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NILPOTENT MATRICES AND SPECTRALLY ARBITRARY SIGN PATTERNS*

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Abstract. It is shown that all potentially nilpotent full sign patterns are spectrally arbitrary. A related result for sign patterns all of whose zeros lie on the main diagonal is also given.

Key words. Nilpotent matrices, Potentially nilpotent sign patterns, Spectrally arbitrary sign patterns.

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1. Full Spectrally Arbitrary Patterns. In what follows, M_n denotes the topological vector space of all $n \times n$ matrices with real entries and P_n denotes the set of all polynomials with real coefficients of degree n or less. The superdiagonal of an $n \times n$ matrix consists of the $n-1$ elements that are in the ith row and $(i+1)$st column for some i, $1 \le i \le n-1$.

A sign pattern is a matrix with entries in $\{+, 0, -\}$. Given two $n \times n$ sign patterns $\mathcal{A}$ and $\mathcal{B}$, we say that $\mathcal{B}$ is a superpattern of $\mathcal{A}$ if $b_{ij} = a_{ij}$ whenever $a_{ij} \neq 0$. Note that a sign pattern is always a superpattern of itself. We define the function $sign : \mathbb{R} \to \{+, 0, -\}$ in the obvious way: $sign(x) = +$ if $x > 0$, $sign(0) = 0$, and $sign(x) = -$ if $x < 0$. Given a real matrix A, $sign(A)$ is the sign pattern with the same dimensions as A whose (i,j)th entry is $sign(a_{ij})$. For every sign pattern $\mathcal{A}$, we define its associated sign pattern class to be the inverse image $Q(\mathcal{A}) = sign^{-1}(\mathcal{A})$. A sign pattern is said to be full if none of its entries are zero [8]. A sign pattern class $Q(\mathcal{A})$ is an open set in the topology of M_n if and only if $\mathcal{A}$ is a full sign pattern; this observation will be very useful in proving our first theorem.

Recall that a square matrix N is nilpotent if there exists a $k \in \mathbb{N}$ such that $N^k = 0$. We define the index of a nilpotent matrix N to be the smallest $k \in \mathbb{N}$ such that $N^k = 0$. A sign pattern $\mathcal{A}$ is said to be potentially nilpotent if there exists a nilpotent matrix $N \in Q(\mathcal{A})$. (Allows nilpotence is sometimes used in place of potentially nilpotent). Potentially nilpotent sign patterns were studied in [6] and [10]. An $n \times n$ sign pattern matrix $\mathcal{A}$ is said to be spectrally arbitrary if every nth degree monic polynomial with real coefficients is a characteristic polynomial of some matrix in $Q(\mathcal{A})$. The study of spectrally arbitrary sign patterns was initiated in [5].

It is clear that any spectrally arbitrary sign pattern is potentially nilpotent. The converse can easily be seen to be false. For instance, any potentially nilpotent sign pattern with only zeros on the main diagonal is not spectrally arbitrary. For $n \le 3$, these are the only counterexamples to the converse as any 2×2 or 3×3 potentially nilpotent sign pattern whose main diagonal does not consist entirely of zeros is spec-

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trally arbitrary [2]. For $n \geq 4$, there exist potentially nilpotent sign patterns whose main diagonals do not consist entirely of zeros yet are still not spectrally arbitrary. The following example is given in [3].

EXAMPLE 1.1. The following 4×4 nilpotent matrix belongs to a sign pattern which is not spectrally arbitrary:

$$(1.1) \qquad \begin{bmatrix} 1 & 1 & -1 & 0 \\ -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}.$$

It is noted in both [3] and [4] that it is the location of the zero entries in (1.1) which prevents this sign pattern from being spectrally arbitrary. (In the sense that any other sign pattern with zeros in the exact same locations would not be spectrally arbitrary). Another collection of potentially nilpotent but not spectrally arbitrary sign patterns can be obtained by following an argument of [4] and noting that any reducible sign pattern having all of its irreducible blocks being potentially nilpotent and at least two of its irreducible blocks being of odd order would be potentially nilpotent but not spectrally arbitrary (since it would require two real eigenvalues). Note that in this case as well it is the location of the zero entries which prevent the sign patterns in this collection from being spectrally arbitrary. This would suggest that one should look at sign patterns with no or few zero entries to see if potential nilpotence implies being spectrally arbitrary in these cases. In page 14 of [1] (Question 2.14), it was asked whether any potentially nilpotent full sign pattern is spectrally arbitrary. We answer this question in the affirmative.

