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# MAPS PRESERVING SPECTRAL RADIUS, NUMERICAL RADIUS, SPECTRAL NORM*

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**Abstract.** Characterizations are obtained for Schur (Hadamard) multiplicative maps on complex matrices preserving the spectral radius, numerical radius, or spectral norm. Similar results are obtained for maps under weaker assumptions. Furthermore, a characterization is given for maps $f$ satisfying $\|A \circ B\| = \|f(A) \circ f(B)\|$ for all matrices $A$ and $B$.

**Key words.** Schur product, Spectral radius, Numerical radius, Spectral norm, $l_p$ norm.

**AMS subject classifications.** 15A04, 15A18, 15A60.

**1. Introduction.** Let $M_{m,n}$ be the set of $m \times n$ complex matrices, and let $M_n = M_{n,n}$. There has been a great deal of interest in studying maps $f : M_{m,n} \to M_{m,n}$ preserving a given function $\Phi$ on $M_{m,n}$, i.e.,

$$\Phi(f(A)) = \Phi(A) \quad \text{for all } A \in M_{m,n}. \tag{1.1}$$

Early research on the topic focused mainly on linear maps $f$; see [14] and its references. Recently, researchers have also considered additive maps, multiplicative maps, differentiable maps; see [18] and its references.

The *Schur product* (also known as the *Hadamard product* or the *entrywise product*) of two matrices $A = [a_{ij}], B = [b_{ij}]$ is defined by $A \circ B = [a_{ij}b_{ij}]$. The study of Schur product is related to many pure and applied areas; see [10]. A map $f : M_{m,n} \to M_{m,n}$ is *Schur multiplicative* if

$$f(A \circ B) = f(A) \circ f(B) \quad \text{for all } A, B \in M_{m,n}.$$

The structure of Schur multiplicative maps $f : M_{m,n} \to M_{m,n}$ can be quite complicated. Nevertheless, if one imposes some mild additional assumptions on $f$, say, $f^{-1}[\{0_{m,n}\}] = \{0_{m,n}\}$, then (e.g., see [4]) $f$ will satisfy the following condition.

(†) There is a map $\mathcal{P} : M_{m,n} \to M_{m,n}$ such that $\mathcal{P}(A)$ is obtained from $A$ by permuting its entries in a fixed pattern, and a family of maps $f_{ij} : \mathbb{C} \to \mathbb{C}$ such that

$$f([a_{ij}]) = \mathcal{P}\left([f_{ij}(a_{ij})]\right). \tag{1.2}$$

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The purpose of this paper is to study Schur multiplicative maps on complex matrices which preserve the spectral radius, the numerical radius, or the spectral norm; see the definitions below. It is worth pointing out that preservers of the spectral radius, numerical radius, and the spectral norm have been studied under other assumptions by researchers; see [1, 2, 3, 5, 6, 7, 8, 9, 13, 15, 16, 17, 19]. Even though the Schur product is very different from the other algebraic operations on matrix spaces such as linear combination and the usual product, the preservers are always real linear maps as shown in the following discussion.

To describe our results, we introduce some notations and definitions. For $A \in M_{m,n}$, let

$$\|A\| = \max\{(x^*A^*Ax)^{1/2} : x \in \mathbb{C}^n, x^*x = 1\}$$

be the *spectral norm* of $A$; if $m = n$ let

$$w(A) = \max\{|x^*Ax| : x \in \mathbb{C}^n, x^*x = 1\} \quad \text{and} \quad r(A) = \max\{|\lambda| : \det(A - \lambda I) = 0\}$$

be the *numerical radius* and *spectral radius* of $A$, respectively.

We prove that a Schur multiplicative map satisfies $\|A\| = \|f(A)\|$ for all $A \in M_{m,n}$ if and only if there are permutation matrices $P \in M_m$ and $Q \in M_n$ such that one of the following holds.

(a) $f$ has the form $A \mapsto PAQ$ or $A \mapsto P\overline{A}Q$.

(b) $m = n$ and $f$ has one of the forms $A \mapsto PA^tQ$ or $A \mapsto PA^*Q$.

We obtain a similar result under the weaker assumption that $f$ has the form (†). Moreover, we get similar characterizations for maps $f : M_n \to M_n$ satisfying $w(A) = w(f(A))$ for all $A \in M_n$, or $r(A) = r(f(A))$ for all $A \in M_n$. Furthermore, we study maps $f$ such that $\Phi(f(A) \circ f(B)) = \Phi(A \circ B)$ for $\Phi(A) = r(A), w(A)$ or $\|A\|$. In [15], a characterization was given for maps $f : M_n \to M_n$ satisfying $w(A \circ B) = w(f(A) \circ f(B))$ for all $A, B \in M_n$. We characterize $f : M_{m,n} \to M_{m,n}$ such that $\|A \circ B\| = \|f(A) \circ f(B)\|$ for all $A, B \in M_{m,n}$. It is shown that maps $f : M_n \to M_n$ satisfying $r(A \circ B) = r(f(A) \circ f(B))$ for all $A, B \in M_n$ do not have nice structure.

In our discussion, let $J_{m,n}$ denote the $m \times n$ matrix with all entries equal to 1, and let $0_{m,n}$ be the $m \times n$ matrix with all entries equal to 0. Denote by $\mathcal{B} = \{E_{11}, E_{12}, \ldots, E_{mn}\}$ the standard basis for $M_{m,n}$. When $m = n$, we use the notation $J_n, 0_n$, etc. A square matrix is a *monomial matrix* if it is a product of a permutation matrix and a diagonal matrix.

## 2. Schur multiplicative preservers.

### 2.1. Spectral radius preservers.

THEOREM 2.1. *Let $r(A)$ be the spectral radius of $A \in M_n$. Suppose $f : M_n \to M_n$ satisfies (†) with $f_{ij}(0) = 0$. Then $r(f(A)) = r(A)$ for all $A \in M_n$ if and only if there is a complex unit $\gamma$ and an invertible monomial matrix $Q$ such that $f$ has one of the following forms:*

$$A \mapsto \gamma Q^{-1}AQ, \quad A \mapsto \gamma Q^{-1}A^tQ, \quad A \mapsto \gamma Q^{-1}\overline{A}Q, \quad A \mapsto \gamma Q^{-1}A^*Q. \tag{2.1}$$

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*Proof.* The sufficiency part is clear. We consider the necessity part. Since $f_{ij}(0) = 0$ for all $(i, j)$ pairs, $f(E_{jj})$ has only one nonzero entry. As $r(f(E_{jj})) = r(E_{jj}) = 1$, we may assume that $f(E_{jj}) = f_{jj}(1)E_{s_j,s_j}$ for some permutation $(s_1, \dots, s_n)$ of $(1, \dots, n)$. We may replace $f$ by a map of the form $A \mapsto Q^t f(A)Q$ for a suitable permutation matrix $Q$ and assume that $f(E_{jj}) = f_{jj}(1)E_{jj}$ for $j = 1, \dots, n$. Evidently,

$$|f_{jj}(1)| = 1 \quad \text{for } j = 1, \dots, n. \tag{2.2}$$

We may replace $f$ by the map $A \mapsto f(A)/f_{11}(1)$ and assume that $f(E_{11}) = 1$. Now for $X = E_{11} + E_{12} - E_{21} - E_{22}$ we have $r(f(X)) = r(X) = 0$. Note that $f(X) = E_{11} + f_{12}(1)E_{pq} + f_{22}(-1)E_{22} + f_{21}(-1)E_{uv}$. It is easy to check that $r(f(X)) = 0$ if and only if $f_{22}(-1) = -1$ and $f(E_{12} - E_{21}) = \mu E_{12} - \mu^{-1}E_{21}$ for some nonzero $\mu \in \mathbb{C}$. Similarly, we can show that for each $j > 1$, $f_{jj}(-1) = -1$ and $f(E_{1j} - E_{j1}) = \mu_j E_{1j} - \mu_j^{-1} E_{j1}$ for some nonzero $\mu_j \in \mathbb{C}$ for $j = 2, \dots, n$. Let $D = \operatorname{diag}(1, \mu_2, \dots, \mu_n)$. We may replace $f$ by the map $A \mapsto Df(A)D^{-1}$, and assume that $f(E_{1j} - E_{j1}) = E_{1j} - E_{j1}$ for all $j = 1, \dots, n$. So for each $j > 1$ we have two possibilities

$$\text{(i) } f(E_{1j}) = E_{1j},\ f(-E_{j1}) = -E_{j1}, \text{ or (ii) } f(E_{1j}) = -E_{j1},\ f(-E_{j1}) = E_{1j}. \tag{2.3}$$

Suppose $j > 1$. For any $a \in \mathbb{C}$, we have $r(f(E_{1j} + aE_{j1})) = r(E_{1j} + aE_{j1}) = |a|^{1/2}$. Similarly, $r(f(aE_{1j} - E_{j1})) = r(aE_{1j} - E_{j1}) = |a|^{1/2}$. Thus,

$$|f_{1j}(a)| = |f_{j1}(a)| = |a| \quad \text{for } j = 2, \dots, n. \tag{2.4}$$

Together with (2.2) and (2.3), we have $f(X) = E_{11} + aE_{1j} + bE_{j1} + cE_{jj}$, where $X = E_{11} + E_{1j} + E_{j1} + E_{jj}$ and $1 = |a| = |b| = |c|$ with either $a = 1$ or $b = -1$. Furthermore, since $r(f(X)) = r(X) = 2$, which is the Frobenius norm of $f(X)$, it follows that $f(X)$ is a rank one normal matrix with trace equal to 2. Hence, we have

$$f_{jj}(1) = 1, \ \ f(E_{1j} + E_{j1}) = a_j(E_{1j} + E_{j1}) \quad \text{for } j = 2, \dots, n, \tag{2.5}$$

where $a_j$ are real units. We may assume $a_j = 1$ for all $j$; otherwise, replace $f$ with the map $A \to Df(A)D$, where $D$ is the diagonal matrix with $D_{jj} = a_j$. We may also assume $f(E_{12}) = E_{12}$ by replacing $f$ with $A \to f(A)^t$ if needed.

