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PRINCIPAL EIGENVECTORS OF IRREGULAR GRAPHS*

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Abstract. Let G be a connected graph. This paper studies the extreme entries of the principal eigenvector x of G, the unique positive unit eigenvector corresponding to the greatest eigenvalue λ_1 of the adjacency matrix of G. If G has maximum degree Δ, the greatest entry $x_{\max}$ of x is at most $1/\sqrt{1+\lambda_1^2/\Delta}$. This improves a result of Papendieck and Recht. The least entry $x_{\min}$ of x as well as the principal ratio $x_{\max}/x_{\min}$ are studied. It is conjectured that for connected graphs of order $n \geq 3$, the principal ratio is always attained by one of the lollipop graphs obtained by attaching a path graph to a vertex of a complete graph.

Key words. Spectral radius, Irregular graph, Eigenvectors.

AMS subject classifications. 05C50, 15A18.

1. Introduction. The study of the eigenvectors of the adjacency matrix of a graph has led to many applications. The principal eigenvectors of graphs form the basis of the PageRank algorithm used by Google (see [2]). The eigenvector corresponding to the second largest eigenvalue of a connected graph has been used in spectral partitioning algorithms (see [14]). The ratio of a non-negative irreducible matrix has been studied by Ostrowski [9, 10], Minc [5], De Oliveira [8], Latham [4] and Zhang [17] among others.

In this paper, we study principal eigenvectors of connected irregular graphs. Our graph theoretic notation is standard, see West [15]. The *eigenvalues* of a graph G are the eigenvalues λ_i of its adjacency matrix A, indexed so that $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$. The greatest eigenvalue, λ_1, is also called the *spectral radius*. If G is connected, then the multiplicity of λ_1 is 1. The positive eigenvector of length 1 corresponding to λ_1 will be called the *principal* eigenvector of G. Note that if G is k-regular and connected, then $k = \lambda_1 > \lambda_2$ and the principal eigenvector of G is $\frac{1}{\sqrt{n}}\mathbf{1}$, where $\mathbf{1}$ is the all one vector.

If $x \in \mathbb{R}^n$, we denote by $x_{\max}$ the greatest entry of x and by $x_{\min}$ the least entry of x. The *ratio* of x is defined as $x_{\max}/x_{\min}$. If A is a non-negative irreducible matrix and ρ is the positive eigenvalue of A of maximum modulus, then $\sigma(A)$ denotes the ratio of x, where x is a positive eigenvector corresponding to ρ. The *principal ratio*, $\gamma(G)$, of a connected graph G is the ratio of the principal eigenvector of G. It is well known that G is regular if and only if $\gamma(G) = 1$. Thus, one can regard $\gamma(G)$ as a measure of the irregularity of the graph G.

Schneider [12] has proved that if G is a connected graph, then $\gamma(G) \leq \lambda_1^{n-1}$. In Section 2, we improve Schneider's result as well as some recent results of Nikiforov

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[7]. We also present some lower bounds on $\gamma(G)$ which improve previous results of Ostrowski [9]and Zhang [17].

If G is a connected graph and x is the principal eigenvector of G, then Papendieck and Recht [11] proved that $x_{\max} \le \frac{1}{\sqrt{2}}$ with equality if and only if $G = K_{1,n-1}$. In Section 3, we generalize and extend this result and we also prove some lower and upper bounds for the extreme entries of the principal eigenvector of a connected graph.

2. The principal ratio of a graph. In this section, we study the principal ratio $\gamma(G)$ of a connected graph. As stated before, this parameter can be regarded as a measure of the irregularity of a graph.

2.1. Upper bounds for $\gamma(\mathbf{G})$. Let G be a connected graph with vertex set $\{1, 2, \dots, n\}$. Suppose that the path P_r on the first r vertices $1, 2, \dots, r$ is a subgraph of G. Subgraphs found by taking the shortest path between two vertices in G and renumbering the vertices will have this property and will be induced paths. Such examples will have length at most D, the diameter of G.

