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SPECTRUM OF INFINITE BLOCK MATRICES AND π-TRIANGULAR OPERATORS*

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Abstract. The paper deals with infinite block matrices having compact off diagonal parts. Bounds for the spectrum are established and estimates for the norm of the resolvent are proposed. Applications to matrix integral operators are also discussed. The main tool is the π-triangular operators defined in the paper.

Key words. Infinite block matrices, Spectrum localization, Integral operators.

AMS subject classifications. 47A10, 47A55, 15A09, 15A18.

1. Introduction and definitions. Many books and papers are devoted to the spectrum of finite block matrices, see [4, 5, 6, 14, 16] and references therein.

At the same time the spectral theory of infinite block matrices is developed considerably less than the one of finite block matrices, although infinite block matrices arise in numerous applications. To the best of our knowledge, mainly the Toeplitz and Hankel infinite block matrices were investigated, cf. [3, 16, 17] and references given therein. In the interesting paper [13], variational principles and eigenvalue estimates for a class of unbounded block operator matrices are explored. The paper [15] also should be mentioned, it is devoted to inequalities on singular values of block triangular matrices.

Recall that in the finite case the generalized Hadamard criterion for the invertibility was established, cf. [4, 5]. That criterion does not assert that any finite block triangular matrix with nonsingular diagonal blocks is invertible. But it is not hard to check that such a matrix is always invertible. Moreover the generalized Hadamard theorem can be extended to the infinite case under rather strong restrictions, only.

In the present paper we consider infinite block matrices whose off diagonal parts are compact. We propose bounds for the spectrum and invertibility conditions which in the finite case improve the Hadamard criterion for matrices that are "close" to block triangular matrices. Moreover, we derive an estimate for the norm of the resolvent of a block matrix. Besides, some results from the papers [7, 8, 10] are generalized. Applications to matrix integral operators are also discussed. Our main tool is the π-triangular operators defined below.

Let H be a separable complex Hilbert space, with the norm $\|.\|$ and unit operator I. All the operators considered in this paper are linear and *bounded*. For an operator A, $\sigma(A)$ and $R_\lambda(A) = (A - \lambda I)^{-1}$ denote the spectrum and resolvent, respectively.

Recall that a linear operator V is called quasinilpotent, if $\sigma(V) = 0$. A linear operator is called *a Volterra operator*, if it is compact and quasinilpotent.

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The paper is organized as follows. In this section we define the π-triangular operators. In Section 2 some properties of Volterra operators are considered. In Section 3 we establish the norm estimates and multiplicative representation for the resolvents of π-triangular operators. Section 4 is devoted to perturbations of block triangular matrices. The main result of the paper-Theorem 5.1 on the spectrum of infinite block matrices is presented in Section 5. Section 6 deals with diagonally dominant block matrices. Besides we generalize the Hadamard criterion to infinite block matrices. Some examples are presented in Section 7.

In what follows

$$\pi = \{P_k,\ k = 0, 1, 2, ...\}$$

is an infinite chain of orthogonal projections P_k in H, such that

$$0 = P_0H \subset P_1H \subset P_2H \subset .... \ (\sup_{k\geq 1} dim\ \Delta\ P_kH < \infty)$$

and $P_n \to I$ strongly as $n \to \infty$. Here and below $\Delta P_k = P_k - P_{k-1}$.

Let a linear operator A acting in H satisfy the relations

$$AP_k = P_kAP_k,\ \ (k = 1, 2, ...). \tag{1.1}$$

That is, P_k are invariant projections for A. Put

$$D := \sum_{k=1}^{\infty} \Delta P_k A \Delta P_k$$

and $V = A - D$. Then

$$A = D + V, \tag{1.2}$$

and

$$DP_k = P_kD,\ \ (k = 1, 2, ...), \tag{1.3}$$

and

$$P_{k-1}VP_k = VP_k,\ (k = 2, 3, ...);\ VP_1 = 0. \tag{1.4}$$

Definition 1.1. Let relations (1.1)-(1.4) hold with a compact operator V. Then we will say that A is a *π-triangular operator*, D is a *π-diagonal operator* and V is a *π-Volterra operator*.

Besides, relation (1.2) will be called the *π-triangular representation* of A, and D and V will be called the *π-diagonal part* and *π-nilpotent part* of A, respectively.

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2. Properties of π-Volterra operators.

