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AFFINE TRANSFORMATIONS OF A LEONARD PAIR*

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Abstract. Let $\mathbb{K}$ denote a field and let V denote a vector space over $\mathbb{K}$ with finite positive dimension. An ordered pair is considered of linear transformations $A : V \to V$ and $A^* : V \to V$ that satisfy (i) and (ii) below:

(i) There exists a basis for V with respect to which the matrix representing A is irreducible tridiagonal and the matrix representing A^* is diagonal.
(ii) There exists a basis for V with respect to which the matrix representing A^* is irreducible tridiagonal and the matrix representing A is diagonal.

Such a pair is called a *Leonard pair* on V. Let $\xi, \zeta, \xi^*, \zeta^*$ denote scalars in $\mathbb{K}$ with ξ, ξ^* nonzero, and note that $\xi A + \zeta I$, $\xi^* A^* + \zeta^* I$ is a Leonard pair on V. Necessary and sufficient conditions are given for this Leonard pair to be isomorphic to A, A^*. Also given are necessary and sufficient conditions for this Leonard pair to be isomorphic to the Leonard pair A^*, A.

Key words. Leonard pair, Tridiagonal pair, q-Racah polynomial, Orthogonal polynomial.

AMS subject classifications. 05E35, 05E30, 33C45, 33D45.

1. Leonard pairs. We begin by recalling the notion of a Leonard pair. We will use the following terms. A square matrix X is said to be *tridiagonal* whenever each nonzero entry lies on either the diagonal, the subdiagonal, or the superdiagonal. Assume X is tridiagonal. Then X is said to be *irreducible* whenever each entry on the subdiagonal is nonzero and each entry on the superdiagonal is nonzero. We now define a Leonard pair. For the rest of this paper $\mathbb{K}$ will denote a field.

Definition 1.1. [37] Let V denote a vector space over $\mathbb{K}$ with finite positive dimension. By a *Leonard pair* on V we mean an ordered pair A, A^* where $A : V \to V$ and $A^* : V \to V$ are linear transformations that satisfy (i) and (ii) below:

(i) There exists a basis for V with respect to which the matrix representing A is irreducible tridiagonal and the matrix representing A^* is diagonal.
(ii) There exists a basis for V with respect to which the matrix representing A^* is irreducible tridiagonal and the matrix representing A is diagonal.

Note 1.2. It is a common notational convention to use A^* to represent the conjugate-transpose of A. We are not using this convention. In a Leonard pair A, A^* the linear transformations A and A^* are arbitrary subject to (i) and (ii) above.

We refer the reader to [9,22,25–31,35–37,39–46,48–50] for background on Leonard pairs. We especially recommend the survey [46]. See [1–8, 10–21, 23, 24, 32–34, 38, 47] for related topics.

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In this paper we consider the following situation. Let V denote a vector space over $\mathbb{K}$ with finite positive dimension and let A, A^* denote a Leonard pair on V. Let $\xi, \zeta, \xi^*, \zeta^*$ denote scalars in $\mathbb{K}$ with ξ, ξ^* nonzero, and note that $\xi A + \zeta I$, $\xi^* A^* + \zeta^* I$ is a Leonard pair on V. We give necessary and sufficient conditions for this Leonard pair to be isomorphic to A, A^*. We also give necessary and sufficient conditions for this Leonard pair to be isomorphic to the Leonard pair A^*, A.

2. Leonard systems. When working with a Leonard pair, it is convenient to consider a closely related object called a *Leonard system*. To prepare for our definition of a Leonard system, we recall a few concepts from linear algebra. Let d denote a nonnegative integer and let $\mathrm{Mat}_{d+1}(\mathbb{K})$ denote the $\mathbb{K}$-algebra consisting of all $d+1$ by $d+1$ matrices that have entries in $\mathbb{K}$. We index the rows and columns by $0, 1, \ldots, d$. We let $\mathbb{K}^{d+1}$ denote the $\mathbb{K}$-vector space of all $d+1$ by 1 matrices that have entries in $\mathbb{K}$. We index the rows by $0, 1, \ldots, d$. We view $\mathbb{K}^{d+1}$ as a left module for $\mathrm{Mat}_{d+1}(\mathbb{K})$. We observe this module is irreducible. For the rest of this paper, let $\mathcal{A}$ denote a $\mathbb{K}$-algebra isomorphic to $\mathrm{Mat}_{d+1}(\mathbb{K})$ and let V denote an irreducible left $\mathcal{A}$-module. We remark that V is unique up to isomorphism of $\mathcal{A}$-modules, and that V has dimension $d+1$. Let $\{v_i\}_{i=0}^d$ denote a basis for V. For $X \in \mathcal{A}$ and $Y \in \mathrm{Mat}_{d+1}(\mathbb{K})$, we say Y *represents* X *with respect to* $\{v_i\}_{i=0}^d$ whenever $Xv_j = \sum_{i=0}^d Y_{ij} v_i$ for $0 \le j \le d$. For $A \in \mathcal{A}$ we say A is *multiplicity-free* whenever it has $d+1$ mutually distinct eigenvalues in $\mathbb{K}$. Assume A is multiplicity-free. Let $\{\theta_i\}_{i=0}^d$ denote an ordering of the eigenvalues of A, and for $0 \le i \le d$ put

$$E_i = \prod_{\substack{0 \le j \le d \\ j \ne i}} \frac{A - \theta_j I}{\theta_i - \theta_j}, \tag{2.1}$$

where I denotes the identity of $\mathcal{A}$. We observe (i) $AE_i = \theta_i E_i$ $(0 \le i \le d)$; (ii) $E_i E_j = \delta_{i,j} E_i$ $(0 \le i, j \le d)$; (iii) $\sum_{i=0}^d E_i = I$; (iv) $A = \sum_{i=0}^d \theta_i E_i$. Let $\mathcal{D}$ denote the subalgebra of $\mathcal{A}$ generated by A. Using (i)–(iv) we find the sequence $\{E_i\}_{i=0}^d$ is a basis for the $\mathbb{K}$-vector space $\mathcal{D}$. We call E_i the *primitive idempotent* of A associated with θ_i. It is helpful to think of these primitive idempotents as follows. Observe

$$V = E_0 V + E_1 V + \cdots + E_d V \qquad \text{(direct sum)}.$$

For $0 \le i \le d$, $E_i V$ is the (one dimensional) eigenspace of A in V associated with the eigenvalue θ_i, and E_i acts on V as the projection onto this eigenspace.

By a *Leonard pair in* $\mathcal{A}$ we mean an ordered pair of elements taken from $\mathcal{A}$ that act on V as a Leonard pair in the sense of Definition 1.1. We call $\mathcal{A}$ the *ambient algebra* of the pair and say the pair is *over* $\mathbb{K}$. We now define a Leonard system.

DEFINITION 2.1. [37] By a *Leonard system* in $\mathcal{A}$ we mean a sequence

$$\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$$

that satisfies (i)–(v) below.

(i) Each of A, A^* is a multiplicity-free element in $\mathcal{A}$.

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(ii) $\{E_i\}_{i=0}^d$ is an ordering of the primitive idempotents of A.
(iii) $\{E_i^*\}_{i=0}^d$ is an ordering of the primitive idempotents of A^*.
(iv) For $0 \leq i, j \leq d$,

$$E_i A^* E_j = \begin{cases} 0 & \text{if } |i-j| > 1, \\ \neq 0 & \text{if } |i-j| = 1. \end{cases} \tag{2.2}$$

(v) For $0 \leq i, j \leq d$,

$$E_i^* A E_j^* = \begin{cases} 0 & \text{if } |i-j| > 1, \\ \neq 0 & \text{if } |i-j| = 1. \end{cases} \tag{2.3}$$

We refer to d as the *diameter* of Φ and say Φ *is over* $\mathbb{K}$. We call $\mathcal{A}$ the *ambient algebra* of Φ.

Leonard systems are related to Leonard pairs as follows. Let $(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a Leonard system in $\mathcal{A}$. Then A, A^* is a Leonard pair in $\mathcal{A}$ [45, Section 3]. Conversely, suppose A, A^* is a Leonard pair in $\mathcal{A}$. Then each of A, A^* is multiplicity-free [37, Lemma 1.3]. Moreover there exists an ordering $\{E_i\}_{i=0}^d$ of the primitive idempotents of A, and there exists an ordering $\{E_i^*\}_{i=0}^d$ of the primitive idempotents of A^*, such that $(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ is a Leonard system in $\mathcal{A}$ [45, Lemma 3.3]. We say this Leonard system is *associated* with the Leonard pair A, A^*.

We recall the notion of *isomorphism* for Leonard pairs and Leonard systems.

Definition 2.2. Let A, A^* and B, B^* denote Leonard pairs over $\mathbb{K}$. By an *isomorphism of Leonard pairs from* A, A^* *to* B, B^* we mean an isomorphism of $\mathbb{K}$-algebras from the ambient algebra of A, A^* to the ambient algebra B, B^* that sends A to B and A^* to B^*. The Leonard pairs A, A^* and B, B^* are said to be *isomorphic* whenever there exists an isomorphism of Leonard pairs from A, A^* to B, B^*.

Let Φ denote the Leonard system from Definition 2.1 and let $\sigma : \mathcal{A} \to \mathcal{A}'$ denote an isomorphism of $\mathbb{K}$-algebras. We write $\Phi^\sigma := (A^\sigma; \{E_i^\sigma\}_{i=0}^d; A^{*\sigma}; \{E_i^{*\sigma}\}_{i=0}^d)$ and observe Φ^σ is a Leonard system in $\mathcal{A}'$.

Definition 2.3. *Let* Φ *and* Φ' *denote Leonard systems over* $\mathbb{K}$. *By an* isomorphism of Leonard systems from Φ to Φ' *we mean an isomorphism of* $\mathbb{K}$*-algebras* σ *from the ambient algebra of* Φ *to the ambient algebra of* Φ' *such that* $\Phi^\sigma = \Phi'$. *The Leonard systems* Φ *and* Φ' *are said to be* isomorphic *whenever there exists an isomorphism of Leonard systems from* Φ *to* Φ'.

3. The D_4 action.

Let $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a Leonard system in $\mathcal{A}$. Then each of the following is a Leonard system in $\mathcal{A}$:

$$\Phi^* := (A^*; \{E_i^*\}_{i=0}^d; A; \{E_i\}_{i=0}^d),$$
$$\Phi^\downarrow := (A; \{E_i\}_{i=0}^d; A^*; \{E_{d-i}^*\}_{i=0}^d),$$
$$\Phi^\Downarrow := (A; \{E_{d-i}\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d).$$

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Viewing $*$, $\downarrow$, $\Downarrow$ as permutations on the set of all the Leonard systems,

$$*^2 = \downarrow^2 = \Downarrow^2 = 1, \tag{3.1}$$

$$\Downarrow * = * \downarrow, \quad \downarrow * = * \Downarrow, \quad \downarrow\Downarrow = \Downarrow\downarrow. \tag{3.2}$$

The group generated by symbols $*$, $\downarrow$, $\Downarrow$ subject to the relations (3.1), (3.2) is the dihedral group D_4. We recall D_4 is the group of symmetries of a square, and has 8 elements. Apparently $*$, $\downarrow$, $\Downarrow$ induce an action of D_4 on the set of all Leonard systems. Two Leonard systems will be called *relatives* whenever they are in the same orbit of this D_4 action. The relatives of Φ are as follows:

name	relative
Φ	$(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$
$\Phi^{\downarrow}$	$(A; \{E_i\}_{i=0}^d; A^*; \{E_{d-i}^*\}_{i=0}^d)$
$\Phi^{\Downarrow}$	$(A; \{E_{d-i}\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$
$\Phi^{\downarrow\Downarrow}$	$(A; \{E_{d-i}\}_{i=0}^d; A^*; \{E_{d-i}^*\}_{i=0}^d)$
Φ^*	$(A^*; \{E_i^*\}_{i=0}^d; A; \{E_i\}_{i=0}^d)$
$\Phi^{\downarrow *}$	$(A^*; \{E_{d-i}^*\}_{i=0}^d; A; \{E_i\}_{i=0}^d)$
$\Phi^{\Downarrow *}$	$(A^*; \{E_i^*\}_{i=0}^d; A; \{E_{d-i}\}_{i=0}^d)$
$\Phi^{\downarrow\Downarrow *}$	$(A^*; \{E_{d-i}^*\}_{i=0}^d; A; \{E_{d-i}\}_{i=0}^d)$

4. The parameter array. In this section we recall the parameter array of a Leonard system.

Definition 4.1. Let $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a Leonard system over $\mathbb{K}$. For $0 \le i \le d$ we let θ_i (resp. θ_i^*) denote the eigenvalue of A (resp. A^*) associated with E_i (resp. E_i^*). We refer to $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta_i^*\}_{i=0}^d$) as the *eigenvalue sequence* (resp. *dual eigenvalue sequence*) of Φ. We observe $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta_i^*\}_{i=0}^d$) are mutually distinct and contained in $\mathbb{K}$.

Definition 4.2. [26, Theorem 4.6] Let $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a Leonard system with eigenvalue sequence $\{\theta_i\}_{i=0}^d$ and dual eigenvalue sequence $\{\theta_i^*\}_{i=0}^d$. For $1 \le i \le d$ we define

$$\varphi_i := (\theta_0^* - \theta_i^*) \frac{\mathrm{tr}(E_0^* \prod_{h=0}^{i-1}(A - \theta_h I))}{\mathrm{tr}(E_0^* \prod_{h=0}^{i-2}(A - \theta_h I))}, \tag{4.1}$$

$$\phi_i := (\theta_0^* - \theta_i^*) \frac{\mathrm{tr}(E_0^* \prod_{h=0}^{i-1}(A - \theta_{d-h} I))}{\mathrm{tr}(E_0^* \prod_{h=0}^{i-2}(A - \theta_{d-h} I))}, \tag{4.2}$$

where tr means trace. In (4.1), (4.2) the denominators are nonzero by [26, Corollary 4.5]. The sequence $\{\varphi_i\}_{i=1}^d$ (resp. $\{\phi_i\}_{i=1}^d$) is called the *first split sequence* (resp. *second split sequence*) of Φ.