Now, we show that

$$f(E_{1j}) = E_{1j} \quad \text{for } j = 3, \dots, n. \tag{2.6}$$

By the fact that $f(\pm E_{jj}) = \pm E_{jj}$ for $j = 2, \dots, n$, and $r(f(E_{22} + E_{2j} - E_{j2} - E_{jj})) = r(E_{22} + E_{2j} - E_{j2} - E_{jj}) = 0$, we have $f(E_{2j} - E_{j2}) = \mu E_{2j} - E_{j2}/\mu$ for some nonzero $\mu \in \mathbb{C}$. By (2.5), $f(E_{1j}) = E_{1j}$ or $f(E_{1j}) = E_{j1}$. If the latter holds, then for $Y = E_{12} + E_{1j} + E_{22} + E_{2j} - E_{j2} - E_{jj}$,

$$r(Y) = 0 \neq |\mu|^{1/3} = r(E_{12} + E_{22} + \mu E_{2j} + E_{j1} - E_{j2}/\mu - E_{jj}) = r(f(Y)),$$

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which is a contradiction. Thus, (2.6) and (2.3i) hold (and so $f(\pm E_{j1}) = \pm E_{j1}$ too). Since $0 = r(X) = r(f(X))$ for $X = -E_{11} - E_{1j} + E_{j1} + E_{jj}$, we see that $f(-E_{11}) = -E_{11}$ and $f(-E_{1j}) = -E_{1j}$.

Next, we prove that if $j \neq k$, then for any $a \in \mathbb{C}$, $f(aE_{jk}) = bE_{jk}$ with $b \in \{a, \bar{a}\}$. Note that $|a|^{1/3} = r(f(X)) = r(X)$ for $X = E_{1j} + aE_{jk} + E_{k1}$ (with $j, k \neq 1$). So, the first part of the assertion follows, i.e. $f(aE_{jk}) = bE_{jk}$ with $|b| = |a|$. Now for $Y = E_{11} + E_{1j} + E_{1k} + E_{j1} + E_{jj} + E_{jk} + E_{k1} + E_{kj} + E_{kk}$, $r(f(Y)) = r(Y) = 3$ is the Frobenius norm of $Y$. So, $f(Y)$ is a rank one normal matrix. It follows that $f(E_{jk}) = E_{jk}$ and $f(E_{kj}) = E_{kj}$. Applying a similar argument to $3 = r(Z) = r(f(Z))$ for $Z = E_{11} + E_{1j} - E_{1k} + E_{j1} + E_{jj} - E_{jk} - E_{k1} - E_{kj} + E_{kk}$, we see that $f(-E_{jk}) = -E_{jk}$ and $f(-E_{kj}) = -E_{kj}$. For any complex $a$, since $r(X) = |1-a|^{1/2} = r(f(X))$ and $r(Y) = |1+a|^{1/2} = r(f(Y))$ for $X = E_{jj} + aE_{jk} - E_{kj} - E_{kk}$ and $Y = E_{jj} + aE_{jk} + E_{kj} - E_{kk}$, we conclude that $f_{jk}(a) = a$ or $f_{jk}(a) = \bar{a}$.

Next, we show that $f_{kj}(a^{-1}) = f_{jk}(a)^{-1}$ for any nonzero $a \in \mathbb{C}$. To see this, let $X = E_{jj} + E_{kk} + aE_{jk} + a^{-1}E_{kj}$ with $r(X) = 2$. Then $f(X) = E_{jj} + E_{kk} + bE_{jk} + cE_{kj}$ with $|b| = |a|$ and $|c| = |a|^{-1}$. Then $f(X)$ has characteristic polynomial of the form $\lambda^2 - 2\lambda + (1 - bc)$ so that $r(f(X)) = \max\{|1 \pm \sqrt{bc}|\}$. Thus, $r(f(X)) = 2$ implies that $bc = 1$.

We may assume that $f_{12}(i) = i$. Otherwise, we may replace $f$ by the map $A \mapsto \overline{f(A)}$. It then follows that $f_{21}(-i) = -i$ by the argument in the preceding paragraph. Consider now the matrix $X = E_{11} - iE_{21} + E_{22} + aE_{12}$. We see that $f(X) = E_{11} - iE_{21} + E_{22} + aE_{12}$ or $E_{11} - iE_{21} + E_{22} + \bar{a}E_{12}$. However, if $f_{12}(a) = \bar{a}$, then $r(f(X)) \neq r(X)$. Therefore, $f_{12}(a) = a$, and by the assertion in the preceding paragraph, we have $f_{21}(a)^{-1} = a^{-1}$, i.e., $f_{21}(a) = a$. Consider $Y = aE_{11} + aE_{12} - aE_{21} - aE_{22}$, with $r(Y) = 0$, and note that $f(Y) = f_{11}(a)E_{11} + aE_{12} - aE_{21} + f_{22}(-a)E_{22}$, also with $r(f(Y)) = 0$. So $f_{11}(a) + f_{22}(-a) = 0$ and $f_{11}(a)f_{22}(-a) = -a^2$. This gives $(f_{11}(a), f_{22}(-a)) = \pm(a, -a)$. Suppose $f_{11}(a) = -a$. Consider $Z_b = aE_{11} + bE_{12} + E_{21} + E_{22}$. Then $Z_b$ has eigenvalues $\left\{(1+a) \pm \sqrt{(1-a)^2 + 4b}\right\}/2$ and $f(Z_b)$ has eigenvalues $\left\{(1-a) \pm \sqrt{(1+a)^2 + 4b}\right\}/2$. If $|1 + a| > |1 - a|$, let $b = -(1 + a)^2/4$ so that $r(f(Z_b)) = |1 - a|/2 < r(Z_b)$. If $|1 + a| \leq |1 - a|$ and $a \neq 0$, let $b = -(1 - a)^2/4$ so that $r(f(Z_b)) > |1 + a|/2 = r(Z_b)$. Therefore we must have $f_{11}(a) = f_{22}(a) = a$.

We now consider indices of the form $(1, j)$. Recall that $f_{ij}(a) = a$ or $\bar{a}$; we can see that $f_{1j}(i) = i$, otherwise $r(Y) \neq r(f(Y))$ for the matrix $Y = (1 + 2e^{i\pi/4})E_{11} + iE_{1j} + E_{j1} + E_{jj}$. Now, we can use the previous arguments to conclude that $f_{1j}(a) = f_{j1}(a) = a = f_{jj}(a)$. Similarly, we can then prove that for arbitrary $k$, we have $f_{jk}(a) = a$ for all $a \in \mathbb{C}$. □

Corollary 2.2. *Suppose $f : M_n \to M_n$ satisfies (†). The following conditions are equivalent.*

(a) *$f(A)$ and $A$ always have the same spectrum.*
(b) *$f(A)$ and $A$ always have the same eigenvalues counting multiplicities.*
(c) *$f$ has one of the first two forms in* (2.1) *with $\gamma = 1$ and $Q$ being an invertible monomial matrix.*

Next, we consider Schur multiplicative maps.

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THEOREM 2.3. *Suppose $n \geq 3$ and $f : M_n \to M_n$ is Schur multiplicative. Then $r(f(A)) = r(A)$ for all $A \in M_n$ if and only if $f$ has one of the forms in (2.1) with $\gamma = 1$ and $Q$ being a permutation matrix.*

*Proof.* The sufficiency part is clear. Consider the necessity part. We begin by showing that the hypotheses imply $f(0_n) = 0_n$ and $f(\mathcal{B}) = \mathcal{B}$, where $\mathcal{B} = \{E_{ij} : 1 \leq i, j \leq n\}$.

Note that if $A \circ A = A$ then $f(A) \circ f(A) = f(A)$, so 0-1 matrices are mapped into 0-1 matrices.