LEMMA 2.1. *Let $\pi_m = \{Q_k, k = 1, ..., m;\ m < \infty\}$, $Q_m = I$ be a finite chain of projections. Then any operator V satisfying the condition $Q_{k-1}VQ_k = VQ_k$ $(k = 2, ..., m)$, $VQ_1 = 0$ is a nilpotent operator. Namely, $V^m = 0$.*

Proof. Since

$$V^m = V^m Q_m = V^{m-1}Q_{m-1}V = V^{m-2}Q_{m-2}VQ_{m-1}V = ... =$$

$$VQ_1...VQ_{m-2}VQ_{m-1}V,$$

we have $V^m = 0$. As claimed. □

LEMMA 2.2. *Let V be a π-Volterra operator (i.e. it is compact and satisfies (1.4)). Then V is quasinilpotent and hence Volterra.*

Proof. Thanks to the definition of a π-Volterra operator and the previous lemma, V is a limit of nilpotent operators in the operator norm, cf. [1, 2] and [11]. This proves the lemma. □

LEMMA 2.3. *Let V be a π-Volterra operator and B be π-triangular. Then VB and BV are π-Volterra operators.*

Proof. It is obvious that

$$P_{k-1}BVP_k = P_{k-1}BP_{k-1}VP_k = BP_{k-1}VP_k = BVP_k.$$

Similarly $P_{k-1}VBP_k = VBP_k$. This proves the lemma. □

LEMMA 2.4. *Let A be a π-triangular operator. Let V and D be the π-nilpotent and π-diagonal parts of A, respectively. Then for any regular point λ of D, the operators $VR_\lambda(D)$ and $R_\lambda(D)V$ are π-Volterra ones.*

Proof. Since $P_kR_\lambda(D) = R_\lambda(D)P_k$, the previous lemma ensures the required result. □

Let Y be *a norm ideal of compact linear operators in H*. That is, Y is algebraically a two-sided ideal, which is complete in an auxiliary norm $|\cdot|_Y$ for which $|CB|_Y$ and $|BC|_Y$ are both dominated by $\|C\||B|_Y$.

In the sequel we suppose that there are positive numbers θ_k $(k \in \mathbb{N})$, with

$$\theta_k^{1/k} \to 0 \text{ as } k \to \infty,$$

such that

$$\|V^k\| \le \theta_k |V|_Y^k \tag{2.1}$$

for an arbitrary Volterra operator

$$V \in Y.$$

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Recall that C_{2p} $(p = 1, 2, ...)$ is the von Neumann-Schatten ideal of compact operators with the finite ideal norm

$$N_{2p}(K) \equiv [Trace\ (K^*K)^p]^{1/2p}, \quad (K \in C_{2p}).$$

Let $V \in C_{2p}$ be a Volterra operator. Then due to Corollary 6.9.4 from [9], we get

$$\|V^j\| \leq \theta_j^{(p)} N_{2p}^j(V), \quad (j = 1, 2, ...) \tag{2.2}$$

where

$$\theta_j^{(p)} = \frac{1}{\sqrt{[j/p]!}}$$

and $[x]$ means the integer part of a positive number x. Inequality (2.2) can be written as

$$\|V^{kp+m}\| \leq \frac{N_{2p}^{pk+m}(V)}{\sqrt{k!}}, \quad (k = 0, 1, 2, ...;\ m = 0, ..., p-1). \tag{2.3}$$

In particular, if V is a Hilbert-Schmidt operator, then

$$\|V^j\| \leq \frac{N_2^j(V)}{\sqrt{j!}}, \quad (j = 0, 1, 2, ...).$$

3. Resolvents of π-triangular operators.

LEMMA 3.1. *Let A be a π-triangular operator. Then $\sigma(A) = \sigma(D)$, where D is the π-diagonal part of A. Moreover,*

$$R_\lambda(A) = R_\lambda(D) \sum_{k=0}^{\infty} (VR_\lambda(D))^k (-1)^k. \tag{3.1}$$

Proof. Let λ be a regular point of operator D. According to the triangular representation (1.2) we obtain

$$R_\lambda(A) = (D + V - \lambda I)^{-1} = R_\lambda(D)(I + VR_\lambda(D))^{-1}.$$

Operator $VR_\lambda(D)$ for a regular point λ of operator D is a Volterra one due to Lemma 2.3. Therefore,

$$(I + VR_\lambda(D))^{-1} = \sum_{k=0}^{\infty} (VR_\lambda(D))^k (-1)^k$$

and the series converges in the operator norm. Hence, it follows that λ is a regular point of A.

