r_i of them nontrivial, and let $D_j^i = d(\alpha_{n_i-j+1}^i)$, $j = 1, \ldots, r_i$, $i = 2, \ldots, m$. Then the matrix (5.8) is feedback equivalent to the following one

$$\left[\begin{array}{ccccc|c} A_c & 0 & 0 & \ddots & 0 & B_c \\ X_1' & N(A_2) & 0 & \ddots & 0 & 0 \\ 0 & X_2' & N(A_3) & \ddots & 0 & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & X_{m-1}' & N(A_m) & 0 \end{array}\right] \tag{5.9}$$

where (A_c, B_c) is the Brunovsky canonical form of the pair (A_1, B_1) and

$$N(A_i) = C(\alpha_{n_i-r_i+1}^i) \oplus \cdots \oplus C(\alpha_{n_i}^i),$$

is the normal form for similarity of the matrix A_i, $i = 2, \ldots, m$, see, e.g. [5].

Now, our problem is equivalent to the problem of defining matrices X_i', $i = 1, \ldots, m-1$, such that the matrix (5.9) is controllable, and such that $\operatorname{rank} X_i' \leq l_i$, $i = 1, \ldots, m-1$.

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Let $l'_i := \min\{s, l_1, \ldots, l_i, n_i, n_{i+1}\}$, $i = 1, \ldots, m-1$. We shall define inductively matrices X'_i such that $\operatorname{rank} X'_i = l'_i$, $i = 1, \ldots, m-1$. From the condition (ii) we have

$$l_i \geq \max\{r_{i+1}, \ldots, r_m\}, \quad i = 1, \ldots, m-1. \tag{5.10}$$

First we define X'_1. Let $b^1_j = \sum_{i=j}^{r_2} D^2_i$, $j = 1, \ldots, r_2$. Put r_2 units in the matrix X'_1 at the positions

$$(b^1_j, \sum_{i=1}^{j-1} c_i + 1), \quad j = 1, \ldots, r_2.$$

Moreover, let $b^1_{r_2+1}, \ldots, b^1_{l'_1}$ be any $l'_1 - r_2$ distinct numbers from the set $\{1, \ldots, n_2\} \setminus \{b^1_1, \ldots, b^1_{r_2}\}$. Then put $l'_1 - r_2$ units at the positions

$$(b^1_j, \sum_{i=1}^{j-1} c_i + 1), \quad j = r_2 + 1, \ldots, l'_1,$$

while all other entries in X'_1 we put to be zeros. Obviously $\operatorname{rank} X'_1 = l'_1$.

Inductively, we define matrices X'_j, $j = 2, \ldots, m-1$:

Let $b^j_k = \sum_{i=k}^{r_{j+1}} D^{j+1}_i$, $k = 1, \ldots, r_{j+1}$. Moreover, let $b^j_{r_{j+1}+1}, \ldots, b^j_{l'_j}$ be any $l'_j - r_{j+1}$ distinct numbers from the set $\{1, \ldots, n_j\} \setminus \{b^j_1, \ldots, b^j_{r_{j+1}}\}$. Now, put l'_j units in the matrix X'_j in the rows $b^j_1, \ldots, b^j_{l'_j}$ such that they belong to any l'_j different columns among the following ones:

$$\{b^{j-1}_i + 1, \quad i = 2, \ldots, l'_{j-1}\} \cup \{1\},$$

while all other entries in X'_j, $j = 2, \ldots, m-1$, we put to be zeros.