Let $\mathbf{S} = \mathbf{S}_1 \cup \mathbf{S}_2$ with

$$\mathbf{S}_1 = \{J_n - E_{jj} : 1 \leq j \leq n\} \quad \text{and} \quad \mathbf{S}_2 = \{J_n - E_{pq} : p \neq q\}.$$

If $A \in \mathbf{S}_1$ then $A$ is unitarily similar to $A_1 = J_n - E_{11}$. If $A \in \mathbf{S}_2$ then $A$ is unitarily similar to $A_2 = J_n - E_{12}$. Recall that the $k \times k$ Fourier matrix $F_k$ has $(r, s)$ entry equal to $e^{i2\pi(r-1)(s-1)/k}/\sqrt{k}$ for $1 \leq r, s \leq k$. Let $U = [1] \oplus F_{n-1}$. Then

$$U^* A_1 U = \begin{bmatrix} 0 & \sqrt{n-1} \\ \sqrt{n-1} & n-1 \end{bmatrix} \oplus 0_{n-2}$$

and

$$U^* A_2 U = \begin{bmatrix} 1 & \sqrt{n-1} \\ \sqrt{n-1} & n-1 \end{bmatrix} \oplus 0_{n-2} - U^* E_{12} U$$

so that $U^* A_2 U$ is unitarily similar to

$$\begin{bmatrix} 1 & (n-2)/\sqrt{(n-1)} & \sqrt{(n-2)/(n-1)} \\ \sqrt{n-1} & n-1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \oplus 0_{n-3}.$$

Thus,

$$r_1 = r(A_1) = \frac{n-1}{2} + \frac{\sqrt{n^2+2n-3}}{2} \quad \text{and} \quad r_2 = r(A_2) = \frac{n}{2} + \frac{\sqrt{n^2-4}}{2}.$$

Now, we claim that all other $A \in M_n$ satisfying $A \circ A = A$ with at least two zeros will satisfy

$$r(A) < r_2 < r_1.$$

Suppose $A \circ A = A$ with at least two zeros. We consider two cases.

Case 1. If $A$ has a zero on the off-diagonal position, then by Theorem 8.4.5 in [11], we see that $r(A) < r(A_2) < r(A_1)$.

Case 2. Suppose $A$ has two or more zeros on the diagonal positions. It suffices to show that $r(A_3) < r_2$ for $A_3 = J_n - E_{11} - E_{22}$. To this end, let $V = F_2 \oplus F_{n-2}$. Then

$$V^* A_3 V = \begin{bmatrix} 1 & 0 & \sqrt{2(n-2)} \\ 0 & -1 & 0 \\ \sqrt{2(n-2)} & 0 & n-2 \end{bmatrix} \oplus 0_{n-3}.$$

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Thus,

$$r(A_3) = \frac{n-1}{2} + \frac{\sqrt{n^2+2n-7}}{2} < r_2 < r_1.$$

Note that the inequality above holds if and only if $n \geq 3$. Since $r(f(A)) = r(A)$ for all $A$, we see that $f(\mathbf{S}) \subseteq \mathbf{S}$. Moreover, for any two $X \neq Y$ in $\mathbf{S}$, we see that $f(X) \neq f(Y)$. Otherwise, $r(f(X \circ Y)) \geq r_2 > r(X \circ Y)$. So, $f(\mathbf{S}) = \mathbf{S}$.

Let $X_{ij} = J_n - E_{ij}$. Since $f(\mathbf{S}) = \mathbf{S}$, we have $f(X_{ij}) = X_{p(i,j)}$ for some permutation $p$ of $\{(i,j) : 1 \leq i, j \leq n\}$. Then

$$\begin{aligned} f(0_n) &= f(X_{11} \circ X_{12} \circ \cdots \circ X_{nn}) = f(X_{11}) \circ f(X_{12}) \circ \cdots \circ f(X_{nn}) \\ &= X_{p(1,1)} \circ X_{p(1,2)} \circ \cdots \circ X_{p(n,n)} = 0_n. \end{aligned}$$

Similarly, for $E_{ij}$, we have

$$f(E_{ij}) = f\left(\prod_{(a,b)\neq(i,j)} X_{ab}\right) = \prod_{(a,b)\neq(i,j)} X_{p(a,b)} = E_{p(i,j)},$$

where the product intended is the Schur product. Thus, we conclude that $f(\mathcal{B}) = \mathcal{B}$.

Now, for each $(i,j)$ pair and for each $a \in \mathbb{C}$,

$$f(aE_{ij}) = f(aE_{ij} \circ E_{ij}) = f(aE_{ij}) \circ f(E_{ij}) = bf(E_{ij})$$

for a certain $b \in \mathbb{C}$. Define $f_{ij}(a) = b$ using this relation. If $A = (a_{ij})$ and $f(A) = B = (b_{ij})$, we see that

$$b_{ij} f(E_{ij}) = B \circ f(E_{ij}) = f(A \circ E_{ij}) = f(a_{ij} E_{ij}).$$

Thus, $b_{ij} = f_{ij}(a_{ij})$. Hence, $f$ has the form (†). We may apply Theorem 2.1 to conclude that $f$ has the asserted form. The assertion on $Q$ is easy to verify in view of the fact that $f(\mathcal{B}) = \mathcal{B}$. □

Corollary 2.4. *Suppose $n \geq 3$ and $f : M_n \to M_n$ is Schur multiplicative. Then the following are equivalent.*

(a) *$f(A)$ and $A$ always have the same spectrum.*

(b) *$f(A)$ and $A$ always have the same eigenvalues counting multiplicities.*

(c) *$f$ has one of the first two forms in* (2.1) *with $\gamma = 1$ and $Q$ being a permutation matrix.*

Note that Theorem 2.3 and Corollary 2.4 are not valid if $n = 2$. For example, define $f : M_2 \to M_2$ by $f(A) = \mathrm{diag}\,(a_{11}, a_{22})$ if $A = (a_{ij})$ such that $a_{21} = 0$, and $f(A) = A$ otherwise. Then $f(A)$ and $A$ always have the same eigenvalues, but $f$ is not of any of the forms in (2.1). Actually, it is not hard to show that a Schur multiplicative map $f : M_2 \to M_2$ preserves the spectral radius or spectrum if and only if $f(aE_{11} + bE_{22}) = aE_{jj} + bE_{kk}$ with $\{1,2\} = \{j,k\}$ and $f(aE_{12} + bE_{21}) = c(a,b)E_{12} + d(a,b)E_{21}$ such that $c(a,b), d(a,b)$ are any multiplicative maps $c, d : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ satisfying $c(a,b)d(a,b) = ab$.

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**2.2. Numerical radius preservers.** The *numerical range* of $A \in M_n$ is defined by

$$W(A) = \{x^*Ax : x \in \mathbb{C}^n,\ x^*x = 1\}.$$

Clearly, $w(A) = \max\{|\mu| : \mu \in W(A)\}$. The following properties of the numerical range and numerical radius will be used in our discussion; see for example [12, Chapter 1].

1. Suppose $A \in M_2$ has eigenvalues $\lambda_1, \lambda_2$. Then $W(A)$ is an elliptical disk with foci $\lambda_1, \lambda_2$ and minor axis of length

$$\left\{\operatorname{tr} A^*A - |\lambda_1|^2 - |\lambda_2|^2\right\}^{1/2}.$$

   If $A$ has real eigenvalues, then $W(A)$ has major axis along the real axis and consequently $w(A) = r(A + A^*)/2$.
2. Suppose $A \in M_n$ is unitarily similar to $A_1 \oplus A_2$. Then $W(A)$ is the convex hull of $W(A_1) \cup W(A_2)$. Consequently, $w(A) = \max\{w(A_1), w(A_2)\}$.
3. Suppose $A \in M_n$ is unitarily similar to $A_0 \oplus 0_{n-2}$. Then the following conditions are equivalent:

   (a) $\|A\| = w(A)$, (b) $\|A\| = r(A)$, (c) $w(A) = r(A)$, (d) $A$ is normal.

   Moreover, $\|A\| = 2w(A)$ if and only if $A_0$ is nilpotent.

THEOREM 2.5. *Let $w(A)$ be the numerical radius of $A \in M_n$. Suppose $f : M_n \to M_n$ satisfies (†). Then $f$ satisfies $w(f(A)) = w(A)$ for all $A \in M_n$ if and only if there is a complex unit $\gamma$ and a unitary monomial matrix $Q$ such that $f$ has one of the following forms:*

$$A \mapsto \gamma Q^{-1}AQ,\ \ A \mapsto \gamma Q^{-1}A^tQ,\ \ A \mapsto \gamma Q^{-1}\overline{A}Q,\ \ A \mapsto \gamma Q^{-1}A^*Q. \tag{2.7}$$

*Consequently, $g : M_n \to M_n$ is a Schur multiplicative map satisfying $w(g(A)) = w(A)$ for all $A \in M_n$ if and only if $g$ has one of the forms in (2.7) with $\gamma = 1$ and $Q$ a permutation matrix.*

*Proof.* The sufficiency part is clear. We divide the proof of the necessity part into several assertions.

**Assertion 1** There are complex units $\gamma_1, \dots, \gamma_n \in \mathbb{C}$ and a permutation matrix $P$ such that $f(E_{jj}) = \gamma_j PE_{jj}P^t$ for $j = 1, \dots, n$.

Let $f(E_{jj}) = \gamma_j E_{r_js_j}$. If $r_j = s_j$ then $|\gamma_j| = 1$, else, $|\gamma_j| = 2$. Now, consider $f(I)$. Note that each row and each column of $f(I)$ can have only one nonzero entry. So, $f(I)$ is permutationally similar to a direct sum of a diagonal unitary, and a monomial matrix so that each row and each column has a nonzero entry with modulus 2. Since $w(f(I)) = 1$, the second part of the direct sum cannot exist. So, $f(I)$ is a diagonal unitary matrix. We may apply a permutation similarity so that $Pf(I)P^t = \operatorname{diag}(\gamma_1, \dots, \gamma_n)$.

We may modify $f$ by $A \mapsto P^tf(A)P/\gamma_1$ and assume the following.