THEOREM 1.2. *Any potentially nilpotent full sign pattern is spectrally arbitrary.*

Proof. Let $\mathcal{A}$ be an $n \times n$ full sign pattern and let $N \in Q(\mathcal{A})$ be a nilpotent matrix. Let $N = PJP^{-1}$ be the Jordan decomposition of N. Since J is a direct sum of Jordan blocks; it is a matrix with ones and zeros on the superdiagonal and zero entries everywhere else. We now show that there is a nilpotent matrix of index n in $Q(\mathcal{A})$. If all the elements on the superdiagonal of J are one, then N is nilpotent of index n. Otherwise, let J_ϵ be the $n \times n$ matrix obtained by replacing all of the zeros on the superdiagonal of J by ϵ and leaving all other entries unchanged. Then $N_\epsilon = PJ_\epsilon P^{-1}$ is a nilpotent matrix of index n for $\epsilon \neq 0$ and further $N_\epsilon \in Q(\mathcal{A})$ for some non-zero ϵ sufficiently small. Since N_ϵ is a nilpotent matrix of index n, it has a Jordan decomposition $N_\epsilon = SJ_1S^{-1}$ where S is an invertible matrix and J_1 is the matrix with ones along the superdiagonal and zeros everywhere else. Now consider matrices of the form SC_pS^{-1} where C_p is the companion matrix with characteristic polynomial $p(x) = x^n + a_{n-1}x^{n-1} + ... + a_1x + a_0$.

$$C_p = \begin{bmatrix} 0 & 1 & 0 & ... & 0 \\ 0 & 0 & 1 & ... & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & ... & 1 \\ -a_0 & -a_1 & -a_2 & ... & -a_{n-1} \end{bmatrix}.$$

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There exists an $\epsilon > 0$, such that if $|a_i| < \epsilon$ for $0 \le i \le n - 1$, $SC_pS^{-1} \in Q(\mathcal{A})$. Now let $p(x)$ be an arbitrary nth degree monic polynomial with real coefficients, Then there exists a positive $k \in \mathbb{R}$ such that the polynomial $q(x) = k^np(x/k)$ is a monic polynomial all of whose nonleading coefficients have modulus less than ϵ. Then $\frac{1}{k}SC_qS^{-1}$ is in $Q(\mathcal{A})$ and has characteristic polynomial $p(x)$. Hence $\mathcal{A}$ is spectrally arbitrary. $\square$

An $n \times n$ sign pattern A is called p-striped if it has p columns with only positive entries and $n - p$ columns with only negative entries. In [9], it was shown that all $n \times n$ p-striped sign patterns where $1 \le p \le n - 1$ are spectrally arbitrary. Since it is easy to construct a rank-one nilpotent matrix in any $n \times n$ p-striped sign pattern class with $1 \le p \le n - 1$, we now have an alternate and shorter proof of this result.

Theorem 1.2 suggests the following random algorithm to find most (or possibly even all) of the full spectrally arbitrary $n \times n$ sign patterns for small n. Randomly choose an invertible $P \in M_n$, calculate PJ_1P^{-1}. If all the entries of PJ_1P^{-1} are far enough from zero to avoid sign errors, add the sign pattern containing PJ_1P^{-1} to the list of $n \times n$ spectrally arbitrary full sign patterns to generate a (not necessarily complete) list of full spectrally arbitrary sign patterns.

2. Patterns with zeros only on the main diagonal. In this section we give a variant of Theorem 1.2 in the case of a sign pattern whose zeros are restricted to the main diagonal. The adjugate of an $n \times n$ matrix A, denoted $adj(A)$ is the $n \times n$ matrix whose (i, j)th entry is $(-1)^{i+j}det(A[j|i])$ where $A[j|i]$ is the $(n - 1) \times (n - 1)$ matrix obtained by deleting the jth row and ith column of A. If A is invertible then $A^{-1} = \frac{adj(A)}{det(A)}$. If the (i, j)th entry of A, a_{ij}, is considered to be a variable; then $\frac{\partial det(A)}{\partial a_{ij}}$ is equal to the (j, i)th entry of $adj(A)$. More about the adjugate can be found in [7]. We also will use $\mathcal{D}_n$ to denote the subspace of M_n consisting of the $n \times n$ diagonal matrices; $M_n/\mathcal{D}_n$ is the quotient vector space.