Such obtained matrix

$$\left[\begin{array}{ccccc|c} A_c & 0 & 0 & \ddots & 0 & B_c \\ X'_1 & N(A_2) & 0 & \ddots & 0 & 0 \\ 0 & X'_2 & N(A_3) & \ddots & 0 & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & X'_{m-1} & N(A_m) & 0 \end{array}\right] \tag{5.11}$$

is controllable, and $\operatorname{rank} X'_i = l'_i \leq l_i$, $i = 1, \ldots, m-1$, as wanted. $\square$

In order to clarify the way of defining the matrices $X'_1, \ldots, X'_{m-1}$ in the previous theorem we give the following example:

Example 5.3. Let $m = 3$, $n_1 = 4$, $n_2 = 5$ and $n_3 = 4$. Let $l_1 = 4$, $l_2 = 2$ and $s = 2$. Let (A_1, B_1) be a controllable pair of matrices with $2 \geq 2$ as nonzero controllability indices. Then

$$(A_c, B_c) = \left(\left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right], \left[\begin{array}{cc} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{array}\right] \right).$$

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Let $r_2 = 1$, and let α be the only nontrivial invariant polynomial of A_2, $d(\alpha) = 5$, i.e. $\alpha = \alpha_5^2$, while $\alpha_1^2 = \cdots = \alpha_4^2 = 1$. Thus,

$$N(A_2) = C(\alpha).$$

Let $r_3 = 2$, and let $\beta|\gamma$ be the nontrivial invariant polynomials of A_3, $d(\beta) = 1$ and $d(\gamma) = 3$, i.e. $\alpha_3^3 = \beta$ and $\alpha_4^3 = \gamma$, while $\alpha_1^3 = \alpha_2^3 = 1$. Thus

$$N(A_3) = C(\beta) \oplus C(\gamma).$$

Then both conditions (i) and (ii) from Theorem 5.2 are satisfied. Now define matrices X_1' and X_2' as explained in the theorem:

Since $l_1' = 2$, $c_1 = 2$ and $b_1^1 = D_1^2 = d(\alpha) = 5$, put a unit in the matrix X_1' at the position $(5, 1)$. Let $b_2^1 = 4$ $(4 \in \{1, \ldots, 5\} \setminus \{5\})$. Then put a unit at the position $(4, 3)$, and all other entries in the matrix X_1' put to be zeros. Thus,

$$X_1' = \left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right].$$

Moreover, since $l_2' = 2$ and $b_1^2 = D_1^3 + D_2^3 = d(\gamma) + d(\beta) = 4$, $b_2^2 = D_2^3 = d(\beta) = 1$, put units in the matrix X_2' on the positions $(4, 1)$ and $(1, b_2^1 + 1) = (1, 5)$, and all other entries in the matrix X_2' put to be zeros. Hence,

$$X_2' = \left[\begin{array}{ccccc} 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{array}\right].$$

For such defined X_1' and X_2' the matrix

$$\left[\begin{array}{ccc|c} A_c & 0 & 0 & B_c \\ X_1' & C(\alpha) & 0 & 0 \\ 0 & X_2' & C(\beta) \oplus C(\gamma) & 0 \end{array}\right] \tag{5.12}$$

is controllable and $\operatorname{rank} X_1' = 2 \leq 4$, $\operatorname{rank} X_2' = 2 \leq 2$, as wanted.

As a direct consequence of the previous result we obtain the following theorem:

Theorem 5.4. *Let $\mathbb{F}$ be a field. Let $A_i \in \mathbb{F}^{n_i \times n_i}$, $i = 1, \ldots, m$, and $B_1 \in \mathbb{F}^{n_1 \times m_1}$ be such that the pair (A_1, B_1) is controllable,* $\operatorname{rank} B_1 = s$. *There exist matrices $X_i \in \mathbb{F}^{n_{i+1} \times n_i}$, $i = 1, \ldots, m - 1$, such that*

$$\left[\begin{array}{ccccc|c} A_1 & 0 & 0 & \ddots & 0 & B_1 \\ X_1 & A_2 & 0 & \ddots & 0 & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & X_{m-1} & A_m & 0 \end{array}\right] \tag{5.13}$$

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is controllable if and only if the following conditions are valid:

$$(i) \quad s \geq \max\{r_2, r_3, \dots, r_m\}$$
$$(ii) \quad n_i \geq \max\{r_i, \dots, r_m\}, \quad i = 2, \dots, m.$$

Here r_i is the number of the nontrivial invariant factors of $\lambda I - A_i$, $i = 2, \dots, m$.

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