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**(A1)** $f(E_{jj}) = \gamma_j E_{jj}$ for $j = 1, \ldots, n$ with $\gamma_1 = 1$.

**Assertion 2** Assume (A1). Then for $j = 1, \ldots, n$ and for any $a > 0$, $f(aE_{jj}) = aE_{jj}$. Moreover, for any $(i, j)$ pairs with $i \neq j$ and for any $\mu \in \mathbb{C}$, there is $\nu \in \mathbb{C}$ with $|\mu| = |\nu|$ such that $f(\mu E_{ij} + \bar{\mu} E_{ji}) = \nu E_{ij} + \bar{\nu} E_{ji}$.

Note that for any $E_{ij}$ with $i \neq j$ and $\mu \in \mathbb{C}$, we have $f(\mu E_{ij}) = \nu E_{rs}$ for some $r \neq s$ and $\nu \in \mathbb{C}$ such that $|\mu| = |\nu|$.

For any $a > 0$, let $X = E_{11} + aE_{jj} + \sqrt{a}(E_{1j} + E_{j1})$. Then $w(f(X)) = w(X) = 1 + a$ equals the Frobenius norm of $f(X)$, so $f(X)$ is normaloid, which implies that $f(X) = E_{11} + aE_{jj} + \sqrt{a}(e^{it}E_{1j} + e^{-it}E_{j1})$. So, for $j = 2, \ldots, n$, we have $f_{jj}(a) = a$ for all $a > 0$. Consider $f(X)$ for $X = aE_{11} + E_{jj} + \sqrt{a}(E_{1j} + E_{j1})$. We see that $f_{11}(a) = a$ for all $a > 0$.

For any $\mu \in \mathbb{C}$ and $i \neq j$, let $|\mu| = a$ and $X = a(E_{ii} + E_{jj}) + \mu E_{ij} + \bar{\mu} E_{ji}$. Again $w(f(X)) = w(X) = 2a$ equals the Frobenius norm of $f(X)$, so we have $f(X) = a(E_{ii} + E_{jj}) + \nu E_{ij} + \bar{\nu} E_{ji}$ and hence $f(\mu E_{ij} + \bar{\mu} E_{ji}) = \nu E_{ij} + \bar{\nu} E_{ji}$, where $|\nu| = |\mu|$.

Replace $f$ by $A \mapsto Df(A)D^*$ with $D$ a diagonal unitary matrix so that $f(E_{1j} + E_{j1}) = E_{1j} + E_{j1}$ for $j = 2, \ldots, n$. Furthermore, we may assume that $f(X) = X$ for $X \in \{E_{12}, E_{21}\}$. Otherwise, replace $f$ by $A \mapsto f(A)^t$.

In the rest of the proof, we assume the following.

**(A2)** The conclusion of Assertion 2 holds, and $f(X) = X$ for $X = E_{12}, E_{21}$ or $E_{1j} + E_{j1}$ with $j = 3, \ldots, n$.

**Assertion 3** Assume (A2) holds. Then $f(E_{pq}) = E_{pq}$ and $f(-E_{pq}) = -E_{pq}$ for any $(p, q)$ pairs.

Suppose $j \in \{2, \ldots, n\}$. Since $f(E_{1j} + E_{j1}) = E_{1j} + E_{j1}$, we see that $f(X) = X$ or $X^t$ for $X \in \{E_{1j}, E_{j1}\}$. Thus, for $X = E_{11} + E_{1j} - E_{j1} - E_{jj}$, with $w(X) = 1$, there are some complex units $\mu, \nu \in \mathbb{C}$ such that $f(X) = E_{11} + E_{1j} - \mu E_{j1} - \nu E_{jj}$ or $f(X) = E_{11} + E_{j1} - \mu E_{1j} - \nu E_{jj}$. In both cases, we must have $\mu = 1$. Otherwise, for $Y = f(X)$ we have $w(Y) \geq w(Y + Y^*)/2 > 1$, which is a contradiction. Furthermore, we have $\nu = \pm 1$. Otherwise, for $Z = f(X)/\nu$, we have $w(Z) \geq w(Z + Z^*)/2 > 1$, which is a contradiction. For $\nu = -1$, we have $w(f(X)) = \sqrt{2}$, a contradiction, so $\nu = 1$. Thus $f(-E_{jj}) = -E_{jj}$ for all $j$ (consider $f(-X)$ for the case $j = 1$).

Now, for any $2 \leq p < q \leq n$, consider $X = E_{11} + E_{1p} + E_{1q} + E_{p1} + E_{pp} + E_{pq} + E_{q1} + E_{qp} + E_{qq}$. Then $f(X) = E_{11} + E_{1p} + E_{1q} + E_{p1} + E_{pp} + \nu E_{pq} + E_{q1} + \bar{\nu} E_{qp} + E_{qq}$. Since $w(f(X)) = w(X) = 3$ equals the Frobenius norm of $f(X)$, $f(X)$ is normaloid and we see that $\nu = 1$. So, $f(X) = X$ or $X^t$ for $X \in \{E_{pq}, E_{qp}\}$.

Let $X = -I + E_{12} + E_{2j} + E_{j1}$ for $j > 2$. Then $w(X) = \|X\| = |3 - i\sqrt{3}|/2 = \sqrt{3}$. Assume one or both of the following holds: $f(E_{2j}) = E_{j2}$, $f(E_{j1}) = E_{1j}$. Then $f(X)$ is not normal. Let $f(X) = H + iG$, where $H = H^*$, $G = G^*$. Then $\|H\| = 3/2$ and $\|G\| = \sqrt{3}/2$, and there is no unit vector $x$ such that $|x^*Hx| = 3/2$ and $|x^*Gx| = \sqrt{3}/2$. Thus,

$$|x^*(H + iG)x| < \{(3/2)^2 + (\sqrt{3}/2)^2\}^{1/2} = \sqrt{3} = w(X),$$

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which is a contradiction. Thus, we have $f(E_{2j}) = E_{2j}$ and $f(E_{j1}) = E_{j1}$. It will then follow that $f(Y) = Y$ for $Y \in \{E_{j2}, E_{1j}\}$.

Suppose $1 < j < k$. Then we can use $X = -I + E_{1j} + E_{jk} + E_{k1}$ to prove that $f(E_{jk}) = E_{jk}$. It follows that $f(E_{kj}) = E_{kj}$ and hence $f(E_{pq}) = E_{pq}$ for all $(p, q)$ pairs.

Now, by the first part of the proof of this assertion, we see that $f(-E_{pq}) = -E_{pq}$.

**Assertion 4** Assume (A2) and the conclusion of Assertion 3 hold. Then
(1) $f(\mu E_{pq}) = \mu E_{pq}$ for all $(p, q)$ pairs and for any $\mu \in \mathbb{C}$, or
(2) $f(\mu E_{pq}) = \bar{\mu} E_{pq}$ for all $(p, q)$ pairs and for any $\mu \in \mathbb{C}$.

Let $X = i(E_{11} - E_{22}) + E_{12} + E_{21}$. Then $f(X) = \mu E_{11} + \nu E_{22} + E_{12} + E_{21}$ for some complex units $\mu, \nu$. Since $1 = w(Y) \ge w(Y + Y^*)/2$ for $Y = f(X)$, it follows that $\overline{(\mu, \nu)} = \pm(i, -i)$. We may assume that $\mu = i$. Otherwise, replace $f$ by the map $A \mapsto \overline{f(A)}$. Under this assumption, we can consider $X = i(E_{11} - E_{jj}) + E_{1j} + E_{j1}$ and conclude that $f(-iE_{jj}) = -iE_{jj}$ for $j > 1$. If $Y = -i(E_{11} + E_{1j} + E_{j1} + E_{jj})$, then $w(f(Y)) = 2$ is the Frobenius norm of $f(Y)$, so the trace of $f(Y)$ has modulus 2 and thus $f(-iE_{11}) = -iE_{11}$. Now, repeating the above arguments to $X = i(E_{jj} - E_{11}) + E_{1j} + E_{j1}$, we see that $f(iE_{jj}) = iE_{jj}$.

Suppose $j \in \{2, \ldots, n\}$. For any $a > 0$, let $X = a(E_{11} + E_{1j} - E_{j1} - E_{jj})$. Then $f(X) = a(E_{11} + bE_{1j} + cE_{j1} + dE_{jj})$ for some norm one $b, c, d \in \mathbb{C}$. Using the same argument as that following assertion 3, it follows that $d = -1$, and $bc = -1$. Hence, $f(-aE_{jj}) = -aE_{jj}$. Now, consider $X = -a(E_{11} + E_{1j} - E_{j1} - E_{jj})$. Similarly, we see that $f(-aE_{11}) = -aE_{11}$. In summary, we have $f(aE_{jj}) = aE_{jj}$ for all $j = 1, \ldots, n$, and for all $a \in \mathbb{R}$.

**(A4)** At this point, we have $f(aE_{jj}) = aE_{jj}$ for all $a \in \mathbb{R}$ and $a = \pm i$. Also, we have $f(\pm E_{pq}) = \pm E_{pq}$ for all $(p, q)$ pairs.

Now, suppose $p \neq q$. For any $a > 0$, let $X = \sqrt{a}(E_{pp} - E_{qq}) \pm (aE_{pq} - E_{qp})$. Then $f(X) = \sqrt{a}(E_{pp} - E_{qq}) \pm (\mu E_{pq} - E_{qp})$ for some $\mu \in \mathbb{C}$ with $|\mu| = a$. Since $w(f(X)) = w(X) = (1 + a)/2$, we conclude that $f(\pm aE_{pq}) = \pm aE_{pq}$.