Definition 4.3. Let $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a Leonard system over $\mathbb{K}$. By the *parameter array of* Φ we mean the sequence $(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\varphi_i\}_{i=1}^d;$

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$\{\phi_i\}_{i=1}^d$), where the θ_i, θ_i^* are from Definition 4.1 and the φ_i, ϕ_i are from Definition 4.2.

THEOREM 4.4. [37, Theorem 1.9] *Let d denote a nonnegative integer and let*

$$(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d) \tag{4.3}$$

denote a sequence of scalars taken from $\mathbb{K}$. Then there exists a Leonard system Φ over $\mathbb{K}$ with parameter array (4.3) *if and only if* (PA1)–(PA5) *hold below.*

(PA1) $\varphi_i \neq 0$, $\phi_i \neq 0$ $(1 \leq i \leq d)$.

(PA2) $\theta_i \neq \theta_j$, $\theta_i^* \neq \theta_j^*$ *if* $i \neq j$ $(0 \leq i, j \leq d)$.

(PA3) *For* $1 \leq i \leq d$,

$$\varphi_i = \phi_1 \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} + (\theta_i^* - \theta_0^*)(\theta_{i-1} - \theta_d).$$

(PA4) *For* $1 \leq i \leq d$,

$$\phi_i = \varphi_1 \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} + (\theta_i^* - \theta_0^*)(\theta_{d-i+1} - \theta_0).$$

(PA5) *The expressions*

$$\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i}, \qquad \frac{\theta_{i-2}^* - \theta_{i+1}^*}{\theta_{i-1}^* - \theta_i^*} \tag{4.4}$$

are equal and independent of i for $2 \leq i \leq d-1$.

Suppose (PA1)–(PA5) *hold. Then Φ is unique up to isomorphism of Leonard systems.*

The D_4 action affects the parameter array as follows.

LEMMA 4.5. [37, Theorem 1.11] *Let $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a Leonard system with parameter array $(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d)$. For each relative of Φ the parameter array is given below.*

relative	parameter array
Φ	$(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d)$
$\Phi^{\downarrow}$	$(\{\theta_i\}_{i=0}^d; \{\theta_{d-i}^*\}_{i=0}^d; \{\phi_{d-i+1}\}_{i=1}^d; \{\varphi_{d-i+1}\}_{i=1}^d)$
$\Phi^{\Downarrow}$	$(\{\theta_{d-i}\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\phi_i\}_{i=1}^d; \{\varphi_i\}_{i=1}^d)$
$\Phi^{\downarrow\Downarrow}$	$(\{\theta_{d-i}\}_{i=0}^d; \{\theta_{d-i}^*\}_{i=0}^d; \{\varphi_{d-i+1}\}_{i=1}^d; \{\phi_{d-i+1}\}_{i=1}^d)$
Φ^*	$(\{\theta_i^*\}_{i=0}^d; \{\theta_i\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_{d-i+1}\}_{i=1}^d)$
$\Phi^{\downarrow *}$	$(\{\theta_{d-i}^*\}_{i=0}^d; \{\theta_i\}_{i=0}^d; \{\phi_{d-i+1}\}_{i=1}^d; \{\varphi_i\}_{i=1}^d)$
$\Phi^{\Downarrow *}$	$(\{\theta_i^*\}_{i=0}^d; \{\theta_{d-i}\}_{i=0}^d; \{\phi_i\}_{i=1}^d; \{\varphi_{d-i+1}\}_{i=1}^d)$
$\Phi^{\downarrow\Downarrow *}$	$(\{\theta_{d-i}^*\}_{i=0}^d; \{\theta_{d-i}\}_{i=0}^d; \{\varphi_{d-i+1}\}_{i=1}^d; \{\phi_i\}_{i=1}^d)$

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5. Affine transformations of a Leonard system. In this section we consider the affine transformations of a Leonard system. We start with an observation.

Lemma 5.1. *Let* $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ *denote a Leonard system in* $\mathcal{A}$. *Let* $\xi, \zeta, \xi^*, \zeta^*$ *denote scalars in* $\mathbb{K}$ *with* ξ, ξ^* *nonzero. Then the sequence*

$$(\xi A + \zeta I; \{E_i\}_{i=0}^d; \xi^* A^* + \zeta^* I; \{E_i^*\}_{i=0}^d) \tag{5.1}$$

is a Leonard system in $\mathcal{A}$.

Definition 5.2. Referring to Lemma 5.1, we call (5.1) the *affine transformation* of Φ associated with $\xi, \zeta, \xi^*, \zeta^*$.

Definition 5.3. Let Φ and Φ' denote Leonard systems over $\mathbb{K}$. We say Φ and Φ' are *affine isomorphic* whenever Φ is isomorphic to an affine transformation of Φ'. Observe that affine isomorphism is an equivalence relation.

Let Φ denote a Leonard system. We now consider how the set of relatives of Φ is partitioned into affine isomorphism classes. In order to avoid trivialities we assume the diameter of Φ is at least 1. The following is our main result on this topic.

Theorem 5.4. *Let* Φ *denote a Leonard system with first split sequence* $\{\varphi_i\}_{i=1}^d$ *and second split sequence* $\{\phi_i\}_{i=1}^d$. *Assume* $d \geq 1$.

(i) *Assume* $\varphi_1 = \varphi_d = -\phi_1 = -\phi_d$. *Then all eight relatives of* Φ *are mutually affine isomorphic.*

(ii) *Assume* $\varphi_1 = \varphi_d$, $\phi_1 = \phi_d$ *and* $\varphi_1 \neq -\phi_1$. *Then the relatives of* Φ *form exactly two affine isomorphism classes, consisting of*
$$\{\Phi, \Phi^{\downarrow\Downarrow}, \Phi^*, \Phi^{\downarrow\Downarrow *}\}, \quad \{\Phi^{\downarrow}, \Phi^{\Downarrow}, \Phi^{\downarrow *}, \Phi^{\Downarrow *}\}.$$

(iii) *Assume* $\varphi_1 = \varphi_d$ *and* $\phi_1 \neq \phi_d$. *Then the relatives of* Φ *form exactly four affine isomorphism classes, consisting of*
$$\{\Phi, \Phi^{\downarrow\Downarrow *}\}, \quad \{\Phi^{\downarrow}, \Phi^{\downarrow *}\}, \quad \{\Phi^{\Downarrow}, \Phi^{\Downarrow *}\}, \quad \{\Phi^{\downarrow\Downarrow}, \Phi^*\}.$$

(iv) *Assume* $\phi_1 = \phi_d$ *and* $\varphi_1 \neq \varphi_d$. *Then the relatives of* Φ *form exactly four affine isomorphism classes, consisting of*
$$\{\Phi, \Phi^*\}, \quad \{\Phi^{\downarrow}, \Phi^{\Downarrow *}\}, \quad \{\Phi^{\Downarrow}, \Phi^{\downarrow *}\}, \quad \{\Phi^{\downarrow\Downarrow}, \Phi^{\downarrow\Downarrow *}\}.$$

(v) *Assume* $\varphi_1 = -\phi_1$, $\varphi_d = -\phi_d$ *and* $\varphi_1 \neq \varphi_d$. *Then the relatives of* Φ *form exactly four affine isomorphism classes, consisting of*
$$\{\Phi, \Phi^{\Downarrow}\}, \quad \{\Phi^{\downarrow}, \Phi^{\downarrow\Downarrow}\}, \quad \{\Phi^*, \Phi^{\Downarrow *}\}, \quad \{\Phi^{\downarrow *}, \Phi^{\downarrow\Downarrow *}\}.$$

(vi) *Assume* $\varphi_1 = -\phi_d$, $\varphi_d = -\phi_1$ *and* $\varphi_1 \neq \varphi_d$. *Then the relatives of* Φ *form exactly four affine isomorphism classes, consisting of*
$$\{\Phi, \Phi^{\downarrow}\}, \quad \{\Phi^{\Downarrow}, \Phi^{\downarrow\Downarrow}\}, \quad \{\Phi^*, \Phi^{\downarrow *}\}, \quad \{\Phi^{\Downarrow *}, \Phi^{\downarrow\Downarrow *}\}.$$

(vii) *Assume none of* (i)–(vi) *hold above. Then* $\varphi_1 \neq \varphi_d$, $\phi_1 \neq \phi_d$, *at least one of* $\varphi_1 \neq -\phi_1$, $\varphi_d \neq -\phi_d$, *and at least one of* $\varphi_1 \neq -\phi_d$, $\varphi_d \neq -\phi_1$. *In this case the eight relatives of* Φ *are mutually non affine isomorphic.*

The proof of Theorem 5.4 will be given in Section 9. In Sections 6–8 we obtain some results that will be used in this proof.

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6. How the parameter array is affected by affine transformation. Let Φ denote a Leonard system. In this section we consider how the parameter array of Φ is affected by affine transformation.

LEMMA 6.1. *Referring to Lemma* 5.1*, let* $(\{\theta_i\}_{i=0}^d; \{\theta^*_i\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d)$ *denote the parameter array of* Φ*. Then the parameter array of the Leonard system* (5.1) *is*

$$(\{\xi\theta_i + \zeta\}_{i=0}^d; \{\xi^*\theta^*_i + \zeta^*\}_{i=0}^d; \{\xi\xi^*\varphi_i\}_{i=1}^d; \{\xi\xi^*\phi_i\}_{i=1}^d). \tag{6.1}$$

Proof. By Definition 4.1, for $0 \le i \le d$ the scalar θ_i is the eigenvalue of A associated with E_i, so $\xi\theta_i + \zeta$ is the eigenvalue of $\xi A + \zeta I$ associated with E_i. Thus $\{\xi\theta_i + \zeta\}_{i=0}^d$ is the eigenvalue sequence of (5.1). Similarly $\{\xi^*\theta^*_i + \zeta^*\}_{i=0}^d$ is the dual eigenvalue sequence of (5.1). In the right-hand side of (4.1), if we replace A by $\xi A+\zeta I$, and if we replace θ_j, θ^*_j by $\xi\theta_j+\zeta$, $\xi^*\theta^*_j+\zeta^*$ $(0 \le j \le d)$ and simplify the result we get $\xi\xi^*\varphi_i$. Therefore $\{\xi\xi^*\varphi_i\}_{i=1}^d$ is the first split sequence of (5.1). Similarly $\{\xi\xi^*\phi_i\}_{i=1}^d$ is the second split sequence of (5.1) and the result follows. □

7. Some equations. In this section we obtain some equations that will be useful in the proof of Theorem 5.4.

NOTATION 7.1. Let $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E^*_i\}_{i=0}^d)$ denote a Leonard system over $\mathbb{K}$, with parameter array $(\{\theta_i\}_{i=0}^d; \{\theta^*_i\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d)$. To avoid trivialities we assume $d \ge 1$.

LEMMA 7.2. [37, Lemma 9.5] *Referring to Notation* 7.1*,*

$$\frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} = \frac{\theta^*_h - \theta^*_{d-h}}{\theta^*_0 - \theta^*_d} \qquad (0 \le h \le d).$$

DEFINITION 7.3. Referring to Notation 7.1, for $1 \le i \le d$ we have

$$\sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} = \sum_{h=0}^{i-1} \frac{\theta^*_h - \theta^*_{d-h}}{\theta^*_0 - \theta^*_d}.$$

We denote this common value by ϑ_i. We observe that $\vartheta_1 = 1$ and $\vartheta_i = \vartheta_{d-i+1}$ for $1 \le i \le d$.

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Lemma 7.4. *Referring to Notation* 7.1 *and Definition* 7.3*, the following hold for* $1 \leq i \leq d$*.*

$$\varphi_i = \phi_1 \vartheta_i + (\theta^*_i - \theta^*_0)(\theta_{i-1} - \theta_d), \tag{7.1}$$
$$\varphi_{d-i+1} = \phi_1 \vartheta_i + (\theta^*_{d-i+1} - \theta^*_0)(\theta_{d-i} - \theta_d), \tag{7.2}$$
$$\varphi_i = \phi_d \vartheta_i + (\theta_i - \theta_0)(\theta^*_{i-1} - \theta^*_d), \tag{7.3}$$
$$\varphi_{d-i+1} = \phi_d \vartheta_i + (\theta_{d-i+1} - \theta_0)(\theta^*_{d-i} - \theta^*_d), \tag{7.4}$$
$$\phi_i = \varphi_1 \vartheta_i + (\theta^*_i - \theta^*_0)(\theta_{d-i+1} - \theta_0), \tag{7.5}$$
$$\phi_{d-i+1} = \varphi_1 \vartheta_i + (\theta^*_{d-i+1} - \theta^*_0)(\theta_i - \theta_0), \tag{7.6}$$
$$\phi_i = \varphi_d \vartheta_i + (\theta_{d-i} - \theta_d)(\theta^*_{i-1} - \theta^*_d), \tag{7.7}$$
$$\phi_{d-i+1} = \varphi_d \vartheta_i + (\theta_{i-1} - \theta_d)(\theta^*_{d-i} - \theta^*_d). \tag{7.8}$$

Proof. Apply D_4 to the equation (PA3) from Theorem 4.4, and use Lemma 4.5. □

8. The relatives and affine transformations of a Leonard system. Let Φ denote a Leonard system in $\mathcal{A}$. In this section we give, for each relative of Φ, necessary and sufficient conditions for it to be affine isomorphic to Φ. Recall that by Theorem 4.4, two Leonard systems are isomorphic if and only if they have the same parameter array.