Conversely let $\lambda \notin \sigma(A)$. According to the triangular representation (1.2) we obtain

$$R_\lambda(D) = (A - V - \lambda I)^{-1} = R_\lambda(A)(I - VR_\lambda(A))^{-1}.$$

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Since V is a π-Volterra, for a regular point λ of A, operator $VR_\lambda(A)$ is a Volterra one due to Lemmas 2.2 and 2.3. So

$$(I - VR_\lambda(A))^{-1} = \sum_{k=0}^{\infty}(VR_\lambda(A))^k$$

and the series converges in the operator norm. Thus,

$$R_\lambda(D) = R_\lambda(A)\sum_{k=0}^{\infty}(VR_\lambda(A))^k.$$

Hence, it follows that λ is a regular point of D. This finishes the proof. $\square$

With $V \in Y$, introduce the function

$$\zeta_Y(x, V) := \sum_{k=0}^{\infty}\theta_k|V|_Y^k x^{k+1}, \quad (x \geq 0).$$

COROLLARY 3.2. *Let A be a π-triangular operator and let its π-nilpotent part V belong to a norm ideal Y with the property* (2.1). *Then*

$$\|R_\lambda(A)\| \leq \zeta_Y(\|R_\lambda(D)\|, V) \equiv \sum_{k=0}^{\infty}\theta_k|V|_Y^k\|R_\lambda(D)\|^{k+1}$$

for all regular λ of A.

Indeed, according to Lemma 2.3 and (2.1)

$$\|(VR_\lambda(D))^k\| \leq \theta_k|VR_\lambda(D)|_Y^k.$$

But

$$|VR_\lambda(D)|_Y \leq |V|_Y\|R_\lambda(D)\|.$$

Now the required result is due to (3.1).

Corollary 3.2 and inequality (2.2) yield

COROLLARY 3.3. *Let A be a π-triangular operator and its π-nilpotent part $V \in C_{2p}$ for some integer $p \geq 1$. Then*

$$\|R_\lambda(A)\| \leq \sum_{k=0}^{\infty}\theta_k^{(p)}N_{2p}^k(V)\|R_\lambda(D)\|^{k+1}, \quad (\lambda \notin \sigma(A))$$

where D is the π-diagonal part of A. In particular, if V is a Hilbert-Schmidt operator, then

$$\|R_\lambda(A)\| \leq \sum_{k=0}^{\infty}\frac{N_2^k(V)}{\sqrt{k!}}\|R_\lambda(D)\|^{k+1}, \quad (\lambda \notin \sigma(A)).$$

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Note that under the condition $V \in C_{2p},\ p > 1$, inequality (2.3) implies

$$\|R_\lambda(A)\| \le \sum_{j=0}^{p-1} \sum_{k=0}^{\infty} \frac{N_{2p}^{pk+j}(V)}{\sqrt{k!}} \|R_\lambda(D)\|^{pk+j+1}. \tag{3.2}$$

Thanks to the Schwarz inequality, for all $x > 0$ and $a \in (0,1)$,

$$[\sum_{k=0}^{\infty} \frac{x^k}{\sqrt{k!}}]^2 = [\sum_{k=0}^{\infty} \frac{x^k a^k}{a^k \sqrt{k!}}]^2 \le$$

$$\sum_{k=0}^{\infty} a^{2k} \ \sum_{k=0}^{\infty} \frac{x^{2k}}{a^{2k} k!} = (1-a^2)^{-1} e^{x^2/a^2}.$$

In particular, take $a^2 = 1/2$. Then

$$\sum_{k=0}^{\infty} \frac{x^k}{\sqrt{k!}} \le \sqrt{2} e^{x^2}.$$

Now (3.2) implies

COROLLARY 3.4. *Let A be a π-triangular operator and its π-nilpotent part $V \in C_{2p}$ for some integer $p \ge 1$. Then*

$$\|R_\lambda(A)\| \le \zeta_p(\|R_\lambda(D)\|, V), \quad (\lambda \notin \sigma(A)).$$

where

$$\zeta_p(x,V) := \sqrt{2} \sum_{j=0}^{p-1} N_{2p}^j(V) x^{j+1}\ exp\ [\ N_{2p}^{2p}(V) x^{2p}], \ \ (x > 0). \tag{3.3}$$

LEMMA 3.5. *Let A be a π-triangular operator, whose π-nilpotent part V belongs to a norm ideal Y with the property* (2.1). *Then for any $\mu \in \sigma(B)$, either $\mu \in \sigma(D)$ or*

$$\|A - B\| \zeta_Y(\|R_\mu(D)\|, V) \ge 1.$$

In particular, if $V \in C_{2p}$ for some integer $p \ge 1$, then this inequality holds with $\zeta_Y = \zeta_p$.

Indeed this result follows from Corollaries 3.2 and 3.4.