Assume that $\lambda_1 = \lambda_1(G) > 2$. Let x be the principal eigenvector of G. Because $Ax = \lambda_1 x$, it follows that, for each vertex i,

$$\lambda_1 x_i = \sum_{j \sim i} x_j.$$

Since the entries of x are all positive, if we let $x_0 = 0$, it follows that

$$x_2 \le \lambda_1 x_1 \quad \text{and} \quad x_k \le \lambda_1 x_{k-1} - x_{k-2} \quad \text{for} \quad 2 \le k \le r, \tag{2.1}$$

where $x_2 = \lambda_1 x_1$ if and only if vertex 1 has degree 1 while $x_k = \lambda_1 x_{k-1} - x_{k-2}$ if and only if vertex $k-1$ has degree 2. Thus

$$\begin{bmatrix} x_k \\ x_{k-1} \end{bmatrix} \le \begin{bmatrix} \lambda_1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_{k-1} \\ x_{k-2} \end{bmatrix} \quad \text{for } 2 \le k \le r.$$

Thus

$$\begin{bmatrix} x_k \\ x_{k-1} \end{bmatrix} \le \begin{bmatrix} \lambda_1 & -1 \\ 1 & 0 \end{bmatrix}^{k-1} \begin{bmatrix} x_1 \\ x_0 \end{bmatrix} \quad \text{for } 1 \le k \le r,$$

with equality for $2 \le k \le r$ if and only if vertices $2, 3, \dots, r-1$ all have degree 2.

Let $\sigma, \tau = \frac{1}{2}(\lambda_1 \pm \sqrt{\lambda_1^2 - 4})$, the eigenvalues of the 2×2 matrix above. Because $\lambda_1 > 2$, then $\sigma \ne \tau$ and there is an invertible matrix P such that

$$\begin{bmatrix} \lambda_1 & -1 \\ 1 & 0 \end{bmatrix} = P \begin{bmatrix} \sigma & 0 \\ 0 & \tau \end{bmatrix} P^{-1}.$$

Then

$$x_k \le \begin{bmatrix} 1 & 0 \end{bmatrix} P \begin{bmatrix} \sigma^{k-1} & 0 \\ 0 & \tau^{k-1} \end{bmatrix} P^{-1} \begin{bmatrix} x_1 \\ x_0 \end{bmatrix} \quad \text{for} \quad 1 \le k \le r.$$

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Thus, there are constants a, b such that

$$x_k \le a\sigma^{k-1} + b\tau^{k-1} \quad \text{for} \quad 1 \le k \le r.$$

Solving the recursion in the case of equality gives $a+b = 1$ when $k = 1$ and $\sigma a+\tau b = \lambda_1$ when $k = 2$. Thus we may take $a = (\lambda_1 - \tau)/(\sigma - \tau)$ and $b = (\sigma - \lambda_1)/(\sigma - \tau)$. Substituting these values and noting that $\sigma\tau = 1$ and $\sigma + \tau = \lambda_1$, we get

$$x_k \;\le\; \frac{\sigma^k - \tau^k}{\sigma - \tau}\, x_1 \quad \text{for} \quad 1 \le k \le r, \tag{2.2}$$

with equality at a particular value of $k \ge 2$ if and only if vertex 1 has degree 1 and vertices $2, 3, \ldots, k-1$ all have degree 2.

Theorem 2.1. *Let G be a connected graph of order n with spectral radius $\lambda_1 > 2$ and principal eigenvector x. Let d be the shortest distance from a vertex on which x is maximum to a vertex on which it is minimum. Then*

$$\gamma(G) \le \frac{\sigma^{d+1} - \tau^{d+1}}{\sigma - \tau}, \tag{2.3}$$

where $\sigma = \frac{1}{2}(\lambda_1 + \sqrt{\lambda_1^2 - 4})$ and $\tau = \sigma^{-1}$.

Equality is attained if and only if either G is regular or there is an induced path of length $d > 0$ whose endpoints index $x_{\min}$ and $x_{\max}$ and the degrees of the endpoints are 1 and 3 or more, respectively, while all other vertices of the path have degree 2 in G.

Proof. Let x denote the principal eigenvector of G. If G is regular, then $d = 0$ and the bound equals 1. If G is irregular, relabel the vertices so that x_1 is a smallest entry of x and x_{d+1} a largest and let P_{d+1} be a path of length d connecting vertex 1 and vertex $d+1$. Because $\lambda_1 > 2$, the upper bound (2.3) follows from (2.2) with $r = d+1$.

If $d = 0$, then equality is attained if and only if $\gamma(G) = 1$, that is, if and only if G is regular. If $d > 0$ and equality occurs then, by (2.2), vertex 1 has degree 1 and vertices $2, 3, \ldots r-1$ all have degree 2. Vertex r cannot have degree 1, otherwise $x_{r-1} = \lambda_1 x_r > x_{r-1}$, a contradiction. If vertex r has just one more neighbour, x_{r+1} say, then $\lambda_1 x_r = x_{r+1} + x_r < 2x_r$ and so $\lambda_1 < 2$, a contradiction. Thus, x_r has degree at least 3. Conversely, if such a path exists and the endpoints index extreme entries, then all the inequalities in (2.2) are equalities and we have equality for the ratio. ∎