Lemma 8.1. *Let* Φ *and* Φ' *denote Leonard systems over* $\mathbb{K}$ *which are affine isomorphic. Then* Φ^g *and* Φ'^g *are affine isomorphic for all* $g \in D_4$*.*

Proof. Routine. □

Proposition 8.2. *Referring to Notation* 7.1*, let* $\xi, \zeta, \xi^*, \zeta^*$ *denote scalars in* $\mathbb{K}$ *with* ξ, ξ^* *nonzero. Then* Φ *is isomorphic to the Leonard system* (5.1) *if and only if* $\xi = 1$*,* $\zeta = 0$*,* $\xi^* = 1$*,* $\zeta^* = 0$*.*

Proof. Suppose that Φ is isomorphic to the Leonard system (5.1). Then these Leonard systems have the same parameter array. These parameter arrays are given in Notation 7.1 and (6.1); comparing them we find $\xi\theta_i + \zeta = \theta_i$ for $0 \leq i \leq d$. Setting $i = 0$, $i = 1$ in this equation we find $\xi = 1$, $\zeta = 0$. Similarly we find $\xi^* = 1$, $\zeta^* = 0$. This proves the result in one direction and the other direction is clear. □

Lemma 8.3. *Referring to Notation* 7.1*, let* $\xi, \zeta, \xi^*, \zeta^*$ *denote scalars in* $\mathbb{K}$ *with* ξ, ξ^* *nonzero. Then* $\Phi^{\downarrow}$ *is isomorphic to the Leonard system* (5.1) *if and only if*

$$\theta_i = \xi\theta_i + \zeta \qquad (0 \leq i \leq d), \tag{8.1}$$
$$\theta^*_{d-i} = \xi^*\theta^*_i + \zeta^* \qquad (0 \leq i \leq d), \tag{8.2}$$
$$\phi_{d-i+1} = \xi\xi^*\varphi_i \qquad (1 \leq i \leq d), \tag{8.3}$$
$$\varphi_{d-i+1} = \xi\xi^*\phi_i \qquad (1 \leq i \leq d). \tag{8.4}$$

Proof. Compare the parameter array of $\Phi^{\downarrow}$ from Lemma 4.5, with the parameter array (6.1). □

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Proposition 8.4. *Referring to Notation* 7.1*, the following* (i)–(iii) *are equivalent.*

(i) $\Phi^{\downarrow}$ *is affine isomorphic to* Φ.

(ii) $\varphi_1 = -\phi_d$ *and* $\varphi_d = -\phi_1$.

(iii) $\varphi_i = -\phi_{d-i+1}$ *for* $1 \leq i \leq d$ *and* $\theta^*_i + \theta^*_{d-i}$ *is independent of* i *for* $0 \leq i \leq d$.

Suppose (i)–(iii) *hold. Then* $\Phi^{\downarrow}$ *is isomorphic to* (5.1) *with* $\xi = 1$, $\zeta = 0$, $\xi^* = -1$, *and* ζ^* *equal to the common value of* $\theta^*_i + \theta^*_{d-i}$.

Proof. (i)⇒(ii): By Definition 5.3 there exist scalars $\xi, \zeta, \xi^*, \zeta^*$ in $\mathbb{K}$ with ξ, ξ^* nonzero such that $\Phi^{\downarrow}$ is isomorphic to the Leonard system (5.1). Now (8.1)–(8.4) hold by Lemma 8.3. Setting $i = 0$, $i = 1$ in (8.1) we find $\xi = 1$, $\zeta = 0$. Setting $i = 0$, $i = d$ in (8.2) we find $\xi^* = -1$. Setting $i = 1$, $i = d$ in (8.3) and using $\xi = 1$, $\xi^* = -1$ we find $\varphi_1 = -\phi_d$ and $\varphi_d = -\phi_1$.

(ii)⇒(iii): By (7.1), (7.8) and $\varphi_d = -\phi_1$,

$$\varphi_i + \phi_{d-i+1} = (\theta_{i-1} - \theta_d)(\theta^*_i + \theta^*_{d-i} - \theta^*_0 - \theta^*_d) \qquad (1 \leq i \leq d). \tag{8.5}$$

By (7.3), (7.6) and $\varphi_1 = -\phi_d$,

$$\varphi_i + \phi_{d-i+1} = (\theta_i - \theta_0)(\theta^*_{i-1} + \theta^*_{d-i+1} - \theta^*_0 - \theta^*_d) \qquad (1 \leq i \leq d). \tag{8.6}$$

Replacing i by $i+1$ in (8.6) and comparing the result with (8.5) we find

$$\frac{\varphi_i + \phi_{d-i+1}}{\theta_{i-1} - \theta_d} = \frac{\varphi_{i+1} + \phi_{d-i}}{\theta_{i+1} - \theta_0} \qquad (1 \leq i \leq d-1).$$

From this and since $\varphi_1 + \phi_d = 0$ we find $\varphi_i + \phi_{d-i+1} = 0$ for $1 \leq i \leq d$. Evaluating (8.5) using this we find $\theta^*_i + \theta^*_{d-i}$ is independent of i for $0 \leq i \leq d$.

(iii)⇒(i): Let ζ^* denote the common value of $\theta^*_i + \theta^*_{d-i}$, and let $\xi = 1$, $\zeta = 0$, $\xi^* = -1$. Now (8.1)–(8.4) hold so $\Phi^{\downarrow}$ is isomorphic to (5.1) by Lemma 8.3. Now $\Phi^{\downarrow}$ is affine isomorphic to Φ in view of Definition 5.3. □

Proposition 8.5. *Referring to Notation* 7.1*, the following* (i)–(iii) *are equivalent.*

(i) $\Phi^{\Downarrow}$ *is affine isomorphic to* Φ.

(ii) $\varphi_1 = -\phi_1$ *and* $\varphi_d = -\phi_d$.

(iii) $\varphi_i = -\phi_i$ *for* $1 \leq i \leq d$ *and* $\theta_i + \theta_{d-i}$ *is independent of* i *for* $0 \leq i \leq d$.

Suppose (i)–(iii) *hold. Then* $\Phi^{\Downarrow}$ *is isomorphic to* (5.1) *with* $\xi = -1$, ζ *equal to the common value of* $\theta_i + \theta_{d-i}$, $\xi^* = 1$, *and* $\zeta^* = 0$.

Proof. By Lemma 8.1 (with $g = *$) and since $\Downarrow * = * \downarrow$ we find $\Phi^{\Downarrow}$ is affine isomorphic to Φ if and only if $\Phi^{*\downarrow}$ is affine isomorphic to Φ^*. Now apply Proposition 8.4 to Φ^* and use Lemma 4.5. □

Lemma 8.6. *Referring to Notation* 7.1*, let* $\xi, \zeta, \xi^*, \zeta^*$ *denote scalars in* $\mathbb{K}$ *with* ξ, ξ^* *nonzero. Then* Φ^* *is isomorphic to the Leonard system* (5.1) *if and only if*

$$\theta^*_i = \xi\theta_i + \zeta \qquad (0 \leq i \leq d), \tag{8.7}$$

$$\theta_i = \xi^*\theta^*_i + \zeta^* \qquad (0 \leq i \leq d), \tag{8.8}$$

$$\varphi_i = \xi\xi^*\varphi_i \qquad (1 \leq i \leq d), \tag{8.9}$$

$$\phi_{d-i+1} = \xi\xi^*\phi_i \qquad (1 \leq i \leq d). \tag{8.10}$$

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Proof. Compare the parameter array of Φ^* from Lemma 4.5, with the parameter array (6.1). □

PROPOSITION 8.7. *Referring to Notation* 7.1*, the following* (i)–(iv) *are equivalent.*
(i) Φ^* *is affine isomorphic to* Φ*.*
(ii) $\phi_1 = \phi_d$*.*
(iii) $\phi_i = \phi_{d-i+1}$ *for* $1 \le i \le d$*.*
(iv) $(\theta^*_i - \theta^*_0)(\theta_i - \theta_0)^{-1}$ *is independent of* i *for* $1 \le i \le d$*.*
Suppose (i)–(iv) *hold. Then* Φ^* *is isomorphic to* (5.1) *with* ξ *equal to the common value of* $(\theta^*_i - \theta^*_0)(\theta_i - \theta_0)^{-1}$, $\zeta = \theta^*_0 - \xi\theta_0$, $\xi^* = \xi^{-1}$*, and* $\zeta^* = \theta_0 - \xi^*\theta^*_0$*.*

Proof. (i)⇒(ii): By Definition 5.3 there exist scalars $\xi, \zeta, \xi^*, \zeta^*$ in $\mathbb{K}$ with ξ, ξ^* nonzero such that Φ^* is isomorphic to the Leonard system (5.1). Now (8.7)–(8.10) hold by Lemma 8.6. By (8.9) we find $\xi\xi^* = 1$. Setting $i = 1$ in (8.10) and using $\xi\xi^* = 1$ we find $\phi_1 = \phi_d$.

(ii)⇒(iv): For $0 \le i \le d$ define $\eta_i = (\theta^*_i - \theta^*_0)(\theta_d - \theta_0) - (\theta_i - \theta_0)(\theta^*_d - \theta^*_0)$ and observe $\eta_0 = 0$. We show $\eta_i = 0$ for $1 \le i \le d$. By (7.1), (7.3) and since $\phi_1 = \phi_d$,

$$(\theta^*_i - \theta^*_0)(\theta_{i-1} - \theta_d) = (\theta_i - \theta_0)(\theta^*_{i-1} - \theta^*_d) \qquad (1 \le i \le d).$$

In this equation we rearrange terms to get

$$\eta_i(\theta_{i-1} - \theta_d) = \eta_{i-1}(\theta_i - \theta_0) \qquad (1 \le i \le d).$$

By this and since $\eta_0 = 0$ we find $\eta_i = 0$ for $1 \le i \le d$. The result follows.

(iv)⇒(iii): Let i be given. Since $(\theta^*_i - \theta^*_0)(\theta_i - \theta_0)^{-1}$ is independent of i,

$$(\theta^*_i - \theta^*_0)(\theta_{d-i+1} - \theta_0) = (\theta^*_{d-i+1} - \theta^*_0)(\theta_i - \theta_0).$$

Comparing (7.5) and (7.6) using this we find $\phi_i = \phi_{d-i+1}$.

(iii)⇒(ii): Clear.

(iii), (iv)⇒(i): Let ξ denote the common value of $(\theta^*_i - \theta^*_0)(\theta_i - \theta_0)^{-1}$ and set $\xi^* = \xi^{-1}$, $\zeta = \theta^*_0 - \xi\theta_0$, $\zeta^* = \theta_0 - \xi^*\theta^*_0$. Then (8.7)–(8.10) hold so Φ^* is isomorphic to (5.1) by Lemma 8.6. Now Φ^* is affine isomorphic to Φ in view of Definition 5.3. □

PROPOSITION 8.8. *Referring to Notation* 7.1*, the following* (i)–(iv) *are equivalent.*
(i) $\Phi^{\downarrow\Downarrow *}$ *is affine isomorphic to* Φ*.*
(ii) $\varphi_1 = \varphi_d$*.*
(iii) $\varphi_i = \varphi_{d-i+1}$ *for* $1 \le i \le d$*.*
(iv) $(\theta^*_{d-i} - \theta^*_d)(\theta_i - \theta_0)^{-1}$ *is independent of* i *for* $1 \le i \le d$*.*
Suppose (i)–(iv) *hold. Then* $\Phi^{\downarrow\Downarrow *}$ *is isomorphic to* (5.1) *with* ξ *equal to the common value of* $(\theta^*_{d-i} - \theta^*_d)(\theta_i - \theta_0)^{-1}$, $\zeta = \theta^*_d - \xi\theta_0$, $\xi^* = \xi^{-1}$*, and* $\zeta^* = \theta_0 - \xi^*\theta^*_d$*.*

Proof. By Lemma 8.1 (with $g = \downarrow$) and since $\Downarrow * \downarrow = *$ we find that $\Phi^{\downarrow\Downarrow *}$ is affine isomorphic to Φ if and only if $\Phi^{\downarrow *}$ is affine isomorphic to $\Phi^{\downarrow}$. Now apply Proposition 8.7 to $\Phi^{\downarrow}$ and use Lemma 4.5. □

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Lemma 8.9. *Referring to Notation* 7.1*, let* $\xi, \zeta, \xi^*, \zeta^*$ *denote scalars in* $\mathbb{K}$ *with* ξ, ξ^* *nonzero. Then* $\Phi^{\downarrow\Downarrow}$ *is isomorphic to the Leonard system* (5.1) *if and only if*

$$\theta_{d-i} = \xi\theta_i + \zeta \qquad (0 \le i \le d), \tag{8.11}$$

$$\theta^*_{d-i} = \xi^*\theta^*_i + \zeta^* \qquad (0 \le i \le d), \tag{8.12}$$

$$\varphi_{d-i+1} = \xi\xi^*\varphi_i \qquad (1 \le i \le d), \tag{8.13}$$

$$\phi_{d-i+1} = \xi\xi^*\phi_i \qquad (1 \le i \le d). \tag{8.14}$$

Proof. Compare the parameter array of $\Phi^{\downarrow\Downarrow}$ from Lemma 4.5, with the parameter array (6.1). □

Proposition 8.10. *Referring to Notation* 7.1*, the following* (i)–(iv) *are equivalent.*

(i) $\Phi^{\downarrow\Downarrow}$ *is affine isomorphic to* Φ.
(ii) $\varphi_1 = \varphi_d$ *and* $\phi_1 = \phi_d$.
(iii) $\varphi_i = \varphi_{d-i+1}$ *and* $\phi_i = \phi_{d-i+1}$ *for* $1 \le i \le d$.
(iv) *Each of* $(\theta^*_i - \theta^*_0)(\theta_i - \theta_0)^{-1}$, $(\theta^*_{d-i} - \theta^*_d)(\theta_i - \theta_0)^{-1}$ *is independent of* i *for* $1 \le i \le d$.