Let us establish the multiplicative representation for the resolvent of a π-triangular operator. To this end, for bounded linear operators $X_1, X_2, ..., X_m$ and $j < m$, denote

$$\prod_{j \le k \le m}^{\rightarrow} X_k \equiv X_j X_{j+1} ... X_m.$$

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In addition

$$\prod_{j\le k\le\infty}^{\rightarrow} X_k := \lim_{m\to\infty} \prod_{j\le k\le m}^{\rightarrow} X_k$$

if the limit exists in the operator norm.

LEMMA 3.6. *Let $\pi = \{P_k\}_{k=1}^{\infty}$ be a chain of orthogonal projectors, V a π-Volterra operator. Then*

$$(I-V)^{-1} = \prod_{k=2,3,....}^{\rightarrow} (I + V\Delta P_k) \tag{3.4}$$

Proof. First let $\pi = \{P_1, ..., P_m\}$ be finite. According to Lemma 2.1

$$(I-V)^{-1} = \sum_{k=0}^{m-1} V^k. \tag{3.5}$$

On the other hand,

$$\prod_{2\le k\le m}^{\rightarrow} (I + V\Delta P_k) = I + \sum_{k=2}^{m} V_k + \sum_{2\le k_1<k_2\le m} V_{k_1}V_{k_2}$$

$$+... + V_2V_3...V_m.$$

Here, as above, $V_k = V\Delta P_k$. However,

$$\sum_{2\le k_1<k_2\le m} V_{k_1}V_{k_2} = V \sum_{2\le k_1<k_2\le m} \Delta P_{k_1} V \Delta P_{k_2} =$$

$$V \sum_{3\le k_2\le m} P_{k_2-1} V \Delta P_{k_2} = V^2 \sum_{3\le k_2\le m} \Delta P_{k_2} = V^2.$$

Similarly,

$$\sum_{2\le k_1<k_2...<k_j\le m} V_{k_1}V_{k_2}...V_{k_j} = V^j$$

for $j < m$. Thus from (3.5) the relation (3.4) follows. The rest of the proof is left to the reader. □

THEOREM 3.7. *For any π-triangular operator A and a regular $\lambda \in \mathbb{C}$*

$$R_\lambda(A) = (D-\lambda I)^{-1} \prod_{2\le k\le\infty}^{\rightarrow} (I - V\Delta P_k (D-\lambda I)^{-1} \Delta P_k),$$

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where D and V are the π-diagonal and π-nilpotent parts of A, respectively.

Proof. Due to Lemma 2.4, $VR_\lambda(D)$ is π-nilpotent. Now the previous lemma implies

$$(I + VR_\lambda(D))^{-1} = \overrightarrow{\prod_{2\le k\le m}} (I - VR_\lambda(D)\Delta P_k).$$

But $R_\lambda(D)\Delta P_k = \Delta P_k R_\lambda(D)$. This proves the result. □

Let A be a π-triangular operator and

$$\Pi(A,\lambda) := \|R_\lambda(D)\| \prod_{k=2}^{\infty}(1 + \|R_\lambda(D)\Delta P_k\|\|V\Delta P_k\|) < \infty.$$

Then from the previous theorem it follows the inequality $\|R_\lambda(A)\| \le \Pi(A,\lambda)$.

4. Perturbations of block triangular matrices. Let $H = l^2(\mathbb{C}^n)$ be the space of sequences $h = \{h_k \in \mathbb{C}^n\}_{k=1}^{\infty}$ with values in the Euclidean space $\mathbb{C}^n$ and the norm

$$|h|_{l^2(\mathbb{C}^n)} = [\sum_{k=1}^{\infty} \|h_k\|_n^2]^{1/2},$$

where $\|\cdot\|_n$ is the Euclidean norm in $\mathbb{C}^n$.

Consider the operator defined in $l^2(\mathbb{C}^n)$ by the upper block triangular matrix

$$A_+ = \begin{pmatrix} A_{11} & A_{12} & A_{13} & \dots \\ 0 & A_{22} & A_{23} & \dots \\ 0 & 0 & A_{33} & \dots \\ . & . & . & \dots \end{pmatrix}, \tag{4.1}$$

where A_{jk} are $n \times n$-matrices.