In a recent paper [7], Nikiforov proved that if H is a proper subgraph of a connected graph G with n vertices and diameter D, then

$$\lambda_1(G) - \lambda_1(H) > \frac{1}{n\lambda_1^{2D}(G)}. \tag{2.4}$$

In proving (2.4), Nikiforov uses the following inequality. If r is the distance between two vertices i and j of a connected graph G, then

$$\frac{x_i}{x_j} < \lambda_1(G)^r. \tag{2.5}$$

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If $\lambda_1 = \lambda_1(G) > 2$, then (2.2) gives the following slight improvement of (2.5)

$$\frac{x_i}{x_j} \leq \frac{\sigma^{r+1} - \tau^{r+1}}{\sigma - \tau}, \tag{2.6}$$

where $\sigma = \frac{\lambda_1 + \sqrt{\lambda_1^2 - 4}}{2}$ and $\tau = \frac{\lambda_1 - \sqrt{\lambda_1^2 - 4}}{2} = \frac{1}{\sigma}$. Following the arguments from [7], inequality (2.6) can be used to show that if H is a proper subgraph of a connected graph G with n vertices and diameter D, then

$$\lambda_1(G) - \lambda_1(H) > \frac{(\sigma - \tau)^2}{n(\sigma^{D+1} - \tau^{D+1})^2} = \frac{1}{n\left(\sigma^D + \sigma^{D-2} + \ldots + \sigma^{-(D-2)} + \sigma^{-D}\right)^2}.$$

This is a slight improvement of inequality (2.4) since

$$\frac{1}{n\lambda_1^{2D}} = \frac{1}{n(\sigma + \tau)^{2D}} = \frac{1}{n\left(\sigma^D + \binom{D}{1}\sigma^{D-2} + \ldots + \binom{D}{D-1}\sigma^{-(D-2)} + \sigma^{-D}\right)^2}.$$

When $\lambda_1(G) \leq 2$, inequality (2.4) can be further improved by using Smith's classification [13] of the connected graphs with spectral radius at most 2.

2.2. Maximizing $\gamma(\mathbf{G})$ over connected graphs on n vertices. Let $\gamma(n)$ be the maximum value of $\gamma(G)$ taken over all connected graphs of order n. Let $P_r \cdot K_s$ denote the graph of order $n = r + s - 1$ formed by identifying an end vertex of a path P_r on $r \geq 2$ vertices with a vertex of the complete graph K_s on $s \geq 2$ vertices. In [1], Brightwell and Winkler call such graphs *lollipop* graphs and show that, for a graph on n vertices, the maximum expected time for a random walk between two vertices is attained on a lollipop graph $P_r \cdot K_s$ of order n with $s = \lceil (2n-2)/3 \rceil$. A computer search reveals that for $3 \leq n \leq 9$, $\gamma(n)$ is always attained by one of the two lollipop graphs $P_r \cdot K_s$ of order n with $s = \lceil (n+1)/4 \rceil + \epsilon$ where $\epsilon = 1$ or 2. We conjecture that $\gamma(n)$ is always attained by a lollipop graph of order n. As a first step in studying this conjecture, we examine the entries of the principal eigenvectors of the graphs $P_r \cdot K_s$.

2.3. Principal eigenvectors of lollipop graphs. Let the vertices of P_r in $P_r \cdot K_s$ be $\{1, 2, \ldots, r\}$ and the vertices of K_s be $\{r, r+1, \ldots, r+s-1\}$. We continue to assume that $n = r + s - 1$ denotes the order of $P_r \cdot K_s$ and that $r \geq 2$. However, we assume now that $s \geq 3$. This ensures that $\lambda_1 = \lambda_1(P_r \cdot K_s) > 2$ since K_s is a proper induced subgraph. As before, we let $\sigma = \frac{1}{2}(\lambda_1 + \sqrt{\lambda_1^2 - 4}\ )$ and $\tau = \sigma^{-1}$.

Lemma 2.2. *Let λ_1 be the greatest eigenvalue of $P_r \cdot K_s$ and let x be the principal eigenvector. Then*

$$x_k \ = \ \frac{\sigma^k - \tau^k}{\sigma - \tau} x_1 \quad \textit{for} \quad 1 \leq k \leq r,$$

while

$$x_k \ = \ \frac{1}{s-1} \frac{\sigma^{r+1} - \tau^{r+1}}{\sigma - \tau} x_1 \quad \textit{for} \quad r+1 \leq k \leq n.$$

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Proof. Because the vertices $\{1, 2, \ldots, r\}$ are the vertices of a path, the inequalities (2.1) and (2.2) are equalities for $k = 1, 2, \ldots, r$. This gives the first expression.