Suppose (i)–(iv) *hold. Then each of* $\theta_i + \theta_{d-i}$, $\theta^*_i + \theta^*_{d-i}$ *is independent of* i *for* $0 \le i \le d$*. Moreover* $\Phi^{\downarrow\Downarrow}$ *is isomorphic to* (5.1) *with* $\xi = -1$, $\xi^* = -1$*, and* ζ (*resp.* ζ^*) *equal to the common value of* $\theta_i + \theta_{d-i}$ (*resp.* $\theta^*_i + \theta^*_{d-i}$).

Proof. (i)⇒(ii): By Definition 5.3 there exist scalars $\xi, \zeta, \xi^*, \zeta^*$ in $\mathbb{K}$ with ξ, ξ^* nonzero such that $\Phi^{\downarrow\Downarrow}$ is isomorphic to the Leonard system (5.1). Now (8.11)–(8.14) hold by Lemma 8.9. Setting $i = 0$, $i = d$ in (8.11) we find $\xi = -1$. Setting $i = 0$, $i = d$ in (8.12) we find $\xi^* = -1$. Setting $i = 1$ in (8.13), (8.14) and using $\xi = -1$, $\xi^* = -1$ we find $\varphi_1 = \varphi_d$ and $\phi_1 = \phi_d$.

(ii)⇔(iii)⇔(iv): Follows from Propositions 8.7 and 8.8.

(iii), (iv)⇒(i): We first show that $\theta^*_i + \theta^*_{d-i}$ is independent of i for $0 \le i \le d$. By assumption

$$\frac{\theta^*_i - \theta^*_0}{\theta_i - \theta_0} = \frac{\theta^*_d - \theta^*_0}{\theta_d - \theta_0} \qquad (1 \le i \le d), \tag{8.15}$$

and

$$\frac{\theta^*_{d-i} - \theta^*_d}{\theta_i - \theta_0} = \frac{\theta^*_0 - \theta^*_d}{\theta_d - \theta_0} \qquad (1 \le i \le d). \tag{8.16}$$

Adding (8.15), (8.16) we find $\theta^*_i + \theta^*_{d-i} = \theta^*_0 + \theta^*_d$ for $1 \le i \le d$. Therefore $\theta^*_i + \theta^*_{d-i}$ is independent of i for $0 \le i \le d$. Next we show that $\theta_i + \theta_{d-i}$ is independent of i for $0 \le i \le d$. Rearranging the terms in (8.15) we find that for $1 \le i \le d$,

$$\frac{\theta_i + \theta_{d-i} - \theta_0 - \theta_d}{\theta_0 - \theta_d} = \frac{\theta^*_i + \theta^*_{d-i} - \theta^*_0 - \theta^*_d}{\theta^*_0 - \theta^*_d}.$$

In the above equation the numerator on the right is zero so the numerator on the left is zero. Therefore $\theta_i + \theta_{d-i}$ is independent of i for $0 \le i \le d$. Now let ζ (resp. ζ^*)

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denote the common value of $\theta_i + \theta_{d-i}$ (resp. $\theta^*_i + \theta^*_{d-i}$), and let $\xi = -1$, $\xi^* = -1$. Then (8.11)–(8.14) hold so $\Phi^{\downarrow\Downarrow}$ is isomorphic to (5.1) by Lemma 8.9. Now $\Phi^{\downarrow\Downarrow}$ is affine isomorphic to Φ in view of Definition 5.3. □

Lemma 8.11. *Referring to Notation* 7.1*, let $\xi, \zeta, \xi^*, \zeta^*$ denote scalars in $\mathbb{K}$ with ξ, ξ^* nonzero. Then $\Phi^{\downarrow *}$ is isomorphic to the Leonard system* (5.1) *if and only if*

$$\theta^*_{d-i} = \xi\theta_i + \zeta \qquad (0 \le i \le d), \tag{8.17}$$

$$\theta_i = \xi^*\theta^*_i + \zeta^* \qquad (0 \le i \le d), \tag{8.18}$$

$$\phi_{d-i+1} = \xi\xi^*\varphi_i \qquad (1 \le i \le d), \tag{8.19}$$

$$\varphi_i = \xi\xi^*\phi_i \qquad (1 \le i \le d). \tag{8.20}$$

Proof. Compare the parameter array of $\Phi^{\downarrow *}$ from Lemma 4.5, with the parameter array (6.1). □

Proposition 8.12. *Referring to Notation* 7.1*, the following* (i)–(iii) *are equivalent.*

(i) *$\Phi^{\downarrow *}$ is affine isomorphic to Φ.*
(ii) *$\varphi_1 = \varphi_d = -\phi_1 = -\phi_d$.*
(iii) *φ_i, φ_{d-i+1}, $-\phi_i$, $-\phi_{d-i+1}$ coincide for $1 \le i \le d$ and each of $(\theta^*_i - \theta^*_0)(\theta_i - \theta_0)^{-1}$, $(\theta^*_{d-i} - \theta^*_d)(\theta_i - \theta_0)^{-1}$ is independent of i for $1 \le i \le d$.*

Suppose (i)–(iii) *hold. Then $\Phi^{\downarrow *}$ is isomorphic to* (5.1) *with ξ equal to the common value of $(\theta^*_{d-i} - \theta^*_d)(\theta_i - \theta_0)^{-1}$, $\zeta = \theta^*_d - \xi\theta_0$, $\xi^* = -\xi^{-1}$, and $\zeta^* = \theta_0 - \xi^*\theta^*_0$.*

Proof. (i)⇒(ii): By Definition 5.3 there exist scalars $\xi, \zeta, \xi^*, \zeta^*$ in $\mathbb{K}$ with ξ, ξ^* nonzero such that $\Phi^{\downarrow *}$ is isomorphic to the Leonard system (5.1). Now (8.17)–(8.20) hold by Lemma 8.11. Setting $i = 0$, $i = d$ in (8.17) we find $\xi(\theta_0 - \theta_d) = \theta^*_d - \theta^*_0$. Setting $i = 0$, $i = d$ in (8.18) we find $\xi^*(\theta^*_0 - \theta^*_d) = \theta_0 - \theta_d$. By these comments $\xi\xi^* = -1$. Setting $i = 1$, $i = d$ in (8.19) and using $\xi\xi^* = -1$ we find $\varphi_1 = -\phi_d$ and $\varphi_d = -\phi_1$. Setting $i = 1$ in (8.20) and using $\xi\xi^* = -1$ we find $\varphi_1 = -\phi_1$.

(ii)⇔(iii): Follows from Propositions 8.4, 8.5 and 8.10.

(ii), (iii)⇒(i): Let ξ denote the common value of $(\theta^*_{d-i} - \theta^*_d)(\theta_i - \theta_0)^{-1}$, and let $\xi^* = -\xi^{-1}$, $\zeta = \theta^*_d - \xi\theta_0$, $\zeta^* = \theta_0 - \xi^*\theta^*_0$. Then (8.17)–(8.20) hold so $\Phi^{\downarrow *}$ is isomorphic to (5.1) by Lemma 8.11. Now $\Phi^{\downarrow *}$ is affine isomorphic to Φ in view of Definition 5.3. □

Proposition 8.13. *Referring to Notation* 7.1*, the following* (i)–(iii) *are equivalent.*

(i) *$\Phi^{\Downarrow *}$ is affine isomorphic to Φ.*
(ii) *$\varphi_1 = \varphi_d = -\phi_1 = -\phi_d$.*
(iii) *φ_i, φ_{d-i+1}, $-\phi_i$, $-\phi_{d-i+1}$ coincide for $1 \le i \le d$ and each of $(\theta^*_i - \theta^*_0)(\theta_i - \theta_0)^{-1}$, $(\theta^*_{d-i} - \theta^*_d)(\theta_i - \theta_0)^{-1}$ is independent of i for $1 \le i \le d$.*

Suppose (i)–(iii) *hold. Then $\Phi^{\Downarrow *}$ is affine isomorphic to* (5.1) *with ξ equal to the common value of $(\theta^*_i - \theta^*_0)(\theta_i - \theta_0)^{-1}$, $\zeta = \theta^*_0 - \xi\theta_0$, $\xi^* = -\xi^{-1}$, and $\zeta^* = \theta_0 - \xi^*\theta^*_d$.*

Proof. By Lemma 8.1 (with $g = \downarrow$) and since $\Downarrow * \downarrow = * = \downarrow \downarrow *$ we find that $\Phi^{\Downarrow *}$ is affine isomorphic to Φ if and only if $(\Phi^{\downarrow})^{\downarrow *}$ is affine isomorphic to $\Phi^{\downarrow}$. Now apply Proposition 8.12 to $\Phi^{\downarrow}$ and use Lemma 4.5. □

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9. Proof of Theorem 5.4. In this section we prove Theorem 5.4.

(i): Observe that Φ^g is isomorphic to Φ for all $g \in D_4$ by Propositions 8.4, 8.5, 8.7, 8.8, 8.10, 8.12 and 8.13.

(ii): Since $\varphi_1 = \varphi_d$ and $\phi_1 = \phi_d$, the Leonard systems Φ^*, $\Phi^{\downarrow\Downarrow *}$, $\Phi^{\downarrow\Downarrow}$ are affine isomorphic to Φ by Propositions 8.7, 8.8, 8.10 respectively. Therefore Φ, Φ^*, $\Phi^{\downarrow\Downarrow *}$, $\Phi^{\downarrow\Downarrow}$ are contained in a common affine isomorphism class. By this and Lemma 8.1 the Leonard systems $\Phi^{\downarrow}$, $\Phi^{\Downarrow}$, $\Phi^{\downarrow *}$, $\Phi^{\Downarrow *}$ are contained in a common isomorphism class. The above affine isomorphism classes are distinct; indeed $\Phi^{\downarrow}$ is not affine isomorphic to Φ by Proposition 8.4 and since $\varphi_1 \neq -\phi_d$. The result follows.

(iii): By Proposition 8.8 the Leonard system Φ is affine isomorphic to $\Phi^{\downarrow\Downarrow *}$. By Propositions 8.4, 8.5, 8.10, 8.7, 8.12, 8.13, Φ is not affine isomorphic to any of $\Phi^{\downarrow}$, $\Phi^{\Downarrow}$, $\Phi^{\downarrow\Downarrow}$, Φ^*, $\Phi^{\downarrow *}$, $\Phi^{\Downarrow *}$. The result follows from these comments in view of Lemma 8.1 and (3.1), (3.2).

(iv): By Proposition 8.7 the Leonard system Φ is affine isomorphic to Φ^*. By Propositions 8.4, 8.5, 8.10, 8.12, 8.13, 8.8, Φ is not affine isomorphic to any of $\Phi^{\downarrow}$, $\Phi^{\Downarrow}$, $\Phi^{\downarrow\Downarrow}$, $\Phi^{\downarrow *}$, $\Phi^{\Downarrow *}$, $\Phi^{\downarrow\Downarrow *}$. The result follows from these comments in view of Lemma 8.1 and (3.1), (3.2).

(v): By Proposition 8.5 the Leonard system Φ is affine isomorphic to $\Phi^{\Downarrow}$. By Propositions 8.4, 8.10, 8.7, 8.12, 8.13, 8.8, Φ is not affine isomorphic to any of $\Phi^{\downarrow}$, $\Phi^{\downarrow\Downarrow}$, Φ^*, $\Phi^{\downarrow *}$, $\Phi^{\Downarrow *}$, $\Phi^{\downarrow\Downarrow *}$. The result follows from these comments in view of Lemma 8.1 and (3.1), (3.2).

(vi): By Proposition 8.4 the Leonard system Φ is affine isomorphic to $\Phi^{\downarrow}$. By Propositions 8.5, 8.10, 8.7, 8.12, 8.13, 8.8, Φ is not affine isomorphic to any of $\Phi^{\Downarrow}$, $\Phi^{\downarrow\Downarrow}$, Φ^*, $\Phi^{\downarrow *}$, $\Phi^{\Downarrow *}$, $\Phi^{\downarrow\Downarrow *}$. The result follows from these comments in view of Lemma 8.1 and (3.1), (3.2).

(vii): By Propositions 8.4, 8.5, 8.7, 8.10, 8.12, 8.13, 8.8, Φ is not affine isomorphic to any of $\Phi^{\downarrow}$, $\Phi^{\Downarrow}$, Φ^*, $\Phi^{\downarrow\Downarrow}$, $\Phi^{\downarrow *}$, $\Phi^{\Downarrow *}$, $\Phi^{\downarrow\Downarrow *}$. The result follows from this and Lemma 8.1.

10. The parameters a_i and a_i^*. It turns out that for some of the cases of Theorem 5.4 there is a natural interpretation in terms of the parameters a_i and a_i^* [37, Definition 2.5]. In this section we explain the situation. We start with a definition.

DEFINITION 10.1. *[37, Definition 2.5]* Referring to Notation 7.1, for $0 \le i \le d$ we define scalars

$$a_i := \mathrm{tr}(E_i^* A), \qquad\qquad a_i^* := \mathrm{tr}(E_i A^*).$$

LEMMA 10.2. *Referring to Notation* 7.1 *and Definition* 10.1*, the following* (i)*,* (ii) *are equivalent.*

(i) *a_i is independent of i for $0 \le i \le d$.*

(ii) *The equivalent conditions* (i)–(iii) *hold in Proposition 8.5.*

Suppose (i)*,* (ii) *hold. Then the common value of $\theta_i + \theta_{d-i}$ is twice the common value of a_i.*

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Proof. Follows from [25, Theorem 1.5] and Proposition 8.5. □

LEMMA 10.3. *Referring to Notation* 7.1 *and Definition* 10.1*, the following* (i), (ii) *are equivalent.*

(i) a^*_i *is independent of* i *for* $0 \le i \le d$.
(ii) *The equivalent conditions* (i)–(iii) *hold in Proposition* 8.4*.*

Suppose (i), (ii) *hold. Then the common value of* $\theta^*_i + \theta^*_{d-i}$ *is twice the common value of* a^*_i*.*

Proof. Follows from [25, Theorem 1.6] and Proposition 8.4. □

THEOREM 10.4. *Referring to Notation* 7.1 *and Definition* 10.1*, the following* (i)–(iv) *hold.*

(i) *In Case* (i) *of Theorem* 5.4*, each of* a_i, a^*_i *is independent of* i *for* $0 \le i \le d$.
(ii) *In Case* (v) *of Theorem* 5.4*,* a_i *is independent of* i *for* $0 \le i \le d$ *but* a^*_i *is not independent of* i *for* $0 \le i \le d$.
(iii) *In Case* (vi) *of Theorem* 5.4*,* a^*_i *is independent of* i *for* $0 \le i \le d$ *but* a_i *is not independent of* i *for* $0 \le i \le d$.
(iv) *In the remaining cases of Theorem* 5.4*, neither of* a_i, a^*_i *is independent of* i *for* $0 \le i \le d$.