So $A_+ = \tilde{D} + V_+$, where V_+ and $\tilde{D}$ are the strictly upper triangular, and diagonal parts of A_+, respectively:

$$V_+ = \begin{pmatrix} 0 & A_{12} & A_{13} & A_{14} & \dots \\ 0 & 0 & A_{23} & A_{24} & \dots \\ 0 & 0 & 0 & A_{34} & \dots \\ . & . & . & . & \dots \end{pmatrix}$$

and $\tilde{D} = diag\,[A_{11}, A_{22}, A_{33}, ...]$. Put

$$\eta_n(\lambda) := \sup_k \|R_\lambda(A_{kk})\|_n.$$

LEMMA 4.1. *Let A_+ be the block triangular matrix defined by* (4.1) *and V_+ be a compact operator belonging to a norm ideal Y with the property* (2.1) *Then $\sigma(A_+) = \sigma(\tilde{D})$, and*

$$|R_\lambda(A_+)|_{l^2(\mathbb{C}^n)} \le \zeta_Y(\eta_n(\lambda), V_+)$$

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for all regular λ of $\tilde{D}$. Moreover, for any bounded operator B acting in $l^2(\mathbb{C}^n)$ and a $\mu \in \sigma(B)$, either $\mu \in \sigma(\tilde{D})$ or

$$q\zeta_Y(\eta_n(\mu), V_+) \geq 1$$

where $q := |A_+ - B|_{l^2(\mathbb{C}^n)}$. In particular, if

$$V_+ \in C_{2p} \ \ (p = 1, 2, ...) \tag{4.2}$$

then $\zeta_Y = \zeta_p$.

Proof. Let $P_j,\ j = 0, 1, 2, ...$ be projections onto the subspaces of $l^2(\mathbb{C}^n)$ generated by the first nj elements of the standard basis. Then $\pi = \{P_k\}$ is the infinite chain of orthogonal projections in $l^2(\mathbb{C}^n)$, such that (1.1) holds and $P_n \to I$ strongly as $n \to \infty$. Moreover, $dim\ \Delta P_k H \equiv n$ $(k = 1, 2, ..)$ and

$$A_{jk} = \Delta P_k \tilde{A} \Delta P_k, \tilde{D} = \sum_{j=1}^{\infty} A_{jj}.$$

Hence it follows that A_+ is a π triangular operator. Now Corollaries 3.2 and 3.5 prove the result. □

Let

$$\|R_\lambda(A_{kk})\| \leq \phi(\rho(A_{kk}, \lambda)) := \sum_{l=0}^{n-1} \frac{c_l}{\rho^{l+1}(A_{kk}, \lambda)}, \ \ (\lambda \notin \sigma(\tilde{D}))$$

where c_l are nonnegative coefficients, independent of k, and $\rho(A, \lambda)$ is the distance between a complex point λ and $\sigma(A)$. Then

$$\|R_\lambda(A_{kk})\| \leq \phi(\rho(\tilde{D}, \lambda)) := \sum_{l=0}^{n-1} \frac{c_l}{\rho^{l+1}(\tilde{D}, \lambda)}, \ \ (\lambda \notin \sigma(\tilde{D}))$$

and

$$\rho(\tilde{D}, \lambda) = \inf_{k=1,2,....} \min_{j=1,...,n} |\lambda - \lambda_j(A_{kk})|$$

is the distance between a point λ and $\sigma(\tilde{D})$, and

$$\phi(y) = \sum_{k=0}^{n-1} \frac{c_k}{y^{k+1}}, \ \ (y > 0).$$

Then

$$\eta_n(\lambda) = \sup_{j=1,2,...} \|R_\lambda(A_{jj})\| \leq \phi(\rho(\tilde{D}, \lambda)).$$

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Under (4.2), Lemma 4.1 gives us the inequality

$$|R_\lambda(A_+)|_{l^2(\mathbb{C}^n)} \le \zeta_p(\phi(\rho(\tilde{D},\lambda)), V_+)$$

for all regular λ of $\tilde{D}$, provided $V_+ \in Y$. So for a bounded operator B and $\mu \in \sigma(B)$, either $\mu \in \sigma(\tilde{D})$ or

$$q\zeta_p(\phi(\rho(\tilde{D},\mu)), V_+) \ge 1. \tag{4.3}$$

Furthermore, let $C = (c_{jk})_{j,k=1}^n$ be an $n \times n$-matrix. Then as it is proved in [9, Corollary 2.1.2],

$$\|R_\lambda(C)\|_n \le \sum_{k=0}^{n-1} \frac{g^k(C)}{\sqrt{k!}\rho^{k+1}(C,\lambda)},$$

where $\lambda_k(C);\ k = 1,\dots,n$ are the eigenvalues of C including their multiplicities, and

$$g(C) = (N_2^2(C) - \sum_{k=1}^{n} |\lambda_k(C)|^2)^{1/2}.$$

Here $N_2(.)$ is the Hilbert-Schmidt norm in $\mathbb{C}$. In particular, the inequalities

$$g^2(C) \le N_2^2(C) - |Trace\ C^2| \text{ and } g^2(C) \le \frac{1}{2}N_2^2(C^* - C)$$

are true (see [9, Section 2.1]). If C is a normal matrix, then $g(C) = 0$. Thus

$$\|R_\lambda(A_{jj})\|_n \le \sum_{k=0}^{n-1} \frac{g^k(A_{jj})}{\sqrt{k!}\rho^{k+1}(A_{jj},\lambda)}.$$