Because x is unique, by symmetry we have $x_{r+1} = x_{r+2} = \cdots = x_n$. Thus, $\lambda_1 x_r = (Ax)_r = x_{r-1} + (s-1)x_n$. So, for $k = r+1, \ldots, n$,

$$\begin{aligned} x_k &= \frac{1}{s-1}(\lambda_1 x_r - x_{r-1}) \\ &= \frac{1}{s-1} \frac{(\sigma+\tau)(\sigma^r - \tau^r) - (\sigma^{r-1} - \tau^{r-1})}{\sigma - \tau} x_1 \\ &= \frac{1}{s-1} \frac{\sigma^{r+1} - \tau^{r+1}}{\sigma - \tau} x_1. \ \square \end{aligned}$$

LEMMA 2.3. *Let λ_1 be the greatest eigenvalue of $P_r \cdot K_s$ and let x be the principal eigenvector. Then $x_{r+1} = \cdots = x_n$ and $x_1 < x_2 < \cdots < x_{r-1} < x_n < x_r$. Thus*

$$\gamma(P_r \cdot K_s) = \frac{x_r}{x_1} = \frac{\sigma^r - \tau^r}{\sigma - \tau}.$$

Also,

$$\lambda_1^{r-1} - \lambda_1^{-r-1} < \gamma(P_r \cdot K_s) < \lambda_1^{r-1}.$$

em Proof. These results follow from Lemma 2.2, but it is perhaps clearer and shorter to give direct proofs based on the recursion (2.1). Because the graph has maximum degree s and K_s is an induced subgraph, it follows that

$$2 \leq s - 1 < \lambda_1 < s.$$

Thus, $\lambda_1 - s + 2 > 1$, $\lambda_1 > 2$ and σ is real. Because $1, 2, \ldots, r$ are the vertices of a path, the inequalities (2.1) are equalities for $k = 1, 2, \ldots, r$. Thus

$$x_k = \lambda_1 x_{k-1} - x_{k-2} = (\lambda_1 - 1)x_{k-1} + (x_{k-1} - x_{k-2}),$$

and it follows by induction that $x_k < \lambda_1 x_{k-1}$ for $k = 2, \ldots, r$. Thus, $x_1 < x_2 < \cdots < x_r$ and $x_r < \lambda_1^{r-1} x_1$. Because x is unique, by symmetry we have $x_{r+1} = x_{r+2} = \cdots = x_n$. Thus $\lambda_1 x_n = (Ax)_n = x_r + (s-2)x_n$ and so

$$x_r = (\lambda_1 - s + 2)x_n > x_n. \tag{2.7}$$

Also, $(\lambda_1 - s + 2)x_n = x_r > (\lambda_1 - 1)x_{r-1} \geq (\lambda_1 - s + 2)x_{r-1}$ so $x_{r-1} < x_n$. Thus $\gamma = x_r/x_1$. Finally, because $\sigma > \lambda_1$ and $\sigma\tau = 1$, we have $\tau < 1/\lambda_1$ and so

$$\gamma = \frac{x_r}{x_1} = \frac{\sigma^r - \tau^r}{\sigma - \tau} > \frac{\lambda_1^r - \lambda_1^{-r}}{\sqrt{\lambda_1^2 - 4}} > \lambda_1^{r-1} - \lambda_1^{-r-1}. \ \square$$

Note that substituting the expression (2.7) into the formulae in Lemma 2.2 gives

$$\frac{\sigma^{r+1} - \tau^{r+1}}{\sigma^r - \tau^r} = \frac{s-1}{\lambda_1 - s + 2} \tag{2.8}$$

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a relation that determines λ_1 in terms of r and s.

In the next lemma, we see that the parameter r has only a slight effect on the spectral radius of the graphs $P_r \cdot K_s$.

LEMMA 2.4. *For $r \geq 2$ and $s \geq 3$,*

$$s - 1 + \frac{1}{s(s-1)} < \lambda_1(P_r \cdot K_s) < s - 1 + \frac{1}{(s-1)^2}.$$

Proof. Because $P_2 \cdot K_s$ is an induced subgraph of $P_r \cdot K_s$, we have $\lambda_1(P_r \cdot K_s) \geq \lambda_1(P_2 \cdot K_s)$. Thus, it is sufficient to establish the lower bound when $r = 2$.