Proof. Follows from Theorem 5.4 and Lemmas 10.2, 10.3. □

11. Affine transformations of a Leonard pair. Let A, A^* denote a Leonard pair in $\mathcal{A}$ and let $\xi, \zeta, \xi^*, \zeta^*$ denote scalars in $\mathbb{K}$ with ξ, ξ^* nonzero. By Lemma 5.1 and our comments below Definition 2.1 the pair

$$\xi A + \zeta I, \quad \xi^* A^* + \zeta^* I \tag{11.1}$$

is a Leonard pair in $\mathcal{A}$. We call (11.1) the *affine transformation* of A, A^* associated with $\xi, \zeta, \xi^*, \zeta^*$. In this section we find necessary and sufficient conditions for the Leonard pair (11.1) to be isomorphic to A, A^*. We also find necessary and sufficient conditions for the Leonard pair (11.1) to be isomorphic to the Leonard pair A^*, A.

NOTATION 11.1. Let A, A^* denote a Leonard pair in $\mathcal{A}$. Let $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E^*_i\}_{i=0}^d)$ denote a Leonard system associated with A, A^* and let $(\{\theta_i\}_{i=0}^d; \{\theta^*_i\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d)$ denote the parameter array of Φ. To avoid trivialities we assume $d \ge 1$.

PROPOSITION 11.2. *Referring to Notation* 11.1*, the Leonard pair* A, A^* *is isomorphic to the Leonard pair* (11.1) *if and only if at least one of* (i)–(iv) *holds below.*

(i) $\xi = 1$, $\zeta = 0$, $\xi^* = 1$, $\zeta^* = 0$.
(ii) $\varphi_1 = -\phi_d$, $\varphi_d = -\phi_1$, $\xi = 1$, $\zeta = 0$, $\xi^* = -1$, $\zeta^* = \theta^*_0 + \theta^*_d$.
(iii) $\varphi_1 = -\phi_1$, $\varphi_d = -\phi_d$, $\xi = -1$, $\zeta = \theta_0 + \theta_d$, $\xi^* = 1$, $\zeta^* = 0$.
(iv) $\varphi_1 = \varphi_d$, $\phi_1 = \phi_d$, $\xi = -1$, $\zeta = \theta_0 + \theta_d$, $\xi^* = -1$, $\zeta^* = \theta^*_0 + \theta^*_d$.

In this case precisely one of (i)–(iv) *holds.*

Proof. By [40, Lemma 5.4] the Leonard systems associated with A, A^* are Φ, $\Phi^{\downarrow}$, $\Phi^{\Downarrow}$, $\Phi^{\downarrow\Downarrow}$. Therefore the Leonard pair A, A^* is isomorphic to the Leonard pair (11.1) if and only if at least one of Φ, $\Phi^{\downarrow}$, $\Phi^{\Downarrow}$, $\Phi^{\downarrow\Downarrow}$ is isomorphic to the Leonard system (5.1).

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By this and Propositions 8.2, 8.4, 8.5, 8.10 we find A, A^* is isomorphic to (11.1) if and only if at least one of (i)–(iv) holds. Assume that at least one of (i)–(iv) holds. We show that precisely one of (i)–(iv) holds. By way of contradiction assume that at least two of (i)–(iv) hold. Then at least one of 2ξ, $2\xi^*$ is zero, forcing $\mathrm{Char}(\mathbb{K}) = 2$, and at least one of $\theta_0 + \theta_d$, $\theta^*_0 + \theta^*_d$ is zero, forcing $\mathrm{Char}(\mathbb{K}) \neq 2$ and giving a contradiction. Therefore precisely one of (i)–(iv) holds. □

Proposition 11.3. *Referring to Notation* 11.1*, the Leonard pair A^*, A is isomorphic to the Leonard pair* (11.1) *if and only if at least one of* (i)–(iv) *holds below.*

(i) $\phi_1 = \phi_d$, $\xi = (\theta^*_d - \theta^*_0)(\theta_d - \theta_0)^{-1}$, $\zeta = \theta^*_0 - \xi\theta_0$, $\xi^* = \xi^{-1}$, $\zeta^* = \theta_0 - \xi^*\theta^*_0$.

(ii) $\varphi_1 = \varphi_d = -\phi_1 = -\phi_d$, $\xi = (\theta^*_0 - \theta^*_d)(\theta_d - \theta_0)^{-1}$, $\zeta = \theta^*_d - \xi\theta_0$, $\xi^* = -\xi^{-1}$, $\zeta^* = \theta_0 - \xi^*\theta^*_0$.

(iii) $\varphi_1 = \varphi_d = -\phi_1 = -\phi_d$, $\xi = (\theta^*_d - \theta^*_0)(\theta_d - \theta_0)^{-1}$, $\zeta = \theta^*_0 - \xi\theta_0$, $\xi^* = -\xi^{-1}$, $\zeta^* = \theta_0 - \xi^*\theta^*_d$.

(iv) $\varphi_1 = \varphi_d$, $\xi = (\theta^*_0 - \theta^*_d)(\theta_d - \theta_0)^{-1}$, $\zeta = \theta^*_d - \xi\theta_0$, $\xi^* = \xi^{-1}$, $\zeta^* = \theta_0 - \xi^*\theta^*_d$.

In this case precisely one of (i)–(iv) *holds.*

Proof. By [40, Lemma 5.4] the Leonard systems associated with A^*, A are Φ^*, $\Phi^{\downarrow *}$, $\Phi^{\Downarrow *}$, $\Phi^{\downarrow\Downarrow *}$. Therefore the Leonard pair A^*, A is isomorphic to the Leonard pair (11.1) if and only if at least one of Φ^*, $\Phi^{\downarrow *}$, $\Phi^{\Downarrow *}$, $\Phi^{\downarrow\Downarrow *}$ is isomorphic to the Leonard system (5.1). By this and Propositions 8.7, 8.8, 8.12, 8.13 we find A^*, A is isomorphic to (11.1) if and only if at least one of (i)–(iv) holds. Assume that at least one of (i)–(iv) holds. We show that precisely one of (i)–(iv) holds. By way of contradiction assume that at least two of (i)–(iv) hold. Then at least one of $\theta_0 = \theta_d$, $\theta^*_0 = \theta^*_d$ holds, for a contradiction. Therefore precisely one of (i)–(iv) holds. □

The following is the main result of the paper.

Theorem 11.4. *Referring to Notation* 11.1*, we set $\alpha = (\theta^*_d - \theta^*_0)(\theta_d - \theta_0)^{-1}$.*

(i) *Assume $\varphi_1 = \varphi_d = -\phi_1 = -\phi_d$. Then A, A^* is isomorphic to* (11.1) *if and only if the sequence $\xi, \zeta, \xi^*, \zeta^*$ is listed in the following table.*

ξ	ζ	ξ^*	ζ^*
1	0	1	0
1	0	-1	$\theta^*_0 + \theta^*_d$
-1	$\theta_0 + \theta_d$	1	0
-1	$\theta_0 + \theta_d$	-1	$\theta^*_0 + \theta^*_d$

Moreover A^*, A is isomorphic to (11.1) *if and only if the sequence $\xi, \zeta, \xi^*, \zeta^*$ is listed in the following table.*

ξ	ζ	ξ^*	ζ^*
α	$\theta^*_0 - \alpha\theta_0$	α^{-1}	$\theta_0 - \alpha^{-1}\theta^*_0$
$-\alpha$	$\theta^*_d + \alpha\theta_0$	α^{-1}	$\theta_0 - \alpha^{-1}\theta^*_0$
α	$\theta^*_0 - \alpha\theta_0$	$-\alpha^{-1}$	$\theta_0 + \alpha^{-1}\theta^*_d$
$-\alpha$	$\theta^*_d + \alpha\theta_0$	$-\alpha^{-1}$	$\theta_0 + \alpha^{-1}\theta^*_d$

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(ii) *Assume $\varphi_1 = \varphi_d$, $\phi_1 = \phi_d$ and $\varphi_1 \neq -\phi_1$. Then A, A^* is isomorphic to* (11.1) *if and only if the sequence $\xi, \zeta, \xi^*, \zeta^*$ is listed in the following table.*

ξ	ζ	ξ^*	ζ^*
1	0	1	0
-1	$\theta_0 + \theta_d$	-1	$\theta^*_0 + \theta^*_d$

Moreover A^*, A is isomorphic to (11.1) *if and only if the sequence ξ, ζ, ξ^*, ζ^* is listed in the following table.*

ξ	ζ	ξ^*	ζ^*
α	$\theta^*_0 - \alpha\theta_0$	α^{-1}	$\theta_0 - \alpha^{-1}\theta^*_0$
$-\alpha$	$\theta^*_d + \alpha\theta_0$	$-\alpha^{-1}$	$\theta_0 + \alpha^{-1}\theta^*_d$

(iii) *Assume $\varphi_1 = \varphi_d$ and $\phi_1 \neq \phi_d$. Then A, A^* is isomorphic to* (11.1) *if and only if $\xi = 1$, $\zeta = 0$, $\xi^* = 1$, $\zeta^* = 0$. Moreover A^*, A is isomorphic to* (11.1) *if and only if $\xi = -\alpha$, $\zeta = \theta^*_d + \alpha\theta_0$, $\xi^* = -\alpha^{-1}$, $\zeta^* = \theta_0 + \alpha^{-1}\theta^*_d$.*

(iv) *Assume $\phi_1 = \phi_d$ and $\varphi_1 \neq \varphi_d$. Then A, A^* is isomorphic to* (11.1) *if and only if $\xi = 1$, $\zeta = 0$, $\xi^* = 1$, $\zeta^* = 0$. Moreover A^*, A is isomorphic to* (11.1) *if and only if $\xi = \alpha$, $\zeta = \theta^*_0 - \alpha\theta_0$, $\xi^* = \alpha^{-1}$, $\zeta^* = \theta_0 - \alpha^{-1}\theta^*_0$.*

(v) *Assume $\varphi_1 = -\phi_1$, $\varphi_d = -\phi_d$ and $\varphi_1 \neq \varphi_d$. Then A, A^* is isomorphic to* (11.1) *if and only if the sequence $\xi, \zeta, \xi^*, \zeta^*$ is listed in the following table.*

ξ	ζ	ξ^*	ζ^*
1	0	1	0
-1	$\theta_0 + \theta_d$	1	0

Moreover A^*, A is not isomorphic to (11.1) *for any $\xi, \zeta, \xi^*, \zeta^*$.*

(vi) *Assume $\varphi_1 = -\phi_d$, $\varphi_d = -\phi_1$ and $\varphi_1 \neq \varphi_d$. Then A, A^* is isomorphic to* (11.1) *if and only if the sequence $\xi, \zeta, \xi^*, \zeta^*$ is listed in the following table.*

ξ	ζ	ξ^*	ζ^*
1	0	1	0
1	0	-1	$\theta^*_0 + \theta^*_d$

Moreover A^*, A is not isomorphic to (11.1) *for any $\xi, \zeta, \xi^*, \zeta^*$.*

(vii) *Assume none of* (i)–(vi) *hold above. Then A, A^* is isomorphic to* (11.1) *if and only if $\xi = 1$, $\zeta = 0$, $\xi^* = 1$, $\zeta^* = 0$. Moreover A^*, A is not isomorphic to* (11.1) *for any $\xi, \zeta, \xi^*, \zeta^*$.*

Proof. Routine consequence of Propositions 11.2 and 11.3. □

12. The parameter arrays in closed form. In [25] and [44] the parameter array of a Leonard system is given in closed form. For the rest of this paper we consider how the results of previous sections look in terms of this form.

NOTATION 12.1. Let $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E^*_i\}_{i=0}^d)$ denote a Leonard system over $\mathbb{K}$ and let $(\{\theta_i\}_{i=0}^d; \{\theta^*_i\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d)$ denote the corresponding parameter array. We assume $d \geq 3$.

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NOTATION 12.2. Referring to Notation 12.1, let $\overline{\mathbb{K}}$ denote the algebraic closure of $\mathbb{K}$ and let q denote a nonzero scalar in $\overline{\mathbb{K}}$ such that $q + q^{-1} + 1$ is equal to the common value of (4.4). We consider the following types:

type	description
I	$q \neq 1$, $q \neq -1$
II	$q = 1$, $\mathrm{Char}(\mathbb{K}) \neq 2$
III$^+$	$q = -1$, $\mathrm{Char}(\mathbb{K}) \neq 2$, d even
III$^-$	$q = -1$, $\mathrm{Char}(\mathbb{K}) \neq 2$, d odd
IV	$q = 1$, $\mathrm{Char}(\mathbb{K}) = 2$

13. Type I: $q \neq 1$ and $q \neq -1$.

LEMMA 13.1. [25, Theorem 6.1] *Referring to Notation* 12.1*, assume* Φ *is Type I. Then there exists unique scalars* η*,* μ*,* h*,* η^**,* μ^**,* h^**,* τ *in* $\overline{\mathbb{K}}$ *such that*

$$\theta_i = \eta + \mu q^i + h q^{d-i}, \tag{13.1}$$

$$\theta^*_i = \eta^* + \mu^* q^i + h^* q^{d-i} \tag{13.2}$$

for $0 \leq i \leq d$ *and*

$$\varphi_i = (q^i - 1)(q^{d-i+1} - 1)(\tau - \mu\mu^* q^{i-1} - hh^* q^{d-i}), \tag{13.3}$$

$$\phi_i = (q^i - 1)(q^{d-i+1} - 1)(\tau - h\mu^* q^{i-1} - \mu h^* q^{d-i}) \tag{13.4}$$

for $1 \leq i \leq d$*.*

REMARK 13.2. Referring to Lemma 13.1, for $1 \leq i \leq d$ we have $q^i \neq 1$; otherwise $\varphi_i = 0$ by (13.3). For $0 \leq i \leq d-1$ we have $\mu \neq hq^i$; otherwise $\theta_{d-i} = \theta_0$. Similarly $\mu^* \neq h^* q^i$.