Since $\tilde{D}$ is bounded, we have

$$g_0 := \sup_{k=1,2,\dots} g(A_{kk}) < \infty.$$

Then one can take $\phi(y) = \phi_0(y)$ where

$$\phi_0(y) = \sum_{k=0}^{n-1} \frac{g_0^k}{\sqrt{k!}y^{k+1}}.$$

If all the diagonal matrices A_{kk} are normal, then $g(A_{kk}) = 0$ and $\phi_0(y) = 1/y$. Relation (4.3) yields

LEMMA 4.2. *Let A_+ be defined by* (4.1) *and B a linear operator on $l^2(\mathbb{C}^n)$. If, in addition, condition* (4.2) *holds, then for any $\mu \in \sigma(B)$, there is a $\lambda \in \sigma(\tilde{D})$, such that*

$$|\lambda - \mu| \le r_p(q, V_+)$$

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where $r_p(q,V_+)$ is the unique positive root of the equation

$$q\zeta_p(\phi_0(y),V_+)=1.$$

Here y is the unknown.

It is simple to see that

$$r_p(q,V_+)\le y_n(z_p(q,V_+))$$

where $y_n(b)$ is the unique positive root of the equation

$$\phi_0(y)=b,\ \ (b=const>0).$$

and $z_p(q,V_+)$ is the unique positive root of the equation

$$q\zeta_p(x,V_+)\equiv q\sqrt{2}\sum_{j=0}^{p-1}N_{2p}^j(V_+)x^{j+1}\ exp\ [\ N_{2p}^{2p}(V_+)x^{2p}]=1. \tag{4.4}$$

Furthermore, thanks to [9, Lemma 1.6.1] $y_n(b)\le p_n(b)$, where

$$p_n(b)=\begin{cases}\phi_0(1)/b & \text{if } \phi_0(1)\ge b,\\ \sqrt[n]{\phi_0(1)/b} & \text{if } \phi_0(1)<b.\end{cases}$$

We need the following

LEMMA 4.3. *The unique positive root z_a of the equation*

$$\sum_{j=0}^{p-1}z^{j+1}\ exp\ [z^{2p}]=a,\ \ (a\equiv const>0)$$

satisfies the inequality $z_a\ge\delta_p(a)$, where

$$\delta_p(a):=\begin{cases}a/pe & \text{if } a\le pe,\\ [ln\ (a/p)]^{1/2p} & \text{if } a>pe.\end{cases} \tag{4.5}$$

For the proof see [9, Lemma 8.3.2]. Put in (4.4) $N_{2p}(V_+)x=z$. Then we get equation (4.5) with $a=N_{2p}(V_+)/q\sqrt{2}$. The previous lemma implies

$$z_p(q,V_+)\ge\gamma_p(q,V_+)$$

where

$$\gamma_p(q,V_+):=\frac{\delta_p(N_{2p}(V_+)/q\sqrt{2})}{N_{2p}(V_+)}.$$

We thus get

$$r_p(q,V_+)\le p_n(\gamma_p(q,V_+)). \tag{4.6}$$

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5. The main result. Consider in $l^2(\mathbb{C}^n)$ the operator defined by the block matrix

$$\tilde{A} = \begin{pmatrix} A_{11} & A_{12} & A_{13} & \dots \\ A_{21} & A_{22} & A_{23} & \dots \\ A_{31} & A_{32} & A_{33} & \dots \\ . & . & . & \dots \end{pmatrix} \tag{5.1}$$

where A_{jk} are $n \times n$-matrices. Clearly,

$$\tilde{A} = \tilde{D} + V_+ + V_-$$

where V_+ is the strictly upper triangular, part, $\tilde{D}$ is the diagonal part and V_- is the strictly lower triangular part of $\tilde{A}$:

$$V_- = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots \\ A_{21} & 0 & 0 & 0 & \dots \\ A_{31} & A_{32} & 0 & 0 & \dots \\ A_{41} & A_{42} & A_{43} & 0 & \dots \\ . & . & . & \dots & . \end{pmatrix}.$$

Now we get the main result of the paper which is due to (4.3) with $B = \tilde{A}$.