If $r = 2$, then in (2.8),

$$\frac{\sigma^3 - \tau^3}{\sigma^2 - \tau^2} = \frac{\sigma^2 + \sigma\tau + \tau^2}{\sigma + \tau} = \frac{\lambda_1^2 - 1}{\lambda_1} = \frac{s-1}{\lambda_1 - s + 2}.$$

Thus, λ_1 is a root of the polynomial

$$p(x) = x^3 - (s-2)x^2 - sx + s - 2 = (x - s + 1)(x^2 + x - 1) - 1$$

and so

$$\lambda_1 = s - 1 + \frac{1}{\lambda_1^2 + \lambda_1 - 1}. \tag{2.9}$$

Since $\lambda_1 > s - 1$, we deduce that

$$\lambda_1(P_2 \cdot K_s) < s - 1 + \frac{1}{s^2 - s - 1}.$$

Substituting this upper bound for λ_1 in the denominator in (2.9), a lengthy calculation shows that, for $s \geq 4$,

$$\lambda_1(P_2 \cdot K_s) > s - 1 + \frac{1}{s(s-1)}. \tag{2.10}$$

This also holds for $s = 3$ because $p(x) < 0$ when $x = s - 1 + \frac{1}{s(s-1)} = 2 + \frac{1}{6}$.

Suppose now that $r \geq 3$. Then from (2.8), we have

$$\lambda_1(P_r \cdot K_s) = s - 2 + (s-1) \cdot \frac{\sigma^r - \tau^r}{\sigma^{r+1} - \tau^{r+1}} < s - 2 + (s-1)\tau.$$

Thus, to obtain the upper bound, it is sufficient to show that

$$s - 2 + (s-1)\tau < s - 1 + \frac{1}{(s-1)^2},$$

or, equivalently, that

$$2\tau = \lambda_1 - \sqrt{\lambda_1^2 - 4} < \frac{2}{s-1} + \frac{2}{(s-1)^3}.$$

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Isolating and squaring the radical, we find that the above inequality is equivalent to

$$\lambda_1\left(\frac{1}{s-1}+\frac{1}{(s-1)^3}\right)>1+\frac{1}{(s-1)^2}+\frac{2}{(s-1)^4}+\frac{1}{(s-1)^6}.$$

Substituting the lower bound (2.10) for λ_1 shows that the inequality holds. $\square$

Given n, it would be interesting to see if there are bounds that can be used to determine the values of r and s with $r+s=n+1$, for which $\gamma(P_r\cdot K_s)$ is maximum. A computer run indicates that for $n\leq 50$, the maximum is always attained by one of the two graphs $P_r\cdot K_s$ of order n with $s=\lceil (n+1)/4\rceil+\epsilon$ where $\epsilon=1$ or 2.

2.4. The principal ratio of an irreducible matrix. In this subsection, we review some of the previous results regarding the ratio of the principal eigenvector of an irreducible matrix. This parameter has been studied by many researchers. Ostrowski [10] proved the following result.

LEMMA 2.5 (Ostrowski [10]). *If A is an n by n positive, irreducible matrix, then*

$$\gamma(A)\leq \frac{\max_{i,j\in[n]} a_{ij}}{\min_{i,j\in[n]} a_{ij}}. \tag{2.11}$$

Inequality 2.11 was improved by Minc [5].

THEOREM 2.6 (Minc [5]). *If A is an n by n positive irreducible matrix, then*

$$\gamma(A)\leq \max_{j,s,t\in[n]}\frac{a_{sj}}{a_{tj}}.$$

Equality holds if and only if the p-th row of A is a multiple of the q-th row for some pairs of indices p and q satisfying

$$\frac{a_{ph}}{a_{qh}}=\max_{j,s,t}\frac{a_{sj}}{a_{tj}}.$$

From Minc's proof [5], one can deduce the stronger inequality

$$\gamma(A)\leq \max_{j}\frac{a_{1j}}{a_{nj}},$$

where $x_1=x_{\max}$ and $x_n=x_{\min}$. This result along with other refinements of Minc's inequality were obtained by De Oliveira [8] and Latham [4].

Let $k^{(i)}$ be the smallest positive entry of the i-th row of A. Ostrowski [10] also proved the following theorem.

THEOREM 2.7 (Ostrowski [10]). *If A is an irreducible non-negative matrix and assume that $1,\ldots,r$ is a path ($a_{i,i+1}\neq 0$ for $i\in[r-1]$) from 1 to r, where $x_1=x_{\max}$ and $x_{\min}=x_r$. Then*

$$\gamma(A)\leq \prod_{i=1}^{r-1}\frac{\rho-a_{ii}}{k^{(i)}}. \tag{2.12}$$

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The proof of this result is done by using the inequality

$$\frac{x_{i+1}}{x_i} \leq \frac{\rho - a_{ii}}{k^{(i)}} \tag{2.13}$$

for each $i \in [r-1]$. This follows from

$$(\rho - a_{ii})x_i = \sum_{j \neq i} a_{ij} x_j \geq k^{(i)} x_{i+1}.$$

The previous result implies a result of Schneider [12] who proved that

$$\gamma(A) \leq \left(\frac{\rho - k_1}{k} \right)^{n-1},$$

where k is the smallest positive entry of A and $k_1 = \min a_{ii}$. When A is the adjacency matrix of a connected graph G on n vertices, the previous inequality implies

$$\gamma(G) \leq \lambda_1^{n-1}(G)$$

which is weaker than Theorem 2.1.