Next, let $z = |z|\gamma \in \mathbb{C}$ and consider the matrix $X = zE_{pp} + |z|(E_{qp} + E_{qq}) = |z|(\gamma E_{pp} + E_{qp} + E_{qq})$. We see that $f(X) = |z|(\delta E_{pp} + E_{qp} + E_{qq})$ for some complex unit $\delta$, and thus $f(zE_{pp}) = zE_{pp}$ or $\bar{z}E_{pp}$ by [15, Lemma 2.3]. Let $Y = |z|(E_{pp} + E_{pq}) + zE_{qp}$, so $f(Y) = |z|(E_{pp} + E_{pq} + \delta E_{qp})$ and hence $f(zE_{qp}) = zE_{qp}$ or $\bar{z}E_{qp}$ by [15, Lemma 2.4].

Now, for $a > 0$, let $Y = iaE_{pp} - E_{pq} + aE_{qp} + iE_{qq}$ with $w(Y) = (1 + a)/2 + \sqrt{(1 + a^2)/2}$. From the above discussion, we have $f(iaE_{pp}) = iaE_{pp}$ or $-iaE_{pp}$. If the latter case holds, then $f(Y) = -iaE_{pp} - E_{pq} + aE_{qp} + iE_{qq}$ is unitarily similar to $i(-aE_{pp} + E_{pq} + aE_{qp} + E_{qq})$ with numerical radius equal to $|1 - a|/2 + (1 + a)/\sqrt{2}$, which is a contradiction. Thus, we see that $f(Y) = Y$ and $f(iaE_{pp}) = iaE_{pp}$ for all $a > 0$. We can also consider $-Y$ and conclude that $f(-iaE_{pp}) = -iaE_{pp}$.

Up to now we have shown that $f_{pq}(z) = z$ or $\bar{z}$, and $f_{pp}(ia) = ia$ for all real $a$.

Now for any $z = |z|\gamma \in \mathbb{C} \setminus \mathbb{R}$ and any $p \neq q$, let $X = -i|z|E_{pp} + |z|E_{pq} + zE_{qq}$, so $f(X) = |z|(-iE_{pp} + E_{pq} + \delta E_{qq})$, where $\delta = \gamma$ or $\bar{\gamma}$. If $\delta = \bar{\gamma}$ then $w(f(X)) = w(if(X)) = |z|w(E_{pp} + E_{pq} + i\bar{\gamma}E_{qq})$, while $w(X) = w(iX) = |z|w(E_{pp} + E_{pq} +$

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$i\gamma E_{qq})$. By [15, Lemma 2.3], $i\gamma = i\bar{\gamma}$ or $-i\gamma$, which is a contradiction. It follows that $f(zE_{qq}) = zE_{qq}$ for all $z \in \mathbb{C}$.

Let $Y = |z|(e^{-i\pi/4}E_{pp} + E_{pq} + \gamma E_{qp})$, so $w(Y) = w(e^{i\pi/4}Y) = |z|w(E_{pp} + E_{pq} + i\gamma E_{qp})$. If $f(zE_{qp}) = \bar{z}E_{qp}$ we have $w(f(Y)) = |z|w(E_{pp} + E_{pq} + i\bar{\gamma}E_{qp})$, so by [15, Lemma 2.4], $i\gamma = i\bar{\gamma}$ or $-i\gamma$, which is a contradiction. It follows that $f(zE_{qp}) = zE_{qp}$ for all $z \in \mathbb{C}$. Thus the assertion is proved, and we have $f(zE_{pq}) = zE_{pq}$ for any $z \in \mathbb{C}$ and for any $(p, q)$ pairs. This completes the proof, as the function $f$ now has the asserted form.

For a Schur multiplicative map $g : M_n \to M_n$ satisfying $w(g(A)) = w(A)$ for all $A$, we see that $g(A) = 0$ if and only if $A = 0$. Thus, $g$ has the form (†). It is then easy to deduce the conclusion. ☐

We have the following corollary concerning numerical range preservers.

Corollary 2.6. *Let $f : M_n \to M_n$ satisfy (†). Then $f$ satisfies $W(f(A)) = W(A)$ for all $A \in M_n$ if and only if $f$ has one of the first two forms in* (2.7) *with $\gamma = 1$ and $Q$ being a unitary monomial matrix. Consequently, $g : M_n \to M_n$ is a Schur multiplicative map satisfying $W(g(A)) = W(A)$ for all $A \in M_n$ if and only if $g$ has one of the first two forms in* (2.7) *with $\gamma = 1$ and $Q$ being a permutation matrix.*

**2.3. Norm preservers.** In this subsection, we study maps preserving the spectral norm

$$\|A\| = \max\{(v^*A^*Av)^{1/2} : v^*v \le 1\} = \sqrt{\lambda_1(A^*A)},$$

where $\lambda_1(H)$ denotes the largest eigenvalue of a Hermitian matrix $H \in M_n$.

Theorem 2.7. *Let $\|A\|$ denote the spectral norm of $A \in M_{m,n}$. Suppose $f : M_{m,n} \to M_{m,n}$ satisfies (†). Then $\|f(A)\| = \|A\|$ for all $A \in M_{m,n}$ if and only if there are diagonal unitary matrices $P \in M_m$ and $Q \in M_n$ such that one of the following holds.*

(a) *$f$ has the form $A \mapsto PAQ$ or $A \mapsto P\overline{A}Q$,*

(b) *$m = n$ and $f$ has the form $A \mapsto PA^tQ$ or $A \mapsto PA^*Q$.*

*Consequently, a Schur multiplicative map $g : M_{m,n} \to M_{m,n}$ satisfies $\|g(A)\| = \|A\|$ for all $A \in M_{m,n}$ if and only if $g$ has one of the forms in* (a) *or* (b) *such that $P$ and $Q$ are permutation matrices.*

*Proof.* The sufficiency part is clear. Consider the necessity part. Without loss of generality, assume $n \ge m$. Note that $f(E_{jk}) = f_{jk}(1)E_{pq}$ for some $(p, q)$ and $|f_{jk}(1)| = 1$. For $X = E_{jj} + E_{kk}$ with $j \ne k$, we have $\|f(X)\| = \|X\| = 1$, and hence $f(E_{jj} + E_{kk}) = f_{jj}(1)E_{pq} + f_{kk}(1)E_{rs}$ with $\{p, q\} \cap \{r, s\} = \emptyset$. We may assume $f(E_{jj}) = f_{jj}(1)E_{jj}$ for $1 \le j \le m$, otherwise replace $f$ by the map $A \to Pf(A)Q$ for some permutations $P, Q$. By considering the matrices $E_{11} + E_{1j}$ and $E_{jj} + E_{1j}$ we see that

(i) $f(E_{1j}) = \gamma_j E_{1j}$ with $|\gamma_j| = 1$ for $j = 2, \ldots, m$, or
(ii) $f(E_{1j}) = \gamma_j E_{j1}$ with $|\gamma_j| = 1$ for $j = 2, \ldots, m$.
(Consideration of $E_{11} + E_{12} + E_{1j}$ shows that 'mixed' conditions cannot occur.)

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If $n > m$, then we can consider $f(E_{1,m+1})$ and show that condition (ii) cannot hold. If $m = n$ and case (ii) holds, we may replace $f$ by the map $A \mapsto f(A)^t$ and assume that condition (i) holds.

If $j > m$, we can use a similar argument to prove that $f(E_{1j}) = f_{1j}(1)E_{1p}$ for some $p > m$. We may then replace $f$ by a map of the form $A \mapsto f(A)(I_m \oplus Q_0)$ for some permutation $Q_0 \in M_{n-m}$ and assume that $f(E_{1j}) = f_{1j}(1)E_{1j}$ for $j > m$. Now, replace $f$ by the map $A \mapsto f(A)/f_{11}(1)$. We may assume that $f(E_{11}) = E_{11}$. Next, let $D_1 = \operatorname{diag}(1, f_{21}(1), \ldots, f_{m1}(1))$ and $D_2 = \operatorname{diag}(1, f_{12}(1), \ldots, f_{1n}(1))$. Replacing $f$ by the map $A \mapsto D_1^{-1} f(A) D_2^{-1}$, we may assume that $f(X) = X$ for

$$X \in \{E_{11}\} \cup \{E_{j1} : 2 \le j \le m\} \cup \{E_{1j} : 2 \le j \le n\}.$$

Since $\|f(X)\| = \|X\| = 2$ for $X = E_{11} + E_{1k} + E_{j1} + E_{jk}$, we see that $f(E_{jk}) = E_{jk}$ for all $(j, k)$ pairs.