LEMMA 13.3. *Referring to Notation* 12.1*, assume* Φ *is Type I. Then* (i)–(iv) *hold below.*

(i) $\theta_i + \theta_{d-i}$ *is independent of* i *for* $0 \leq i \leq d$ *if and only if* $\mu = -h$*.*
(ii) $\theta^*_i + \theta^*_{d-i}$ *is independent of* i *for* $0 \leq i \leq d$ *if and only if* $\mu^* = -h^*$*.*
(iii) $(\theta^*_i - \theta^*_0)(\theta_i - \theta_0)^{-1}$ *is independent of* i *for* $1 \leq i \leq d$ *if and only if* $\mu h^* = \mu^* h$*.*
(iv) $(\theta^*_{d-i} - \theta^*_d)(\theta_i - \theta_0)^{-1}$ *is independent of* i *for* $1 \leq i \leq d$ *if and only if* $\mu\mu^* = hh^*$*.*

Proof. (i): Using (13.1),

$$\theta_i + \theta_{d-i} - \theta_0 - \theta_d = (q^i - 1)(1 - q^{d-i})(\mu + h)$$

for $0 \leq i \leq d$. The result follows from this and Remark 13.2.

(ii): Similar to the proof of (i).

(iii): Using (13.1) and (13.2),

$$\frac{\theta^*_i - \theta^*_0}{\theta_i - \theta_0} - \frac{\theta^*_d - \theta^*_0}{\theta_d - \theta_0} = \frac{(\mu h^* - \mu^* h)(1 - q^{d-i})}{(\mu - h)(\mu - hq^{d-i})}$$

for $1 \leq i \leq d$. The result follows from this and Remark 13.2.

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(iv): Using (13.1) and (13.2),

$$\frac{\theta^*_{d-i} - \theta^*_d}{\theta_i - \theta_0} - \frac{\theta^*_0 - \theta^*_d}{\theta_d - \theta_0} = \frac{(\mu\mu^* - hh^*)(1 - q^{d-i})}{(\mu - h)(\mu - hq^{d-i})}$$

for $1 \leq i \leq d$. The result follows from this and Remark 13.2. □

Lemma 13.4. *Referring to Notation* 12.1, *assume* Φ *is Type I. Then* (i)–(iv) *hold below.*

(i) $\varphi_i = -\phi_i$ *for* $1 \leq i \leq d$ *if and only if* $\tau = 0$ *and* $\mu = -h$.
(ii) $\varphi_i = -\phi_{d-i+1}$ *for* $1 \leq i \leq d$ *if and only if* $\tau = 0$ *and* $\mu^* = -h^*$.
(iii) $\phi_i = \phi_{d-i+1}$ *for* $1 \leq i \leq d$ *if and only if* $\mu h^* = \mu^* h$.
(iv) $\varphi_i = \varphi_{d-i+1}$ *for* $1 \leq i \leq d$ *if and only if* $\mu\mu^* = hh^*$.

Proof. (i): Using the data in Lemma 13.1 we find

$$\varphi_i + \phi_i = (q^i - 1)(q^{d-i+1} - 1)(2\tau - (\mu + h)(\mu^* q^{i-1} + h^* q^{d-i})) \tag{13.5}$$

for $1 \leq i \leq d$ and

$$\varphi_1 + \phi_1 - \varphi_d - \phi_d = (q - 1)(q^{d-1} - 1)(q^d - 1)(\mu + h)(\mu^* - h^*). \tag{13.6}$$

First assume $\varphi_i = -\phi_i$ for $1 \leq i \leq d$. Then in (13.6) the expression on the left is zero so the expression on the right is zero. In this expression each factor except $\mu + h$ is nonzero by Remark 13.2, so $\mu = -h$. In (13.5) the expression on the left is zero so the expression on the right is zero. Evaluating this expression using $\mu = -h$ and Remark 13.2 we find $2\tau = 0$. Note that $\mathrm{Char}(\mathbb{K}) \neq 2$; otherwise $\mu = h$ and Remark 13.2 is contradicted. Therefore $\tau = 0$. We have now shown $\tau = 0$ and $\mu = -h$. Conversely assume $\tau = 0$ and $\mu = -h$. Then by (13.5) we have $\varphi_i = -\phi_i$ for $1 \leq i \leq d$.

(ii): Similar to the proof of (i). We note that

$$\varphi_i + \phi_{d-i+1} = (q^i - 1)(q^{d-i+1} - 1)(2\tau - (\mu^* + h^*)(\mu q^{i-1} + hq^{d-i}))$$

for $1 \leq i \leq d$ and

$$\varphi_1 + \phi_d - \varphi_d - \phi_1 = (q - 1)(q^{d-1} - 1)(q^d - 1)(\mu - h)(\mu^* + h^*).$$

(iii): Using the data in Lemma 13.1 we find

$$\phi_i - \phi_{d-i+1} = (q^i - 1)(q^{d-i+1} - 1)(q^{i-1} - q^{d-i})(\mu h^* - \mu^* h)$$

for $1 \leq i \leq d$. The result follows from this and Remark 13.2.

(iv): Using the data in Lemma 13.1 we find

$$\varphi_i - \varphi_{d-i+1} = (q^i - 1)(q^{d-i+1} - 1)(q^{d-i} - q^{i-1})(\mu\mu^* - hh^*)$$

for $1 \leq i \leq d$. The result follows from this and Remark 13.2. □

Proposition 13.5. *Referring to Notation* 12.1, *assume* Φ *is Type I. Then* (i)–(vii) *hold below.*

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(i) $\Phi^{\downarrow}$ *is affine isomorphic to* Φ *if and only if* $\mu^* = -h^*$, $\tau = 0$.
(ii) $\Phi^{\Downarrow}$ *is affine isomorphic to* Φ *if and only if* $\mu = -h$, $\tau = 0$.
(iii) $\Phi^{\downarrow\Downarrow}$ *is affine isomorphic to* Φ *if and only if* $\mu = -h$, $\mu^* = -h^*$.
(iv) Φ^* *is affine isomorphic to* Φ *if and only if* $\mu h^* = \mu^* h$.
(v) $\Phi^{\downarrow *}$ *is affine isomorphic to* Φ *if and only if* $\mu = -h$, $\mu^* = -h^*$, $\tau = 0$.
(vi) $\Phi^{\Downarrow *}$ *is affine isomorphic to* Φ *if and only if* $\mu = -h$, $\mu^* = -h^*$, $\tau = 0$.
(vii) $\Phi^{\downarrow\Downarrow *}$ *is affine isomorphic to* Φ *if and only if* $\mu\mu^* = hh^*$.

Proof. Follows from Propositions 8.4, 8.5, 8.7, 8.8, 8.10, 8.12, 8.13, and Lemmas 13.3, 13.4. □

THEOREM 13.6. *Referring to Notation* 12.1*, assume* Φ *is Type I. Then* (i)–(vii) *hold below.*
(i) *Case* (i) *of Theorem* 5.4 *occurs if and only if* $\mu = -h$, $\mu^* = -h^*$ *and* $\tau = 0$.
(ii) *Case* (ii) *of Theorem* 5.4 *occurs if and only if* $\mu = -h$, $\mu^* = -h^*$ *and* $\tau \neq 0$.
(iii) *Case* (iii) *of Theorem* 5.4 *occurs if and only if* $\mu\mu^* = hh^*$ *and* $\mu h^* \neq \mu^* h$.
(iv) *Case* (iv) *of Theorem* 5.4 *occurs if and only if* $\mu\mu^* \neq hh^*$ *and* $\mu h^* = \mu^* h$.
(v) *Case* (v) *of Theorem* 5.4 *occurs if and only if* $\mu = -h$, $\mu^* \neq -h^*$ *and* $\tau = 0$.
(vi) *Case* (vi) *of Theorem* 5.4 *occurs if and only if* $\mu \neq -h$, $\mu^* = -h^*$ *and* $\tau = 0$.
(vii) *Case* (vii) *of Theorem* 5.4 *occurs if and only if* $\mu\mu^* \neq hh^*$, $\mu h^* \neq \mu^* h$, *and at least two of* $\mu \neq -h$, $\mu^* \neq -h^*$, $\tau \neq 0$.

Proof. Combine Theorem 5.4 and Proposition 13.5. □

14. Type II: $q = 1$ and Char$(\mathbb{K}) \neq 2$.

LEMMA 14.1. [25, Theorem 7.1] *Referring to Notation* 12.1*, assume* Φ *is Type II. Then there exists unique scalars* η, μ, h, η^*, μ^*, h^*, τ *in* $\overline{\mathbb{K}}$ *such that*

$$\theta_i = \eta + \mu(i - d/2) + hi(d - i), \tag{14.1}$$
$$\theta^*_i = \eta^* + \mu^*(i - d/2) + h^*i(d - i) \tag{14.2}$$

for $0 \leq i \leq d$ *and*

$$\begin{aligned}\varphi_i = i(d - i + 1)(\tau - \mu\mu^*/2 \\ + (h\mu^* + \mu h^*)(i - (d + 1)/2) + hh^*(i - 1)(d - i)),\end{aligned} \tag{14.3}$$

$$\begin{aligned}\phi_i = i(d - i + 1)(\tau + \mu\mu^*/2 \\ + (h\mu^* - \mu h^*)(i - (d + 1)/2) + hh^*(i - 1)(d - i))\end{aligned} \tag{14.4}$$

for $1 \leq i \leq d$.

REMARK 14.2. Referring to Lemma 14.1, for $0 \leq i \leq d - 1$ we have $\mu \neq -ih$; otherwise $\theta_{d-i} = \theta_0$. Similarly $\mu^* \neq -ih^*$. For any prime i such that $i \leq d$ we have Char$(\mathbb{K}) \neq i$; otherwise $\varphi_i = 0$ by (14.3).

LEMMA 14.3. *Referring to Notation* 12.1*, assume* Φ *is Type II. Then* (i)–(iv) *hold below.*

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(i) $\theta_i + \theta_{d-i}$ *is independent of* i *for* $0 \le i \le d$ *if and only if* $h = 0$.
(ii) $\theta^*_i + \theta^*_{d-i}$ *is independent of* i *for* $0 \le i \le d$ *if and only if* $h^* = 0$.
(iii) $(\theta^*_i - \theta^*_0)(\theta_i - \theta_0)^{-1}$ *is independent of* i *for* $1 \le i \le d$ *if and only if* $\mu h^* = \mu^* h$.
(iv) $(\theta^*_{d-i} - \theta^*_d)(\theta_i - \theta_0)^{-1}$ *is independent of* i *for* $1 \le i \le d$ *if and only if* $\mu h^* = -\mu^* h$.

Proof. (i): Using (14.1),

$$\theta_i + \theta_{d-i} - \theta_0 - \theta_d = 2i(d-i)h$$

for $0 \le i \le d$. The result follows from this and Remark 14.2.
(ii): Similar to the proof of (i).
(iii): Using (14.1) and (14.2),

$$\frac{\theta^*_i - \theta^*_0}{\theta_i - \theta_0} - \frac{\theta^*_d - \theta^*_0}{\theta_d - \theta_0} = \frac{(d-i)(\mu h^* - \mu^* h)}{\mu(\mu + (d-i)h)}$$

for $1 \le i \le d$. The result follows from this and Remark 14.2.
(iv): Using (14.1) and (14.2),

$$\frac{\theta^*_{d-i} - \theta^*_d}{\theta_i - \theta_0} - \frac{\theta^*_0 - \theta^*_d}{\theta_d - \theta_0} = \frac{(d-i)(\mu h^* + \mu^* h)}{\mu(\mu + (d-i)h)}$$

for $1 \le i \le d$. The result follows from this and Remark 14.2. □

Lemma 14.4. *Referring to Notation* 12.1*, assume* Φ *is Type II. Then* (i)–(iv) *hold below.*
(i) $\varphi_i = -\phi_i$ *for* $1 \le i \le d$ *if and only if* $\tau = 0$ *and* $h = 0$.
(ii) $\varphi_i = -\phi_{d-i+1}$ *for* $1 \le i \le d$ *if and only if* $\tau = 0$ *and* $h^* = 0$.
(iii) $\phi_i = \phi_{d-i+1}$ *for* $1 \le i \le d$ *if and only if* $\mu h^* = \mu^* h$.
(iv) $\varphi_i = \varphi_{d-i+1}$ *for* $1 \le i \le d$ *if and only if* $\mu h^* = -\mu^* h$.