Recall that ϕ_0 is defined in the previous section and ζ_p is defined by (3.3).

THEOREM 5.1. *Let $\tilde{A}$ be defined by* (5.1) *and condition* (4.2) *hold. Then for any $\mu \in \sigma(\tilde{A})$, either $\mu \in \sigma(\tilde{D})$ or there is a $\lambda \in \sigma(\tilde{D})$ such that*

$$|V_-|_{l^2(\mathbb{C}^n)} \zeta_p(\phi_0(|\lambda - \mu|), V_+) \geq 1.$$

The theorem is exact in the following sense: if $V_- = 0$, then $\sigma(\tilde{A}) = \sigma(\tilde{D})$. Moreover, Lemma 4.2 with $B = \tilde{A}$ implies

COROLLARY 5.2. *Let $\tilde{A}$ be defined by* (5.1) *and condition* (4.2) *hold. Then for any $\mu \in \sigma(\tilde{D})$, there is a $\lambda \in \sigma(\tilde{D})$, such that*

$$|\lambda - \mu| \leq r_p(\tilde{A}),$$

where $r_p(\tilde{A})$ is is the unique positive root of the equation

$$|V_-|_{l^2(\mathbb{C}^n)} \zeta_p(\phi_0(y), V_+) = 1.$$

Moreover, (4.6) *gives us the bound for $r_p(\tilde{A})$ if we take $q = |V_-|_{l^2(\mathbb{C}^n)}$.*

Note that in Theorem 5.1 it is enough that V_+ is compact. Operator V_- can be noncompact.

Clearly, one can exchange V_+ and V_-.

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6. Diagonally dominant block matrices. Put

$$m_{jk} = \|A_{jk}\|_n, \quad (j,k = 1,2,...)$$

and consider the matrix

$$M = (m_{jk})_{j,k=1}^{\infty}.$$

LEMMA 6.1. *The spectral radius $r_s(\tilde{A})$ of $\tilde{A}$ is less than or is equal to the spectral radius of M.*

Proof. Let $A_{jk}^{(\nu)}$ and $m_{jk}^{(\nu)}$ $(\nu = 2,3,...)$ be the entries of $\tilde{A}^{\nu}$ and M^{ν}, respectively. We have

$$\|A_{jk}^{(2)}\|_n = \|\sum_{l=1}^{\infty} A_{jl}A_{lk}\|_n \le$$

$$\sum_{l=1}^{\infty} \|A_{jl}\|_n \|A_{lk}\|_n = \sum_{l=1}^{\infty} m_{jl}m_{lk} = m_{jk}^{(2)}.$$

Similarly, we get $\|A_{jk}^{(\nu)}\|_n \le m_{jk}^{(\nu)}$.

But for any $h = \{h_k\} \in l^2(\mathbb{C}^n)$, we have

$$|\tilde{A}h|_{l^2(\mathbb{C}^n)}^2 \le \sum_{j=1}^{\infty}(\sum_{k=1}^{\infty} \|A_{jk}h_k\|_n)^2 \le \sum_{j=1}^{\infty}(\sum_{k=1}^{\infty} m_{jk}\|h_k\|_n)^2 = |M\tilde{h}|_{l^2(\mathbb{R})}^2$$

where

$$\tilde{h} = \{\|h_k\|_n\} \in l^2(\mathbb{R}^1).$$

Since

$$|h|_{l^2(\mathbb{C}^n)}^2 = \sum_{k=1}^{\infty} \|h_k\|_n^2 = |\tilde{h}|_{l^2(\mathbb{R})}^2,$$

we obtain $|\tilde{A}^{\nu}|_{l^2(C^n)} \le |M^{\nu}|_{l^2(\mathbb{R})}$ $(\nu = 2,3,...)$. Now the Gel'fand formula for the spectral radius yields the required result. □

Denote

$$S_j := \sum_{k=1,k\ne j}^{\infty} \|A_{jk}\|_n.$$

THEOREM 6.2. *Let A_{jj} be invertible for all integer j. In addition, let*

$$\sup_j S_j < \infty \tag{6.1}$$

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and there be an $\epsilon > 0$, such that

$$\|A_{jj}^{-1}\|_n^{-1} - S_j \geq \epsilon, \quad (j = 1, 2, ...). \tag{6.2}$$

Then $\tilde{A}$ is invertible. Moreover, let

$$\psi(\lambda) := \sup_j \|(A_{jj} - \lambda)^{-1}\| S_j < 1.$$

Then λ is a regular point of $\tilde{A}$, and

$$|(\tilde{A} - \lambda)^{-1}|_{l^2(\mathbb{C}^n)} \leq (1 - \psi(\lambda))^{-1} \sup_j \|(A_{jj} - \lambda)^{-1}\|_n.$$