Recall that $|f_{jk}(a)| = |a|$ for all complex numbers $a$ and all $(j, k)$ pairs. Now

$$\|f(X)\| = \|X\| = \sqrt{2 + 2|a|^2} \quad \text{for} \quad X = aE_{11} + aE_{1k} + E_{j1} + E_{jk}.$$

If $f(X) = f_{11}(a)E_{11} + f_{1k}(a)E_{1k} + E_{j1} + E_{jk}$, we have $f_{11}(a) = f_{1k}(a)$. Similarly, by the fact that

$$\|f(Y)\| = \|Y\| \quad \text{for} \quad Y = aE_{11} + E_{1k} + aE_{j1} + E_{jk},$$

we see that $f_{11}(a) = f_{j1}(a)$. Finally, consider $\|f(Z)\| = \|Z\|$ for $Z = E_{11} + E_{1k} + aE_{j1} + aE_{jk}$, we see that $f_{j1}(a) = f_{jk}(a)$. Consequently, $f_{11}(a) = f_{jk}(a)$ for all $a \in \mathbb{C}$ and all $(j, k)$ pairs. From this we can conclude that there is some function $\tau$ such that for any $(j, k)$, $f_{jk}(a) = \tau(a)$.

Let $Y = E_{11} + E_{12} + E_{21} + aE_{22}$. Then $Y^*Y$ has eigenvalues $\frac{1}{2}(3 + |a|^2 \pm \sqrt{(|a|^2 - 1)^2 + 4|1 + a|^2})$. Here $\|Y\| = \|f(Y)\|$ requires that $\tau(a) = a$ or $\bar{a}$. Now suppose $\tau(i) = i$. Otherwise, replace the map $f(A)$ with $A \to \overline{f(A)}$. For $a \in \mathbb{C}$ with $\operatorname{Re}(a) \neq 0$, let

$$Z = E_{11} + aE_{12} + aE_{21} + iE_{22} \;\; \text{with} \;\; \|Z\| = \sqrt{|a|^2 + 1 + \sqrt{2}|\operatorname{Re}(a) + \operatorname{Im}(a)|}.$$

Since $\|Z\| = \|f(Z)\|$, we see that $\tau(a) = a$. For $a$ purely imaginary, that is $a = ic$ with $c \in \mathbb{R} \setminus \{0\}$, consider $Z = E_{11} + cE_{12} + iE_{21} + icE_{22}$, with $\|Z\| = \sqrt{2 + 2c^2}$. If $\tau(ic) = -ic$, then $\|f(Z)\| = \max(\sqrt{2}, \sqrt{2}|c|) < \|Z\|$, which is a contradiction. So $\tau(a) = a$ for $a$ purely imaginary. Combining with the previous case we obtain $\tau(a) = a$ for all $a \in \mathbb{C}$.

The proof for Schur multiplicative maps is similar and simpler. See the proof of Theorem 2.5. $\square$

**3. Maps without the algebraic assumption.** In this section, we consider maps satisfying $\Phi(A \circ B) = \Phi(f(A) \circ f(B))$ for all $A, B \in M_{m,n}$ for $\Phi(A) = r(A), w(A)$, or $\|A\|$.

First, there is no good result for $\Phi(A) = r(A)$. For instance, let $f(A) = A$ if $A$ is not in triangular form, otherwise let $f(A) = \operatorname{diag}(a_{11}, \ldots, a_{nn})$. The result for $\Phi(A) = w(A)$ was done in [15]. For $\Phi(A) = \|A\|$, we will prove the following result.

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Theorem 3.1. *Let $f : M_{m,n} \to M_{m,n}$. Then*

$$\|A \circ B\| = \|f(A) \circ f(B)\| \tag{3.1}$$

*for all $A, B \in M_{m,n}$ if and only if there exist*

(a) *permutation matrices $P \in M_m$ and $Q \in M_n$,*

(b) *$\Theta \in M_{m,n}$ such that $|\Theta_{ij}| = 1$ for all $i, j$ and $\Theta \circ \Theta$ has rank one,*

(c) *diagonal unitaries $U_X \in M_m$ and $V_X \in M_n$ for each $X \in M_{m,n}$,*

*such that*

$$f(X) = \Theta \circ (U_X P X Q V_X) \quad \text{or} \quad f(X) = \Theta \circ (U_X P \overline{X} Q V_X) \tag{3.2}$$

*or, if $m = n$,*

$$f(X) = \Theta \circ (U_X P X^t Q V_X) \quad \text{or} \quad f(X) = \Theta \circ (U_X P X^* Q V_X). \tag{3.3}$$

Note that if $R \in M_{m,n}$ has rank 1 and is unimodular (that is, $|R_{ij}| = 1$ for all $i, j$) then $R = uv^*$ for some unimodular vectors $u \in \mathbb{C}^m, v \in \mathbb{C}^n$. Then $R \circ A = UAV^*$, where $U = \operatorname{diag}(u_1, \dots, u_m)$ and $V = \operatorname{diag}(v_1, \dots, v_n)$ are diagonal unitaries.

We will find it convenient to define the Schur-inverse of $A$ as the matrix $A^{(-1)}$ having $(i, j)$-entry equal to $(A_{ij})^{-1}$. The support of a matrix $A$ will denote the set of positions $(i, j)$ for which $A_{ij} \neq 0$.

Recall that the Schatten $p$-norm of $A$ is defined by

$$\|A\|_p = \left[\sum_{j=1}^{n} s_j(A)^p\right]^{1/p}, \quad \|A\|_\infty = \|A\| = s_1(A),$$

where $s_1(A) \geq \cdots \geq s_n(A)$ are the singular values of $A$. We believe Theorem 3.1 is probably valid if the spectral norm is replaced by certain Schatten $p$-norms, and so our proof will proceed in this more general context until the last step. We begin with three preliminary lemmas and a proposition giving some basic structure.

Lemma 3.2. *Let $A, B \in M_{m,n}$ and suppose each has at most two nonzero singular values. If $\|A\|_2 = \|B\|_2$ and $\|A\|_p = \|B\|_p$ for some $p \neq 2$, then $A$ and $B$ have the same singular values.*

*Proof.* Let $r = \|A\|_2$, so we may write the singular values of $A$ (respectively, $B$) as $r\cos\alpha$ and $r\sin\alpha$ (respectively, $r\cos\beta$ and $r\sin\beta$) for some $\alpha, \beta \in [0, \pi/4]$. The $r = 0$ case is trivial, so suppose $r \neq 0$. If $p = \infty$ we have $\alpha = \beta$; otherwise, let $f(\theta) = r^p(\cos^p\theta + \sin^p\theta)$, so $f(\alpha) = f(\beta)$. Elementary calculus shows that $f(\theta)$ is strictly monotone on $[0, \pi/4]$, so $\alpha = \beta$ as desired. □

Lemma 3.3. *Suppose $\|A\|_p = \|A\|_2 \neq 0$ for some $p \neq 2$. Then rank $A = 1$.*

*Proof.* Suppose $A$ has nonzero singular values $s_1, \dots, s_k$. Let $f(p) = \|A\|_p = \left[\sum_{j=1}^{k} s_j^p\right]^{1/p}$. If $k > 1$, then $f$ is strictly decreasing on $[1, \infty]$. Thus $A$ has at most 1 nonzero singular value and the result follows. □

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LEMMA 3.4. *Suppose* $|w| = |z| \neq 0$ *and* $p \neq 2$. *Then*

$$\left\| \begin{bmatrix} 1 & 1 \\ 1 & w \end{bmatrix} \right\|_p = \left\| \begin{bmatrix} 1 & 1 \\ 1 & z \end{bmatrix} \right\|_p \iff w = z \text{ or } w = \overline{z}.$$

*Proof.* The singular values of $X(\theta) = \begin{bmatrix} 1 & 1 \\ 1 & re^{i\theta} \end{bmatrix}$ are given by the positive square roots of $\frac{1}{2}(K \pm y(\theta))$, where $K = 3 + r^2$ and $y(\theta) = \sqrt{5 + 2r^2 + r^4 + 8r\cos\theta}$. Let

$$f(\theta) = \|X(\theta)\|_p^p = 2^{-p/2}[(K + y(\theta))^{p/2} + (K - y(\theta))^{p/2}].$$

Since $y(\theta)$ is strictly decreasing on $[0, \pi]$, some calculus shows that $f(\theta)$ is strictly monotone on $[0, \pi]$. Since $f(-\theta) = f(\theta)$, the result follows. □

PROPOSITION 3.5. *Suppose*

$$\|A \circ B\|_p = \|f(A) \circ f(B)\|_p \tag{3.4}$$

*for all* $A, B \in M_{m,n}$. *Then there exist a permutation* $\sigma$ *of* $\{(i,j) : 1 \leq i \leq m, 1 \leq j \leq n\}$ *and a map* $X \mapsto \Gamma_X$ *from* $M_{m,n}$ *to the set of unimodular matrices in* $M_{m,n}$ *such that*

$$f(X) = X_\sigma \circ \Gamma_X, \tag{3.5}$$

*where* $(X_\sigma)_{ij} = X_{\sigma(i,j)}$. *(Clearly if* $p = 2$ *the converse is true.)*