One of the most useful tools in finding spectrally arbitrary sign patterns is the Nilpotent-Jacobian method first used in [5] but first explicitly stated as Lemma 2.1 of [2]. We state it here as follows:

LEMMA 2.1. [2] *Let $\mathcal{A}$ be a $n \times n$ sign pattern, and suppose there exists some nilpotent $A \in Q(\mathcal{A})$ with at least n non-zero entries, say $a_{i_1j_1}, a_{i_2j_2}, ..., a_{i_nj_n}$. Let X be the matrix obtained by replacing these entries in A by the variables $x_1, ..., x_n$ and let*

$$det(xI - X) = x^n + \alpha_1 x^{n-1} + \alpha_2 x^{n-2} + ... + \alpha_{n-1}x + \alpha_n.$$

If the Jacobian $\frac{\partial(\alpha_1, \alpha_2, ..., \alpha_n)}{\partial(x_1, x_2, ..., x_n)}$ is invertible at $(x_1, x_2, ..., x_n) = (a_{i_1j_1}, a_{i_2j_2}, ..., a_{i_nj_n})$, then every superpattern of $\mathcal{A}$ is a spectrally arbitrary.

We will now prove a result for sign patterns whose zeros lie solely on the main diagonal. Since any sign pattern with less than two non-zero elements on the main diagonal is not spectrally arbitrary, we will restrict the statement of our theorem to patterns which have at least two non-zero elements on the main diagonal.

THEOREM 2.2. *Let $\mathcal{A}$ be a $n \times n$ sign pattern having at most $n - 2$ zero entries all of which are on the main diagonal. If there exists a nilpotent matrix of index n in $Q(\mathcal{A})$, then every superpattern of $\mathcal{A}$ is a spectrally arbitrary sign pattern.*

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Proof. Let N be a nilpotent matrix of index n in $Q(\mathcal{A})$. We have $span\{adj(xI - N) : x \neq 0\} = span\{(xI - N)^{-1} : x \neq 0\} = span\{N^k\}_{k=0}^{n-1}$ where the latter equality follows from the Neumann series for N which gives us the identity $(xI - N)^{-1} = \sum_{k=0}^{n-1} x^{-k-1} N^k$. Let V be the image of $span\{N^k\}_{k=0}^{n-1}$ in $M_n/\mathcal{D}_n$. Since $I = N^0$ gets mapped to 0, $dim(V)$ is at most $n-1$. If $dim(V) < n-1$, then there would exist real numbers $\{c_k\}_{i=1}^{n-1}$ not all zero such that $D = \sum_{k=1}^{n-1} c_k N^k$ is a diagonal matrix. Since D would also be nilpotent, it must be zero which is impossible since the index of N is n. So the dimension of V is exactly $n-1$.

Now let $p_{ij}(x)$ be the (j,i)th entry of $adj(xI - N)$. (Taking the (j,i)th entry is intentional and not a misprint; this choice makes the notation simpler later on.) Each polynomial p_{ij} is of degree $n-1$ if $i = j$ and of degree $n-2$ otherwise. The result of the previous paragraph implies that $\dim(span\{p_{ij}(x) : 1 \leq i,j \leq n; i \neq j\}) = n-1$ and hence $\{p_{ij}(x) : 1 \leq i,j \leq n; i \neq j\}$ spans P_{n-2}. We can obtain a basis of P_{n-2} from the elements $\{p_{ij}(x) : 1 \leq i,j \leq n; i \neq j\}$. Now choose any i such that the (i,i)th entry of $Q(\mathcal{A})$ is nonzero. By adjoining $p_{ii}(x)$ to the basis of P_{n-2}, we get a basis of P_{n-1} which we will write as $\{p_{i_k j_k}\}_{k=1}^n$.

We now apply the Nilpotent-Jacobian method to N. Let $X(x_1, x_2, ..., x_n)$ be the $n \times n$ matrix obtained by replacing the (i_k, j_k)th entry of N by the variable x_k for $1 \leq k \leq n$. Let $\vec{x} = (x_1, x_2, ..., x_n)$, and $\vec{\eta}$ be the n-vector whose kth entry is the (i_k, j_k)th entry of N and let $p(x, x_1, x_2, ..., x_n) = det(xI - X(x_1, x_2, ..., x_n))$. Then $\frac{\partial p}{\partial x_k}|_{\vec{x}=\vec{\eta}} = -p_{i_k j_k}(x)$. The kth column of the required Jacobian consists of the coefficients of the polynomial $p_{i_k j_k}(x)$ for all k where $1 \leq k \leq n$ and the invertibility of the Jacobian follows from the linear independence of these polynomials. □

It should be noted that the 4×4 nilpotent matrix in Example 1.1 is of index 4; hence Theorem 2.2 becomes false if we allow too many additional zeros outside of the main diagonal.

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