Using the stronger inequality

$$(\rho - a_{ii})x_i = \sum_{j \neq i} a_{ij} x_j \geq k^{(i)}(x_{i+1} + x_{i-1})$$

and the argument from the proof of Theorem 2.1, one can improve Theorem 2.7.

2.5. Lower bounds for γ. For a matrix A, let $k_1 = \min_{i\in[n]} a_{ii}$ and for $i \in [n]$, let $d_i = \sum_{j=1}^n a_{ij}$. Also, let $\Delta = \max_{i\in[n]} d_i$ and $\delta = \min_{i\in[n]} d_i$.

Ostrowski [9] proved the following result.

THEOREM 2.8 (Ostrowski [9]). *If A is a non-negative irreducible matrix, then*

$$\gamma \geq \max\left(\frac{\Delta - k_1}{\rho - k_1}, \frac{\rho - k_1}{\delta - k_1} \right) \geq \sqrt{\frac{\Delta - k_1}{\delta - k_1}}. \tag{2.14}$$

If we apply this inequality to the adjacency matrix of an irregular graph G, we obtain

$$\gamma(G) \geq \max\left(\frac{\Delta}{\lambda_1}, \frac{\lambda_1}{\delta} \right) \geq \sqrt{\frac{\Delta}{\delta}}.$$

This inequality has been also observed by Zhang [17].

We improve Theorem 2.8 as follows.

THEOREM 2.9. *If A is a non-negative irreducible matrix. Let $T = \{t : d_t > \rho\}$ and $S = \{s : d_s < \rho\}$. Also, let $i \in T$ such that $d_i = \Delta$ and $j \in S$ such that $d_j = \delta$. Then*

$$\gamma \geq \max\left(\frac{\Delta - a_{ii} + \sum_{r\in T\setminus\{i\}} a_{ir} \frac{d_r - \rho}{\rho - a_{rr}}}{\rho - a_{ii}}, \frac{\rho - a_{jj}}{\delta - a_{jj} - \sum_{r\in S\setminus\{j\}} \frac{a_{jr}(\rho - d_r)}{\rho - a_{rr}}} \right). \tag{2.15}$$

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Proof. For each r, we have

$$(\rho - a_{rr})x_r = \sum_{l \neq r} a_{rl}x_l \geq (d_r - a_{rr})x_{\min} > 0,$$

where the final expression is positive since A is nonnegative and irreducible. Thus, for each r,

$$\frac{x_r}{x_{\min}} \geq \frac{d_r - a_{rr}}{\rho - a_{rr}}.$$

Now

$$(\rho - a_{ii})x_i = \sum_{r \in T \setminus \{i\}} a_{ir}x_r + \sum_{s \notin T} a_{is}x_s.$$

Since $d_i = \Delta$, this implies

$$(\rho - a_{ii})x_i \geq x_{\min} \sum_{r \in T \setminus \{i\}} \frac{a_{ir}(d_r - a_{rr})}{\rho - a_{rr}} + (\Delta - \sum_{s \in T} a_{is})x_{\min}.$$

Putting $d_r - a_{rr} = (d_r - \rho) + (\rho - a_{rr})$ and simplifying, we obtain the first lower bound

$$\gamma \geq \frac{x_i}{x_{\min}} \geq \frac{\Delta - a_{ii} + \sum_{r \in T \setminus \{i\}} a_{ir} \frac{d_r - \rho}{\rho - a_{rr}}}{\rho - a_{ii}}.$$

To obtain the second lower bound on γ, we note first that for each r,

$$0 < (\rho - a_{rr})x_r = \sum_{l \neq r} a_{rl}x_l \leq (d_r - a_{rr})x_{\max}$$

and so

$$\frac{x_{\max}}{x_r} \geq \frac{\rho - a_{rr}}{d_r - a_{rr}} > 0.$$

Now

$$(\rho - a_{jj})x_j = \sum_{r \in S \setminus \{j\}} a_{jr}x_r + \sum_{s \notin S} a_{js}x_s.$$

Since $d_j = \delta$, this implies

$$(\rho - a_{jj})x_j \leq x_{\max} \sum_{r \in S \setminus \{j\}} \frac{a_{jr}(d_r - a_{rr})}{\rho - a_{rr}} + (\delta - \sum_{s \in S} a_{js})x_{\max}.$$