Proof. Put $W = \tilde{A} - \tilde{D} = V_+ + V_-$ with an invertible $\tilde{D}$. That is, W is the off-diagonal part of $\tilde{A}$, and

$$\tilde{A} = \tilde{D} + W = \tilde{D}(I + \tilde{D}^{-1}W). \tag{6.3}$$

Clearly,

$$\sum_{k=1, k\neq j}^{\infty} \|A_{jj}^{-1} A_{jk}\|_n \leq S_j \|A_{jj}^{-1}\|.$$

From (6.2) it follows

$$1 - S_j \|A_{jj}^{-1}\|_n \geq \|A_{jj}^{-1}\|_n \epsilon \quad (j = 1, 2, ...).$$

Therefore

$$\sup_j \sum_{k=1, k\neq j}^{\infty} \|A_{jj}^{-1} A_{jk}\|_n < 1.$$

Then thanks to the well-known bound for the spectral radius [12, Section 3.16] and the previous lemma the spectral radius $r_s(\tilde{D}^{-1}W) < 1$. Therefore $I + \tilde{D}^{-1}W$ is invertible. Now (6.3) implies that $\tilde{A}$ is invertible. This proves the theorem. □

Theorem 6.2 extends the above mentioned Hadamard criterion to infinite block matrices.

It should be noted that condition (6.1) implies that the off-diagonal part W of $\tilde{A}$ is compact, since under (6.1) the sequence of the finite dimensional operators

$$W_l := \begin{pmatrix} 0 & A_{12} & A_{13} & \dots & A_{1l} \\ A_{21} & 0 & A_{23} & \dots & A_{2l} \\ A_{31} & A_{32} & 0 & \dots & A_{3l} \\ . & . & . & \dots & . \\ A_{l1} & A_{l2} & A_{l3} & \dots & 0 \end{pmatrix}$$

converges to W in the norm of space $l^2(\mathbb{C}^n)$ as $l \to \infty$.

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7. Examples. Let $n = 2$, then $\tilde{D}$ is the orthogonal sum of the 2×2-matrices

$$A_{kk} = \begin{pmatrix} a_{2k-1,2k-1} & a_{2k-1,2k} \\ a_{2k,2k-1} & a_{2k,2k} \end{pmatrix}.$$

If A_{kk} are real matrices, then due to the above mentioned inequality $g^2(C) \le N_2^2(C^* - C)/2$, we have

$$g(A_{kk}) \le |a_{2k-1,2k} - a_{2k,2k-1}|.$$

So one can take

$$\phi_0(y) = \frac{1}{y}(1 + \frac{\tilde{g}_0}{y})$$

with

$$\tilde{g}_0 := \sup_k |a_{2k-1,2k} - a_{2k,2k-1}|.$$

Besides, $\sigma(\tilde{D}) = \{\lambda_{1,2}(A_{kk})\}_{k=1}^{\infty}$, where

$$\lambda_{1,2}(A_{kk}) = \frac{1}{2}(a_{2k-1,2k-1} + a_{2k,2k} \pm [(a_{2k-1,2k-1} - a_{2k,2k})^2 - a_{2k-1,2k}a_{2k,2k-1}]^{1/2}).$$

Now we can directly apply Theorems 5.1 and 6.2, and Corollary 5.2.

Furthermore, let $L^2(\omega, \mathbb{C}^n)$ be the space of vector valued functions defined on a bounded subset ω of $\mathbb{R}^m$ with the scalar product

$$(f, g) = \int_\omega (f(s), g(s))_{C^n} ds$$

where $(.,.)_{C^n}$ is the scalar product in $\mathbb{C}^n$. Let us consider in $L^2(\omega, \mathbb{C}^n)$ the matrix integral operator

$$(Tf)(x) = \int_\omega K(x, s)f(s)ds$$

with the condition

$$\int_\omega \int_\omega \|K(x, s)\|_n^2 dx\ ds < \infty.$$

That is, T is a Hilbert-Schmidt operator.

Let $\{e_k(x)\}$ be an orthogonal normal basis in $L^2(\omega, \mathbb{C}^n)$ and

$$K(x, s) = \sum_{j,k=1}^{\infty} A_{jk}e_k(s)e_k(x)$$

be the Fourier expansion of K, with the matrix coefficients A_{jk}. Then T is unitarily equivalent to the operator $\tilde{A}$ defined by (1.1). Now one can apply Theorems 5.1 and 6.1, and Corollary 5.2.

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