*Proof.* This result is probably known to experts; we include a proof for completeness. For $(i,j) \neq (r,s)$, $0 = \|E_{ij} \circ E_{rs}\|_p = \|f(E_{ij}) \circ f(E_{rs})\|_p$, so $f(E_{ij}) \circ f(E_{rs}) = 0$. Hence, $f(E_{ij})$ and $f(E_{rs})$ have disjoint support (they cannot both have nonzero entries at the same position). Since $f(E_{ij}) \neq 0$, it follows by the pigeonhole principle that $f(E_{ij}) = c_{ij}E_{\sigma(i,j)}$ for some permutation $\sigma$ of $\{(i,j) : 1 \leq i \leq m, 1 \leq j \leq n\}$. Since

$$1 = \|E_{ij} \circ E_{ij}\|_p = \|f(E_{ij}) \circ f(E_{ij})\|_p = \|c_{ij}^2 E_{\sigma(i,j)}\|_p = |c_{ij}|^2,$$

we have $|c_{ij}| = 1$. Thus, for each $X \in M_{m,n}$,

$$|X_{ij}| = \|X \circ E_{ij}\|_p = \|f(X) \circ f(E_{ij})\|_p = \|f(X) \circ c_{ij}E_{\sigma(i,j)}\|_p = |f(X)_{\sigma(i,j)}|$$

and so there is some unimodular $\Gamma_X$ such that

$$f(X) = \sum_{i,j} X_{ij}(\Gamma_X)_{\sigma(i,j)}E_{\sigma(i,j)} = \sum_{i,j} X_{\sigma^{-1}(i,j)}(\Gamma_X)_{ij}E_{ij} = \Gamma_X \circ X_{\sigma^{-1}}. \quad \square$$

*Proof of Theorem 3.1.* Clearly if $f$ has form (3.2) or (3.3), then (3.4) holds. Now assume (3.4) holds, so $f$ has form (3.5) by Proposition 3.5. Since

$$\|\Gamma_J \circ \Gamma_J\|_p = \|f(J) \circ f(J)\|_p = \|J \circ J\|_p = \sqrt{mn}$$

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and $\|\Gamma_J \circ \Gamma_J\|_2 = \sqrt{mn}$, $\Gamma_J \circ \Gamma_J$ has rank one by Lemma 3.3. Let $\Theta = \Gamma_J$; by replacing $X \mapsto f(X)$ with $X \mapsto f(X) \circ \Theta^{(-1)}$ (where $\Theta^{(-1)}$ is the Schur-inverse), we may suppose $f(J) = J$. Then

$$\|A\|_p = \|A \circ J\|_p = \|f(A) \circ f(J)\|_p = \|f(A)\|_p \tag{3.6}$$

for any $A \in M_{m,n}$, so $f$ is both a $\|\cdot\|_p$ and a $\|\cdot\|_2$ isometry. In particular, if rank $A = 1$, then

$$\|f(A)\|_p = \|A\|_p = \|A\|_2 = \|f(A)\|_2,$$

so rank $f(A) = 1$ by Lemma 3.3. We shall often use this fact in what follows.

**Step 1.** We show that, modulo permutations and transposition, $f$ fixes $E_{ij}$.

By replacing $X \mapsto f(X)$ by $X \mapsto Pf(X)Q$ for some permutations $P \in M_m$ and $Q \in M_n$, we may assume $f$ maps $E_{11}$ back to a multiple of $E_{11}$; that is, $\sigma(1,1) = (1,1)$. Without loss of generality, assume $m \leq n$. Let $A = \sum_{j=1}^{n} p_j E_{1j}$, where $p_j$ denotes the $j$th prime. Since rank $A = 1$, rank $f(A) = 1$ and so all nonzero entries of $f(A)$ must lie in the first row or column. If $m < n$, the nonzero entries of $f(A)$ must lie in the first row. If $m = n$ and the nonzero entries of $f(A)$ lie in the first column, replace $X \mapsto f(X)$ by $X \mapsto f(X)^t$ so that the nonzero entries of $f(A)$ lie in the first row. By replacing $f(X)$ with $f(X)Q'$ for an appropriate permutation $Q'$, we may assume that $\sigma(1,j) = (1,j)$ for all $j$. Applying a similar argument to $B = \sum_{i=1}^{m} p_i E_{i1}$ we may assume that $\sigma(i,1) = (i,1)$ for all $i$. Let $C = E_{11} + 2E_{1j} + 2E_{i1} + 4E_{ij}$. Since rank $C = 1$, we have rank $f(C) = 1$, which implies $\sigma(i,j) = (i,j)$ for all $i,j$. In summary, we may assume that $f(X) = X \circ \Gamma_X$, where $|(\Gamma_X)_{ij}| = 1$ for all $i,j$.

Define an equivalence relation $A \sim B$ if $A = UBV$ for some diagonal unitaries $U$ and $V$. Note that $A \sim B$ if and only if $A = \Gamma \circ B$ for some rank one matrix $\Gamma$ with unimodular entries. Three properties of this equivalence relation are:

(a) $A \sim B \Rightarrow \|A\|_p = \|B\|_p$ for all $p$

(b) $A \sim B \iff \overline{A} \sim \overline{B}$

(c) If $A_1 \sim B_1$ and $A_2 \sim B_2$, then $A_1 \circ A_2 \sim B_1 \circ B_2$.

We shall freely make use of these properties in what follows.

**Step 2.** We show that $f(A) \sim A \sim \overline{A}$ if rank $A = 1$.

If $A$ is a rank one matrix, then so is $f(A)$. We can write $A = \sum_{i,j} x_i y_j E_{ij}$ and $f(A) = \sum_{i,j} x_i y_j \Gamma_{ij} E_{ij}$. Let $r$ (respectively, $s$) be the index corresponding to the first nonzero row (respectively, column) of $A$. Let

$$V = \sum_{j=1}^{n} \Gamma_{rj}^{-1} E_{jj} \quad \text{and} \quad U = E_{rr} + \sum_{i \neq r} \Gamma_{is}^{-1} E_{ii},$$

so that $Uf(A)V$ has the same first nonzero row and column as $A$. Since rank $Uf(A)V$ $= 1$ and $Uf(A)V$ has the same support as $A$, $Uf(A)V = A$, so $f(A) \sim A$. Note also that $A \sim \overline{A}$ since $\overline{A} = (\sum_{i=1}^{m} g(x_i)E_{ii})A(\sum_{j=1}^{n} g(y_j)E_{jj})$ where $g(x) = \overline{x}/x$ if $x \neq 0$ and $g(0) = 1$.

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**Step 3.** We show that $f(A) \sim A$ or $\overline{A}$ if $A$ is supported on a $2 \times 2$ matrix. To simplify notation, we write all matrices as if they were $2 \times 2$. Let

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{and} \quad f(A) = A \circ \Gamma_A.$$

If any $a_{ij}$ is zero, then $f(A) \sim A \sim \overline{A}$, so suppose all $a_{ij}$ are nonzero.

Let $B$ be the rank one matrix $a_{11}^{-1}E_{11} + a_{12}^{-1}E_{12} + a_{21}^{-1}E_{21} + a_{12}^{-1}a_{11}a_{21}^{-1}E_{22}$. Since $f(B) \sim B$,

$$\|A \circ B\|_p = \|f(A) \circ f(B)\|_p = \|f(A) \circ B\|_p.$$

Let $d = a_{22}a_{12}^{-1}a_{11}a_{21}^{-1}$, so

$$A \circ B = \begin{bmatrix} 1 & 1 \\ 1 & d \end{bmatrix} \quad \text{and} \quad f(A) \circ B \sim \begin{bmatrix} 1 & 1 \\ 1 & de^{i\psi} \end{bmatrix}$$

for some $\psi \in [-\pi, \pi]$. By Lemma 3.4, $f(A) \circ B \sim A \circ B$ or $f(A) \circ B \sim \overline{A} \circ \overline{B} \sim \overline{A} \circ B$ (since rank $B = 1$, $B \sim \overline{B}$). Taking the Schur product with the Schur-inverse $B^{(-1)}$, we have $f(A) \sim A$ or $\overline{A}$ as desired.

**Step 4.** Let $C = E_{11} + E_{12} + E_{21} + iE_{22}$. If $f(C) \sim \overline{C}$, replace $X \mapsto f(X)$ by $X \mapsto \overline{f(X)}$, so we may assume $f(C) \sim C$. We show $f(A) \sim A$ for any $A$ supported on a $2 \times 2$ matrix.

Let $B$ be a rank one matrix supported on the same $2 \times 2$ matrix as $C$. By Step 3, $f(C \circ B) \sim C \circ B$ or $\overline{C \circ B}$. In the latter case,

$$\|C \circ (C \circ B)\|_p = \|f(C) \circ f(C \circ B)\|_p = \|C \circ \overline{C} \circ \overline{B}\|_p = \|\overline{B}\|_p = \|B\|_p,$$

so by Lemma 3.2, $C \circ C \circ B$ and $B$ have the same singular values. Since $B$ has rank one, so must $C \circ C \circ B$, which is a contradiction if $B$ has four nonzero entries. Thus we must have $f(C \circ B) \sim C \circ B$ whenever $B$ has rank one, four nonzero entries, and is supported on the same $2 \times 2$ matrix as $C$.

Suppose $A$ is supported on the same $2 \times 2$ matrix as $C$. To show that $f(A) \sim A$, suppose, by way of contradiction, $f(A) \sim \overline{A}$ and $A \not\sim \overline{A}$ (so $A$ has four nonzero entries). Let $B$ be a rank one matrix supported on the same $2 \times 2$ matrix as $C$ such that $A \circ B = E_{11} + E_{12} + E_{21} + dE_{22}$, $d \neq 0$. Since

$$\left\| \begin{bmatrix} 1 & 1 \\ 1 & di \end{bmatrix} \right\|_p = \|A \circ B \circ C\|_p = \|f(A) \circ f(B \circ C)\|_p$$
$$= \|\overline{A} \circ B \circ C\|_p = \|\overline{A} \circ \overline{B} \circ C\|_p = \left\| \begin{bmatrix} 1 & 1 \\ 1 & \overline{d}i \end{bmatrix} \right\|_p,$$

Lemma 3.4 implies $d = \overline{d}$. But this implies $A \circ B = \overline{A} \circ \overline{B} \sim \overline{A} \circ B$, whence $A \sim \overline{A}$ which is a contradiction. Thus, $f(A) \sim A$ as desired.