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Putting $d_r - a_{rr} = (d_r - \rho) + (\rho - a_{rr})$ and simplifying, we obtain

$$(\rho - a_{jj})x_{\min} \le (\rho - a_{jj})x_j \le x_{\max} \sum_{r \in S\setminus\{j\}} \frac{a_{jr}(d_r - \rho)}{\rho - a_{rr}} + (\delta - a_{jj})x_{\max}.$$

The previous inequality implies

$$\gamma = \frac{x_{\max}}{x_{\min}} \ge \frac{\rho - a_{jj}}{\delta - a_{jj} - \sum_{r\in S\setminus\{j\}} \frac{a_{jr}(\rho - d_r)}{\rho - a_{rr}}}. \qquad \square$$

Note that if $T = \{i\}$ and $S = \{j\}$, then the previous result is the same as Ostrowski's bound.

If G is a graph with $\delta = \Delta - 1$ having exactly one vertex of degree $\Delta - 1$, then the previous result implies that

$$\gamma \ge \frac{\Delta + \Delta \cdot \frac{\Delta - \lambda_1(G)}{\lambda_1(G)}}{\lambda_1(G)} = \frac{\Delta^2}{\lambda_1^2(G)}$$

which improves Ostrowski's inequality

$$\gamma \ge \frac{\Delta}{\lambda_1(G)}.$$

3. Extreme entries of principal eigenvectors. In this section, we determine some upper bounds for the entries of x when G is an irregular graph.

Papendieck and Recht [11] obtained the following upper bound on $x_{\max}$. Zhao and Hong [16] generalized Papendieck and Recht's result to symmetric nonnegative matrices with zero trace.

THEOREM 3.1. *If G is a connected graph on n vertices, then*

$$x_{\max} \le \frac{1}{\sqrt{2}} \tag{3.1}$$

with equality if and only if $G = K_{1,n-1}$.

The next result improves Theorem 3.1.

THEOREM 3.2. *Let G be a connected graph on n vertices whose principal eigenvector is x. For $i \in [n]$, if d_i is the degree of vertex i, then*

$$x_i \le \frac{1}{\sqrt{1 + \frac{\lambda_1^2}{d_i}}}. \tag{3.2}$$

Equality is attained if and only if $x_i = x_{\max}$ and G is the join of vertex i and a regular graph on $n-1$ vertices.

Proof. Using the Cauchy-Schwarz inequality, we have

$$\sum_{j\sim i} x_j^2 \ge \frac{(\sum_{j\sim i} x_j)^2}{d_i} = \frac{\lambda_1^2 x_i^2}{d_i}.$$

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It follows that

$$1 = \sum_{l=1}^{n} x_l^2 \geq x_i^2 + \sum_{j \sim i} x_j^2 \geq x_i^2 \left(1 + \frac{\lambda_1^2}{d_i}\right).$$

This proves the inequality.

Equality is attained if and only if $x_j = a$ for $j \sim i$ and $x_l = 0$ for l not adjacent to i. Since G is connected, it follows that every vertex different from i is adjacent to i. For $j \neq i$, we have $\lambda_1 x_j = x_i + (d_j - 1)x_j$. This implies that the graph $G \setminus \{i\}$ is regular. ◻

Theorem 3.1 follows now easily since $\lambda_1 \geq \sqrt{\Delta} \geq \sqrt{d_1}$.

An easy lower bound on $x_{\max}$ can be obtained from the fact that

$$(n-1)x_{\max}^2 + x_{\min}^2 \geq \sum_{i=1}^{n} x_i^2 = 1.$$

This implies the following result.

Lemma 3.3. *If G is a graph on n vertices with maximum degree Δ and minimum degree δ, then*

$$x_{\max} \geq \frac{1}{\sqrt{n - 1 + \frac{1}{\gamma^2(G)}}} \geq \frac{1}{\sqrt{n - \frac{\Delta - \delta}{\Delta}}}$$

with equality if and only if G is regular. If G is irregular, then

$$x_{\max} > \frac{1}{\sqrt{n - \frac{1}{\Delta}}}.$$

Proof. From the previous inequality, we have

$$x_{\max}^2 \left((n-1) + \frac{1}{\gamma^2(G)}\right) \geq 1.$$

Because $\gamma(G) \geq \sqrt{\frac{\Delta}{\delta}}$, we obtain

$$x_{\max} \geq \frac{1}{\sqrt{n - \frac{\Delta - \delta}{\Delta}}}$$

with equality if and only if G is regular.