Proof. (i): Using the data in Lemma 14.1 we find

$$\varphi_i + \phi_i = 2i(d-i+1)(\tau + \mu^* h(i-(d+1)/2) + hh^*(i-1)(d-i)) \tag{14.5}$$

for $1 \le i \le d$ and

$$\varphi_1 + \phi_1 - \varphi_d - \phi_d = 2d(1-d)\mu^* h. \tag{14.6}$$

First assume $\varphi_i = -\phi_i$ for $1 \le i \le d$. Then in (14.6) the expression on the left is zero so the expression on the right is zero. In this expression each factor except h is nonzero by Remark 14.2, so $h = 0$. In (14.5) the expression on the left is zero so the expression on the right is zero. Evaluating this expression using $h = 0$ and Remark 14.2 we find $\tau = 0$. We have now shown $\tau = 0$ and $h = 0$. Conversely assume $\tau = 0$ and $h = 0$. Then by (14.5) we have $\varphi_i = -\phi_i$ for $1 \le i \le d$.
(ii): Similar to the proof of (i). We note that

$$\varphi_i + \phi_{d-i+1} = 2i(d-i+1)(\tau + \mu h^*(i-(d+1)/2) + hh^*(i-1)(d-i))$$

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for $1 \le i \le d$ and

$$\varphi_1 + \phi_d - \varphi_d - \phi_1 = 2d(1-d)\mu h^*.$$

(iii): Using the data in Lemma 14.1 we find

$$\phi_i - \phi_{d-i+1} = i(d-i+1)(d-2i+1)(\mu h^* - \mu^* h)$$

for $1 \le i \le d$. The result follows from this and Remark 14.2.

(iv): Using the data in Lemma 14.1 we find

$$\varphi_i - \varphi_{d-i+1} = -i(d-i+1)(d-2i+1)(\mu h^* + \mu^* h)$$

for $1 \le i \le d$. The result follows from this and Remark 14.2. □

Proposition 14.5. *Referring to Notation* 12.1*, assume* Φ *is Type II. Then* (i)–(vii) *hold below.*

(i) $\Phi^{\downarrow}$ *is affine isomorphic to* Φ *if and only if* $h^* = 0$, $\tau = 0$.
(ii) $\Phi^{\Downarrow}$ *is affine isomorphic to* Φ *if and only if* $h = 0$, $\tau = 0$.
(iii) $\Phi^{\downarrow\Downarrow}$ *is affine isomorphic to* Φ *if and only if* $h = 0$, $h^* = 0$.
(iv) Φ^* *is affine isomorphic to* Φ *if and only if* $\mu h^* = \mu^* h$.
(v) $\Phi^{\downarrow *}$ *is affine isomorphic to* Φ *if and only if* $h = 0$, $h^* = 0$, $\tau = 0$.
(vi) $\Phi^{\Downarrow *}$ *is affine isomorphic to* Φ *if and only if* $h = 0$, $h^* = 0$, $\tau = 0$.
(vii) $\Phi^{\downarrow\Downarrow *}$ *is affine isomorphic to* Φ *if and only if* $\mu h^* = -\mu^* h$.

Proof. Follows from Propositions 8.4, 8.5, 8.7, 8.8, 8.10, 8.12, 8.13, and Lemmas 14.3, 14.4. □

Theorem 14.6. *Referring to Notation* 12.1*, assume* Φ *is Type II. Then* (i)–(vii) *hold below.*

(i) *Case* (i) *of Theorem* 5.4 *occurs if and only if* $h = 0$, $h^* = 0$ *and* $\tau = 0$.
(ii) *Case* (ii) *of Theorem* 5.4 *occurs if and only if* $h = 0$, $h^* = 0$ *and* $\tau \neq 0$.
(iii) *Case* (iii) *of Theorem* 5.4 *occurs if and only if* $\mu h^* \neq \mu^* h$ *and* $\mu h^* = -\mu^* h$.
(iv) *Case* (iv) *of Theorem* 5.4 *occurs if and only if* $\mu h^* = \mu^* h$ *and* $\mu h^* \neq -\mu^* h$.
(v) *Case* (v) *of Theorem* 5.4 *occurs if and only if* $h = 0$, $h^* \neq 0$ *and* $\tau = 0$.
(vi) *Case* (vi) *of Theorem* 5.4 *occurs if and only if* $h \neq 0$, $h^* = 0$ *and* $\tau = 0$.
(vii) *Case* (vii) *of Theorem* 5.4 *occurs if and only if* $\mu h^* \neq \mu^* h$, $\mu h^* \neq -\mu^* h$, *and at least two of* h, h^*, τ *are nonzero.*

Proof. Combine Theorem 5.4 and Proposition 14.5. □

15. Type III$^+$: $q = -1$, Char$(\mathbb{K}) \neq 2$ and d is even.

Lemma 15.1. [25, Theorem 8.1] *Referring to Notation* 12.1*, assume* Φ *is Type III*$^+$*. Then there exists unique scalars* η, h, s, η^*, h^*, s^*, τ *in* $\overline{\mathbb{K}}$ *such that*

$$\theta_i = \begin{cases} \eta + s + h(i - d/2) & \text{if } i \text{ is even,} \\ \eta - s - h(i - d/2) & \text{if } i \text{ is odd,} \end{cases} \tag{15.1}$$

$$\theta^*_i = \begin{cases} \eta^* + s^* + h^*(i - d/2) & \text{if } i \text{ is even,} \\ \eta^* - s^* - h^*(i - d/2) & \text{if } i \text{ is odd} \end{cases} \tag{15.2}$$

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for $0 \leq i \leq d$ *and*

$$\varphi_i = \begin{cases} i(\tau - sh^* - s^*h - hh^*(i-(d+1)/2)) & \text{if } i \text{ is even,} \\ (d-i+1)(\tau + sh^* + s^*h + hh^*(i-(d+1)/2)) & \text{if } i \text{ is odd,} \end{cases} \tag{15.3}$$

$$\phi_i = \begin{cases} i(\tau - sh^* + s^*h + hh^*(i-(d+1)/2)) & \text{if } i \text{ is even,} \\ (d-i+1)(\tau + sh^* - s^*h - hh^*(i-(d+1)/2)) & \text{if } i \text{ is odd} \end{cases} \tag{15.4}$$

for $1 \leq i \leq d$.

REMARK 15.2. Referring to Lemma 15.1, we have $h \neq 0$; otherwise $\theta_0 = \theta_2$ by (15.1). Similarly we have $h^* \neq 0$. For i odd with $0 \leq i \leq d-1$ we have $s \neq ih/2$; otherwise $\theta_{d-i} = \theta_0$. For any prime i such that $i \leq d/2$ we have $\mathrm{Char}(\mathbb{K}) \neq i$; otherwise $\varphi_{2i} = 0$ by (15.3). By this and since $\mathrm{Char}(\mathbb{K}) \neq 2$ we find $\mathrm{Char}(\mathbb{K})$ is either 0 or an odd prime greater than $d/2$. Observe neither of d, $d-2$ vanish in $\mathbb{K}$ since otherwise $\mathrm{Char}(\mathbb{K})$ must divide $d/2$ or $(d-2)/2$.

LEMMA 15.3. *Referring to Notation* 12.1*, assume* Φ *is Type III*$^+$*. Then* (i)–(iv) *hold below.*

(i) $\theta_i + \theta_{d-i}$ *is independent of* i *for* $0 \leq i \leq d$ *if and only if* $s = 0$.
(ii) $\theta^*_i + \theta^*_{d-i}$ *is independent of* i *for* $0 \leq i \leq d$ *if and only if* $s^* = 0$.
(iii) $(\theta^*_i - \theta^*_0)(\theta_i - \theta_0)^{-1}$ *is independent of* i *for* $1 \leq i \leq d$ *if and only if* $hs^* = h^*s$.
(iv) $(\theta^*_{d-i} - \theta^*_d)(\theta_i - \theta_0)^{-1}$ *is independent of* i *for* $1 \leq i \leq d$ *if and only if* $hs^* = -h^*s$.

Proof. (i): Using (15.1),

$$\theta_i + \theta_{d-i} = \begin{cases} 2(\eta + s) & \text{if } i \text{ is even,} \\ 2(\eta - s) & \text{if } i \text{ is odd} \end{cases}$$

for $0 \leq i \leq d$. The result follows from this.

(ii): Similar to the proof of (i).

(iii): Using (15.1) and (15.2),

$$\frac{\theta^*_i - \theta^*_0}{\theta_i - \theta_0} - \frac{\theta^*_d - \theta^*_0}{\theta_d - \theta_0} = \begin{cases} 0 & \text{if } i \text{ is even,} \\ \dfrac{hs^* - h^*s}{h(s - h(d-i)/2)} & \text{if } i \text{ is odd} \end{cases}$$

for $1 \leq i \leq d$. The result follows from this.

(iv): Using (15.1) and (15.2),

$$\frac{\theta^*_{d-i} - \theta^*_d}{\theta_i - \theta_0} - \frac{\theta^*_0 - \theta^*_d}{\theta_d - \theta_0} = \begin{cases} 0 & \text{if } i \text{ is even,} \\ \dfrac{hs^* + h^*s}{h(s - h(d-i)/2)} & \text{if } i \text{ is odd} \end{cases}$$

for $1 \leq i \leq d$. The result follows from this. ∎

LEMMA 15.4. *Referring to Notation* 12.1*, assume* Φ *is Type III*$^+$*. Then* (i)–(iv) *hold below.*

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(i) $\varphi_i = -\phi_i$ *for* $1 \leq i \leq d$ *if and only if* $\tau = 0$ *and* $s = 0$.
(ii) $\varphi_i = -\phi_{d-i+1}$ *for* $1 \leq i \leq d$ *if and only if* $\tau = 0$ *and* $s^* = 0$.
(iii) $\phi_i = \phi_{d-i+1}$ *for* $1 \leq i \leq d$ *if and only if* $hs^* = h^*s$.
(iv) $\varphi_i = \varphi_{d-i+1}$ *for* $1 \leq i \leq d$ *if and only if* $hs^* = -h^*s$.

Proof. (i): Using the data in Lemma 15.1 we find

$$\varphi_i + \phi_i = \begin{cases} 2i(\tau - h^*s) & \text{if } i \text{ is even,} \\ 2(d-i+1)(\tau + h^*s) & \text{if } i \text{ is odd} \end{cases}$$

for $1 \leq i \leq d$. The result follows from this and Remark 15.2.

(ii): Using the data in Lemma 15.1 we find

$$\varphi_i + \phi_{d-i+1} = \begin{cases} 2i(\tau - hs^*) & \text{if } i \text{ is even,} \\ 2(d-i+1)(\tau + hs^*) & \text{if } i \text{ is odd} \end{cases}$$

for $1 \leq i \leq d$. The result follows from this and Remark 15.2.

(iii): Using the data in Lemma 15.1 we find

$$\phi_i - \phi_{d-i+1} = \begin{cases} 2i(hs^* - h^*s) & \text{if } i \text{ is even,} \\ -2(d-i+1)(hs^* - h^*s) & \text{if } i \text{ is odd} \end{cases}$$

for $1 \leq i \leq d$. The result follows from this and Remark 15.2.

(iv): Using the data in Lemma 15.1 we find

$$\varphi_i - \varphi_{d-i+1} = \begin{cases} -2i(hs^* + h^*s) & \text{if } i \text{ is even,} \\ 2(d-i+1)(hs^* + h^*s) & \text{if } i \text{ is odd} \end{cases}$$

for $1 \leq i \leq d$. The result follows from this and Remark 15.2. □

PROPOSITION 15.5. *Referring to Notation* 12.1*, assume* Φ *is Type III*$^+$*. Then* (i)–(vii) *hold below.*

(i) $\Phi^{\downarrow}$ *is affine isomorphic to* Φ *if and only if* $s^* = 0$, $\tau = 0$.
(ii) $\Phi^{\Downarrow}$ *is affine isomorphic to* Φ *if and only if* $s = 0$, $\tau = 0$.
(iii) $\Phi^{\downarrow\Downarrow}$ *is affine isomorphic to* Φ *if and only if* $s = 0$, $s^* = 0$.
(iv) Φ^* *is affine isomorphic to* Φ *if and only if* $hs^* = h^*s$.
(v) $\Phi^{\downarrow *}$ *is affine isomorphic to* Φ *if and only if* $s = 0$, $s^* = 0$, $\tau = 0$.
(vi) $\Phi^{\Downarrow *}$ *is affine isomorphic to* Φ *if and only if* $s = 0$, $s^* = 0$, $\tau = 0$.
(vii) $\Phi^{\downarrow\Downarrow *}$ *is affine isomorphic to* Φ *if and only if* $hs^* = -h^*s$.

Proof. Follows from Propositions 8.4, 8.5, 8.7, 8.8, 8.10, 8.12, 8.13, and Lemmas 15.3, 15.4. □

THEOREM 15.6. *Referring to Notation* 12.1*, assume* Φ *is Type III*$^+$*. Then* (i)–(vii) *hold below.*

(i) *Case* (i) *of Theorem* 5.4 *occurs if and only if* $s = 0$, $s^* = 0$ *and* $\tau = 0$.
(ii) *Case* (ii) *of Theorem* 5.4 *occurs if and only if* $s = 0$, $s^* = 0$ *and* $\tau \neq 0$.