Suppose $f(X) \sim X$ for all $X$ supported on $\{(i,j) : i = p, q; j = r, s\}$, and $f(X) \sim \overline{X}$ for all $X$ supported on $\{(i,j) : i = p, q \text{ and } j = s, t\}$. We show this gives

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a contradiction. Without loss of generality, we take $p = r = 1$, $q = s = 2$, and $t = 3$, and write all matrices as $2 \times 3$.

Let $w = e^{2\pi i/3}$, $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & w & \overline{w} \end{bmatrix}$, $X = \begin{bmatrix} 1 & 1 & 0 \\ 1 & \overline{w} & 0 \end{bmatrix}$, and $Y = \begin{bmatrix} 0 & 1 & 1 \\ 0 & \overline{w} & w \end{bmatrix}$. We may write $f(A) \sim \begin{bmatrix} 1 & 1 & 1 \\ 1 & w\alpha & \overline{w}\beta \end{bmatrix}$, where $|\alpha| = |\beta| = 1$. Using Lemma 3.4 and

$$\left\| \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \right\|_p = \|A \circ X\|_p = \|f(A) \circ f(X)\|_p = \left\| \begin{bmatrix} 1 & 1 & 0 \\ 1 & \alpha & 0 \end{bmatrix} \right\|_p,$$

we have $\alpha = 1$. Using Lemma 3.4 and

$$\begin{aligned} \left\| \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \right\|_p &= \|A \circ Y\|_p = \|f(A) \circ f(Y)\|_p \\ &= \left\| \begin{bmatrix} 0 & 1 & 1 \\ 0 & w^2 & \overline{w}^2\beta \end{bmatrix} \right\|_p = \left\| \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & \overline{w}^4\beta \end{bmatrix} \right\|_p, \end{aligned}$$

we have $\beta = w^4$. Thus $f(A)$ has singular values $\sqrt{3+\sqrt{3}}, \sqrt{3-\sqrt{3}}$, whereas $A$ has singular values $\sqrt{3}, \sqrt{3}$. But $\|A\|_p = \|f(A)\|_p$ so, by Lemma 3.2, $A$ and $f(A)$ have the same singular values, giving a contradiction.

It follows that $f(X) \sim X$ for any $X$ supported on a $2 \times 2$ matrix lying in the first two rows. By taking transposes in the preceding argument, we can conclude that $f(X) \sim X$ for any $X$ supported on a $2 \times 2$ matrix lying in the first two columns, and hence for any $X$ supported on any $2 \times 2$ matrix.

**Step 5.** Suppose $X_{ij} \neq 0 \iff i \in I$ and $j \in J$ for some $I \subset \{1, \ldots, m\}$ and $J \subset \{1, \ldots, n\}$. We shall show $f(X) \sim X$.

Given any $2 \times 2$ submatrix of $X$ with row and column indices in $I$ and $J$ respectively, choose a matrix $B$ supported on the same $2 \times 2$ submatrix such that $B \circ X = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ on that $2 \times 2$ submatrix. Then

$$\|B \circ X\|_p = \|f(B) \circ f(X)\|_p = \|B \circ X \circ \Gamma_X\|_p.$$

By Lemma 3.2, $B \circ X \circ \Gamma_X$ has rank one, and hence the corresponding submatrix of $\Gamma_X$ has rank one. Thus every $2 \times 2$ submatrix of $\Gamma_X$ with row and column indices in $I$ and $J$ respectively has rank one, so the submatrix of $\Gamma_X$ with row indices in $I$ and column indices in $J$ has rank one. By redefining, if necessary, the entries of $\Gamma_X$ which lie outside this submatrix, we have rank $\Gamma_X = 1$, as desired.

**Step 6.** We now restrict to the spectral norm ($p = \infty$), and show $f(X) \sim X$ for all $X$.

First suppose $A$ is an $m \times n$ matrix such that $|A|^t|A|$ is irreducible (the absolute value is entry-wise). Let $x \in \mathbb{R}^n_+$ be the positive unit eigenvector (from Perron-Frobenius theory) such that $\||A|\|^2 = x^t|A|^t|A|x$. Let $B_{ij} = |A_{ij}|/A_{ij}$ if $A_{ij} \neq 0$ and $B_{ij} = 1$ otherwise. Then

$$\||A|\| = \|A \circ B\| = \|f(A) \circ f(B)\| = \|f(A) \circ B\| = \|A \circ B \circ \Gamma_A\| = \||A| \circ \Gamma_A\|.$$

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Write $\tilde{A} = |A| \circ \Gamma_A$, and let $v \in \mathbb{C}^n$ be a unit vector such that $\|\tilde{A}\| = \|\tilde{A}v\|$. Let $D$ be a diagonal unitary such that $D|v| = v$. Then

$$\||A|\| = \|\tilde{A}\| = \|\tilde{A}v\| = \|\tilde{A}D|v|\| \leq \||A||v|\| \leq \||A|x\| = \||A|\|,$$

so all the inequalities are in fact equalities. Equality in the second inequality implies $|v| = x$ has strictly positive entries. Equality in the first inequality implies, for each $i$,

$$\left|\sum_j (\tilde{A}D)_{ij} x_j\right| = \sum_j |A_{ij}| x_j,$$

so $(\tilde{A}D)_{ij}$ has the same argument for each $j$ such that $A_{ij} \neq 0$. By redefining, if necessary, the entries of $\Gamma_A$ corresponding to $A_{ij} = 0$, we may conclude that $\tilde{A}D = |A| \circ R$ for some unimodular rank one matrix $R$. Hence, rank $\Gamma_A = 1$, whence $f(A) \sim A$.

Finally, let $A$ be an arbitrary $m \times n$ matrix. There exist permutations $P$ and $Q$ so that $PAQ$ is a direct sum of matrices $A_1, \ldots, A_k$ with $|A_j|^t|A_j|$ irreducible for $j = 1 \ldots k$. Apply the argument in the preceding paragraph to each $A_j$ to conclude that the submatrix of $\Gamma_A$ corresponding to the supporting submatrix for $A_j$ has rank 1. We can then redefine, if necessary, the entries of $\Gamma_A$ not supported on any of the submatrices $A_j$ so that rank $\Gamma_A = 1$, whence $f(A) \sim A$. $\square$

We have shown for any $p$ that, if $f$ satisfies (3.4), then $f(X)$ has the form (3.2) or (3.3) whenever $X$ has no zero entries (or more generally, if $X$ is permutationally equivalent to $\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}$ for some $A$ with no zero entries). We conjecture that Theorem 3.1 holds if $\|\cdot\|$ is replaced by $\|\cdot\|_p$ if $p$ is not even.

What if $p$ is even? Obviously $p = 2$ is an exceptional case, but consider the following map: Fix any complex unit $e^{i\theta}$ and define $f : M_3 \to M_3$ by

$$f(A) = A \quad \text{if any of } A_{13}, A_{22}, A_{31} \text{ are nonzero,}$$

and

$$f\left(\begin{bmatrix} a & b & 0 \\ c & 0 & d \\ 0 & e & g \end{bmatrix}\right) = \begin{bmatrix} a & b & 0 \\ c & 0 & d \\ 0 & e & ge^{i\theta} \end{bmatrix} \quad \text{otherwise.}$$

Note that if $A = \begin{bmatrix} a & b & 0 \\ c & 0 & d \\ 0 & e & g \end{bmatrix}$, then

$$A^*A = \begin{bmatrix} |a|^2 + |c|^2 & \overline{a}b & \overline{c}d \\ a\overline{b} & |b|^2 + |e|^2 & \overline{e}g \\ c\overline{d} & e\overline{g} & |d|^2 + |g|^2 \end{bmatrix}$$

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has Frobenius norm independent of the argument of $g$, so $\|f(A)\|_4 = \|A\|_4$ regardless of which $e^{i\theta}$ we choose. Thus $f$ satisfies (3.4) when $p = 4$, yet for almost any choice of $e^{i\theta}$, $f$ does not satisfy the conclusion of Theorem 3.1.

More generally, it seems that, for sufficiently large matrices $A$ with a given zero pattern, $\|A\|_p^p = \operatorname{tr}(A^*A)^{p/2}$ will not depend on the arguments of certain entries for certain even values of $p$. An interesting open problem is to determine exactly which zero patterns give rise to these counterexamples for a given even $p \geq 4$.

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$$f([a_{i,j}]) = \left[ \prod_{1\leq p\leq m, 1\leq q\leq n} f_{i,j}^{p,q}(a_{p,q}) \right],$$

where $f_{i,j}^{p,q} : \mathbb{C} \to \mathbb{C}$ is the multiplicative map defined by

$$f_{i,j}^{p,q}(\lambda) = (i,j)\text{-entry of } f(J_{m,n} + (\lambda - 1)E_{p,q}).$$

This comment inspired us to improve Theorem 2.3 and Corollary 2.4 by removing the unnecessary condition $f(0) = 0$ from an earlier version of the paper.

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