If G is irregular, we have $\Delta - \delta \geq 1$ which implies $x_{\max} \geq \frac{1}{\sqrt{n - \frac{1}{\Delta}}}$. The previous inequality is strict because otherwise, it would imply that $n-1$ vertices of the graph have their eigenvector entry equal to $x_{\max}$ and one vertex has its eigenvector entry equal to $x_{\min}$. It can be shown easily that this situation cannot happen when G is irregular. ◻

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Note that the graph on n vertices obtained from K_n by deleting an edge has $x_{\max} < \frac{1}{\sqrt{n-\frac{4}{n-1}}} = \frac{1}{\sqrt{n-\frac{4}{\Delta}}}$ so the second inequality from Lemma 3.3 comes very close to approximating $x_{\max}$ in this case.

Next, we present another lower bound for x_1 in terms of the spectral radius and the degree sequence of the graph.

THEOREM 3.4. *Let G be a connected graph with degrees $d_1, \ldots, d_n$. Then*

$$x_{\max} \geq \frac{\lambda_1}{\sqrt{\sum_{i=1}^{n} d_i^2}} \geq \frac{1}{\sqrt{n}}$$

with equality iff G is regular.

Proof. For each i, we have

$$\lambda_1 x_i = \sum_{j\sim i} x_j \leq d_i x_{\max}.$$

Squaring and summing over all i, we obtain

$$\lambda_1^2 \leq x_{\max}^2 \sum_{i=1}^{n} d_i^2$$

which implies the inequality stated in the theorem.

Equality happens iff $x_{\max} = x_j$ for each j which is equivalent to G being regular. □

One can also apply this inequality to powers of A since the eigenvectors of A^k are the same as the eigenvectors of A. Let $w_l(G)$ denote the number of walks of length l in G.

COROLLARY 3.5. *If G is a graph on n vertices, then*

$$x_{\max} \geq \sqrt{\frac{\lambda_1^{2k}}{w_{2k}(G)}}$$

for each $k \geq 1$.

It follows from the results in [6] that the right hand-side of the previous inequality is monotonically increasing with k. If, in addition, G is connected and not bipartite, it actually tends to $\frac{1}{\sum_{i=1}^{n} x_i}$ as k goes to infinity.

We have provided upper and lower bounds for the maximum entry of the principal eigenvector of a connected graph that are sharp in some cases. We conclude this section with the following result which gives an upper bound for the minimum entry of the principal eigenvector of a connected graph.

THEOREM 3.6. *If G is a graph on n vertices with maximum degree Δ and e edges, then*

$$\left(\Delta - \frac{2e}{n}\right) x_{\min} \leq \frac{\Delta - \lambda_1}{\sqrt{n}} \leq \frac{\Delta - \frac{2e}{n}}{\sqrt{n}}.$$

Equality happens in the first inequality if and only if $d_i = \Delta$ for each vertex i with $x_i > x_{\min}$. The second inequality is equality if and only if G is regular.

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Proof. Since $\lambda_1 \sum_{i=1}^n x_i = \sum_{i=1}^n d_i x_i$, it follows that

$$(\Delta - \lambda_1) \sum_{i=1}^n x_i = \sum_{i=1}^n (\Delta - d_i) x_i \geq x_{\min} \sum_{i=1}^n (\Delta - d_i) = n x_{\min} \left(\Delta - \frac{2e}{n} \right). \tag{3.3}$$

Equality happens if and only if $d_i = \Delta$ for each i with $x_i > x_{\min}$. Since $\lambda_1 \geq \frac{2e}{n}$ with equality if and only if G is regular, this finishes the proof. ∎

Actually, using the inequality $\sum_{i=1}^n x_i \leq x_{\min} + \sqrt{(n-1)(1-x_{\min}^2)}$ in (3.3), one can obtain the following better although more complicated inequality

$$x_{\min} \cdot \sqrt{(\Delta - \lambda_1)^2 + \frac{((n-1)\Delta + \lambda_1 - 2e)^2}{n-1}} \leq \Delta - \lambda_1.$$

Applying Theorem 3.6 and results from [3], we can show that if G is an irregular graph, then

$$x_{\min} \leq \frac{\Delta - \lambda_1}{\Delta - \frac{2e}{n}} \cdot \frac{1}{\sqrt{n}} < \left(1 - \frac{1}{(\Delta + 2)(n\Delta - 2e)} \right) \frac{1}{\sqrt{n}}.$$

Note that while the maximum entry of the principal eigenvector can be as large as $\frac{1}{\sqrt{2}}$, the principal eigenvector can contain entries which are exponentially smaller than $\frac{1}{\sqrt{n}}$ as shown by the graphs of the form $P_r \cdot K_s$.

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