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(iii) *Case* (iii) *of Theorem* 5.4 *occurs if and only if* $hs^* \neq h^*s$ *and* $hs^* = -h^*s$.
(iv) *Case* (iv) *of Theorem* 5.4 *occurs if and only if* $hs^* = h^*s$ *and* $hs^* \neq -h^*s$.
(v) *Case* (v) *of Theorem 5.4 occurs if and only if* $s = 0$, $s^* \neq 0$ *and* $\tau = 0$.
(vi) *Case* (vi) *of Theorem* 5.4 *occurs if and only if* $s \neq 0$, $s^* = 0$ *and* $\tau = 0$.
(vii) *Case* (vii) *of Theorem* 5.4 *occurs if and only if* $hs^* \neq h^*s$, $hs^* \neq -h^*s$, *and at least two of* s, s^*, τ *are nonzero.*

Proof. Combine Theorem 5.4 and Proposition 15.5. □

16. Type III$^-$: $q = -1$, Char($\mathbb{K}$) $\neq 2$ and d is odd.

LEMMA 16.1. [25, Theorem 9.1] *Referring to Notation* 12.1, *assume* Φ *is Type III*$^-$. *Then there exists unique scalars* η, h, s, η^*, h^*, s^*, τ *in* $\overline{\mathbb{K}}$ *such that*

$$\theta_i = \begin{cases} \eta + s + h(i - d/2) & \text{if } i \text{ is even}, \\ \eta - s - h(i - d/2) & \text{if } i \text{ is odd}, \end{cases} \tag{16.1}$$

$$\theta^*_i = \begin{cases} \eta^* + s^* + h^*(i - d/2) & \text{if } i \text{ is even}, \\ \eta^* - s^* - h^*(i - d/2) & \text{if } i \text{ is odd} \end{cases} \tag{16.2}$$

for $0 \leq i \leq d$ *and*

$$\varphi_i = \begin{cases} hh^*i(d - i + 1) & \text{if } i \text{ is even}, \\ \tau - 2ss^* + i(d - i + 1)hh^* & \\ \qquad\qquad -2(hs^* + h^*s)(i - (d + 1)/2) & \text{if } i \text{ is odd}, \end{cases} \tag{16.3}$$

$$\phi_i = \begin{cases} hh^*i(d - i + 1) & \text{if } i \text{ is even}, \\ \tau + 2ss^* + i(d - i + 1)hh^* & \\ \qquad\qquad -2(hs^* - h^*s)(i - (d + 1)/2) & \text{if } i \text{ is odd} \end{cases} \tag{16.4}$$

for $1 \leq i \leq d$.

REMARK 16.2. Referring to Lemma 16.1, we have $h \neq 0$; otherwise $\theta_0 = \theta_2$ by (16.1). Similarly we have $h^* \neq 0$. We have $s \neq 0$; otherwise $\theta_0 = \theta_d$ by (16.1). Similarly we have $s^* \neq 0$. For i even with $0 \leq i \leq d - 1$ we have $s \neq ih/2$; otherwise $\theta_{d-i} = \theta_0$. For any prime i such that $i \leq d/2$ we have Char($\mathbb{K}$) $\neq i$; otherwise $\varphi_{2i} = 0$ by (16.3). By this and since Char($\mathbb{K}$) $\neq 2$ we find Char($\mathbb{K}$) is either 0 or an odd prime greater than $d/2$. Observe $d - 1$ does not vanish in $\mathbb{K}$ since otherwise Char($\mathbb{K}$) must divide $(d - 1)/2$.

LEMMA 16.3. *Referring to Notation* 12.1, *assume* Φ *is Type III*$^-$. *Then* (i)–(iv) *hold below.*
(i) $\theta_0 + \theta_d \neq \theta_1 + \theta_{d-1}$.
(ii) $\theta^*_0 + \theta^*_d \neq \theta^*_1 + \theta^*_{d-1}$.
(iii) $(\theta^*_i - \theta^*_0)(\theta_i - \theta_0)^{-1}$ *is independent of* i *for* $1 \leq i \leq d$ *if and only if* $hs^* = h^*s$.
(iv) $(\theta^*_{d-i} - \theta^*_d)(\theta_i - \theta_0)^{-1}$ *is independent of* i *for* $1 \leq i \leq d$ *if and only if* $hs^* = -h^*s$.

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Proof. (i): Using (16.1),

$$\theta_0 + \theta_d - \theta_1 - \theta_{d-1} = -2(d-1)h.$$

The result follows from this and Remark 16.2.

(ii): Similar to the proof of (i).

(iii): Using (16.1) and (16.2),

$$\frac{\theta^*_i - \theta^*_0}{\theta_i - \theta_0} - \frac{\theta^*_d - \theta^*_0}{\theta_d - \theta_0} = \begin{cases} \dfrac{h^*s - hs^*}{hs} & \text{if } i \text{ is even,} \\[2ex] \dfrac{(d-i)(hs^* - h^*s)}{s(2s - h(d-i))} & \text{if } i \text{ is odd} \end{cases}$$

for $1 \le i \le d$. The result follows from this.

(iv): Using (16.1) and (16.2),

$$\frac{\theta^*_{d-i} - \theta^*_d}{\theta_i - \theta_0} - \frac{\theta^*_0 - \theta^*_d}{\theta_d - \theta_0} = \begin{cases} \dfrac{hs^* + h^*s}{hs} & \text{if } i \text{ is even,} \\[2ex] \dfrac{(d-i)(hs^* + h^*s)}{s(-2s + h(d-i))} & \text{if } i \text{ is odd} \end{cases}$$

for $1 \le i \le d$. The result follows from this. ☐

LEMMA 16.4. *Referring to Notation* 12.1*, assume* Φ *is Type III*$^-$*. Then* (i)–(iv) *hold below.*

(i) $\varphi_2 \ne -\phi_2$*. Moreover if* $\varphi_1 = -\phi_1$ *then* $\varphi_d \ne -\phi_d$*.*

(ii) $\varphi_2 \ne -\phi_{d-1}$*. Moreover if* $\varphi_1 = -\phi_d$ *then* $\varphi_d \ne -\phi_1$*.*

(iii) $\phi_i = \phi_{d-i+1}$ *for* $1 \le i \le d$ *if and only if* $hs^* = h^*s$*.*

(iv) $\varphi_i = \varphi_{d-i+1}$ *for* $1 \le i \le d$ *if and only if* $hs^* = -h^*s$*.*

Proof. (i): Using the data in Lemma 16.1 we find

$$\begin{aligned} \varphi_2 + \phi_2 &= 4(d-1)hh^*, \\ \varphi_1 + \phi_1 - \varphi_d - \phi_d &= 4(d-1)hs^*. \end{aligned}$$

The result follows from this and Remark 16.2.

(ii): Using the data in Lemma 16.1 we find

$$\begin{aligned} \varphi_2 + \phi_{d-1} &= 4(d-1)hh^*, \\ \varphi_1 + \phi_d - \varphi_d - \phi_1 &= 4(d-1)h^*s. \end{aligned}$$

The result follows from this and Remark 16.2.

(iii): Using the data in Lemma 16.1 we find

$$\phi_i - \phi_{d-i+1} = \begin{cases} 0 & \text{if } i \text{ is even,} \\ 2(d-2i+1)(hs^* - h^*s) & \text{if } i \text{ is odd} \end{cases}$$

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for $1 \leq i \leq d$. The result follows from this and Remark 16.2.

(iv): Using the data in Lemma 16.1 we find

$$\varphi_i - \varphi_{d-i+1} = \begin{cases} 0 & \text{if } i \text{ is even,} \\ 2(d-2i+1)(hs^* + h^*s) & \text{if } i \text{ is odd} \end{cases}$$

for $1 \leq i \leq d$. The result follows from this and Remark 16.2. □

Proposition 16.5. *Referring to Notation* 12.1*, assume* Φ *is Type III*$^-$*. Then* (i)–(vii) *hold below.*

(i) $\Phi^{\downarrow}$ *is not affine isomorphic to* Φ*.*
(ii) $\Phi^{\Downarrow}$ *is not affine isomorphic to* Φ*.*
(iii) $\Phi^{\downarrow\Downarrow}$ *is not affine isomorphic to* Φ*.*
(iv) Φ^* *is affine isomorphic to* Φ *if and only if* $hs^* = h^*s$*.*
(v) $\Phi^{\downarrow *}$ *is not affine isomorphic to* Φ*.*
(vi) $\Phi^{\Downarrow *}$ *is not affine isomorphic to* Φ*.*
(vii) $\Phi^{\downarrow\Downarrow *}$ *is affine isomorphic to* Φ *if and only if* $hs^* = -h^*s$*.*

Proof. Follows from Propositions 8.4, 8.5, 8.7, 8.8, 8.10, 8.12, 8.13, and Lemmas 16.3, 16.4. □

Theorem 16.6. *Referring to Notation* 12.1*, assume* Φ *is Type III*$^-$*. Then* (i)–(iv) *hold below.*

(i) *Case* (iii) *of Theorem* 5.4 *occurs if and only if* $hs^* = -h^*s$*.*
(ii) *Case* (iv) *of Theorem* 5.4 *occurs if and only if* $hs^* = h^*s$*.*
(iii) *Case* (vii) *of Theorem* 5.4 *occurs if and only if both* $hs^* \neq h^*s$*,* $hs^* \neq -h^*s$*.*
(iv) *Cases* (i)*,* (ii)*,* (v)*,* (vi) *of Theorem* 5.4 *do not occur.*

Proof. Combine Theorem 5.4 and Proposition 16.5. □

17. Type IV: $q = 1$ and Char$(\mathbb{K}) = 2$.

Lemma 17.1. [25, Theorem 10.1] *Referring to Notation* 12.1*, assume* Φ *is Type IV. Then* $d = 3$*. Moreover there exists unique scalars* h*,* s*,* h^**,* s^**,* r *in* $\overline{\mathbb{K}}$ *such that*

$$\begin{array}{lll} \theta_1 = \theta_0 + h(s+1), & \theta_2 = \theta_0 + h, & \theta_3 = \theta_0 + hs, \\ \theta^*_1 = \theta^*_0 + h^*(s^*+1), & \theta^*_2 = \theta^*_0 + h^*, & \theta^*_3 = \theta^*_0 + h^*s^*, \\ \varphi_1 = hh^*r, & \varphi_2 = hh^*, & \varphi_3 = hh^*(r+s+s^*), \\ \phi_1 = hh^*(r + s(1+s^*)), & \phi_2 = hh^*, & \phi_3 = hh^*(r + s^*(1+s)). \end{array}$$

Remark 17.2. Referring to Lemma 17.1, each of h, h^*, s, s^* is nonzero, and each of s, s^* is not equal to 1.

Lemma 17.3. *Referring to Notation* 12.1*, assume* Φ *is Type IV. Then* (i)–(iv) *hold below.*

(i) $\theta_i + \theta_{d-i} = hs$ *for* $0 \leq i \leq d$*.*
(ii) $\theta^*_i + \theta^*_{d-i} = h^*s^*$ *for* $0 \leq i \leq d$*.*
(iii) $(\theta^*_i - \theta^*_0)(\theta_i - \theta_0)^{-1}$ *is independent of* i *for* $1 \leq i \leq d$ *if and only if* $s = s^*$*.*

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(iv) $(\theta^*_{d-i} - \theta^*_d)(\theta_i - \theta_0)^{-1}$ *is independent of* i *for* $1 \le i \le d$ *if and only if* $s = s^*$.

Proof. (i), (ii): Routine verification using the data in Lemma 17.1.
(iii): Using the data in Lemma 17.1 we find

$$\frac{\theta^*_1 - \theta^*_0}{\theta_1 - \theta_0} = \frac{h^*(s^*+1)}{h(s+1)}, \qquad \frac{\theta^*_2 - \theta^*_0}{\theta_2 - \theta_0} = \frac{h^*}{h}, \qquad \frac{\theta^*_3 - \theta^*_0}{\theta_3 - \theta_0} = \frac{h^* s^*}{hs}.$$

The result follows from this and Remark 17.2.
(iv): Using the data in Lemma 17.1 we find

$$\frac{\theta^*_2 - \theta^*_3}{\theta_1 - \theta_0} = \frac{h^*(s^*+1)}{h(s+1)}, \qquad \frac{\theta^*_1 - \theta^*_3}{\theta_2 - \theta_0} = \frac{h^*}{h}, \qquad \frac{\theta^*_0 - \theta^*_3}{\theta_3 - \theta_0} = \frac{h^* s^*}{hs}.$$

The result follows from this and Remark 17.2. □

Lemma 17.4. *Referring to Notation* 12.1*, assume* Φ *is Type IV. Then* (i)–(iv) *hold below.*
(i) $\varphi_1 \ne -\phi_1$.
(ii) $\varphi_1 \ne -\phi_3$.
(iii) $\phi_1 = \phi_3$ *if and only if* $s = s^*$.
(iv) $\varphi_1 = \varphi_3$ *if and only if* $s = s^*$.

Proof. Using the data in Lemma 17.1 we find

$$\begin{aligned} \varphi_1 + \phi_1 &= hh^* s(s^*+1), & \qquad \varphi_1 + \phi_3 &= hh^* s^*(s+1), \\ \phi_1 - \phi_3 &= hh^*(s+s^*), & \qquad \varphi_1 - \varphi_3 &= hh^*(s+s^*). \end{aligned}$$

Now (i)–(iv) follow from this and Remark 17.2. □

Proposition 17.5. *Referring to Notation* 12.1*, assume* Φ *is Type IV. Then* (i)–(vii) *hold below.*
(i) $\Phi^{\downarrow}$ *is not affine isomorphic to* Φ.
(ii) $\Phi^{\Downarrow}$ *is not affine isomorphic to* Φ.
(iii) $\Phi^{\downarrow\Downarrow}$ *is affine isomorphic to* Φ *if and only if* $s = s^*$.
(iv) Φ^* *is affine isomorphic to* Φ *if and only if* $s = s^*$.
(v) $\Phi^{\downarrow *}$ *is not affine isomorphic to* Φ.
(vi) $\Phi^{\downarrow *}$ *is not affine isomorphic to* Φ.
(vii) $\Phi^{\downarrow\Downarrow *}$ *is affine isomorphic to* Φ *if and only if* $s = s^*$.

Proof. Follows from Propositions 8.4, 8.5, 8.7, 8.8, 8.10, 8.12, 8.13, and Lemmas 17.3, 17.4. □

Theorem 17.6. *Referring to Notation* 12.1*, assume* Φ *is Type IV. Then* (i)–(iii) *hold below.*
(i) *Case* (ii) *of Theorem* 5.4 *occurs if and only if* $s = s^*$.
(ii) *Case* (vii) *of Theorem* 5.4 *occurs if and only if* $s \ne s^*$.
(iii) *Cases* (i)*,* (iii)–(vi) *of Theorem* 5.4 *do not occur.*

Proof. Combine Theorem 5.4 and Proposition 17.5. □

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