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THE WEAK HAWKINS-SIMON CONDITION*

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Abstract. A real square matrix satisfies the weak Hawkins-Simon condition if its leading principal minors are positive (the condition was first studied by the French mathematician Maurice Potron). Three characterizations are given. Simple sufficient conditions ensure that the condition holds after a suitable reordering of columns. A full characterization of this set of matrices should take into account the group of transforms which leave it invariant. A simple algorithm able, in some cases, to implement a suitable permutation of columns is also studied. The nonsingular Stiemke matrices satisfy the WHS condition after reorderings of both rows and columns.

Key words. Hawkins-Simon condition, Linear complementarity problem, LU factorization, Potron, Stiemke matrix.

AMS subject classifications. 15A15, 15A48.

1. Introduction.

A real square matrix is said to satisfy the weak Hawkins-Simon [8] criterion, or to be of the WHS type, if all its *leading* principal minors are positive. When the off-diagonal coefficients are nonpositive, the condition characterizes the semipositivity of the inverse matrix. With no assumption on the signs of the off-diagonal coefficients, three characterizations of the WHS property are given (section 3). Fujimoto and Ranade [6] have recently considered matrices which are of the WHS type after a suitable reordering of columns (these matrices are said to be of the FR type) and shown that an inverse-semipositive matrix has this property. This result is generalized and we show that, since the FR family of matrices is invariant by a group of transforms, the identification of the FR matrices should take into account the associated group (section 4). We define a simple algorithm for reordering the columns of a matrix and wonder when it allows us to find a relevant permutation of columns (section 5). We also consider the case when reorderings of rows and columns are both allowed (section 6). Finally, a historical note does justice to Maurice Potron, an unknown pioneer of the so-called Hawkins-Simon properties (section 7).

2. Generalities.

Let A be a real square $n \times n$ matrix. A is said to be inverse-(semi-) positive if it is non singular and A^{-1} is (semi-) positive. A tilde on a real vector x or a real square matrix denotes transposition. Notations $x \geqq 0$ (or $x \in R^n_+$), $x \geq 0, x > 0$ (or $x \in R^n_{++}$) mean respectively that vector x is nonnegative, semipositive or positive. A bar on a vector or a matrix either denotes truncation of the last components, or suggests a vocation to further extension; a double bar denotes truncation of the first components.

An LU factorization of A is a decomposition $A = LU$, where L is a lower triangular matrix with unit diagonal entries, and $U$is an upper triangular matrix. It is well known (and this results from the ensuing calculations) that such a factorization

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exists when all the leading principal minors ('leading minors', for short) are nonzero and, then, the factorization is unique (Berman and Plemmons, [1]).

We shall consider a classical transform of the system of equations $Ax = y$:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n &= y_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n &= y_2 \\ &... \\ a_{n1}x_1 + a_{n2}x_2 + ... + a_{nn}x_n &= y_n. \end{aligned} \tag{2.1}$$

If $a_{11} \neq 0$, the first equality can be used to eliminate x_1 from the other equations. This section is mainly devoted to the properties of the transformed system and its associated $(n-1)\times(n-1)$ matrix S_1, more generally to those of the $(n-k)\times(n-k)$ matrix S_k obtained after the successive eliminations of $x_1, ..., x_k$.

A fruitful interpretation of the elimination of x_1 is to consider that we have premultiplied both members of the equality $Ax = y$ by the lower triangular matrix

$$L_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -a_{21}/a_{11} & 1 & 0 & 0 \\ ... & 0 & \ddots & 0 \\ -a_{n1}/a_{11} & 0 & 0 & 1 \end{bmatrix}. \tag{2.2}$$

In L_1A, the first row coincides with that of A and the entries 2 to n of the first column are zero. Let us denote $\Delta_{1i1j} = a_{11}a_{ij} - a_{i1}a_{1j}$ the 2×2 minor extracted from rows 1 and i and columns 1 and j of A. The $(n-1)\times(n-1)$ sub-matrix S_1 made of rows and columns 2 to n of L_1A is writtenas

$$S_1 = \begin{bmatrix} \Delta_{1212}/a_{11} & ... & \Delta_{121n}/a_{11} \\ ... & ... & ... \\ \Delta_{1n12}/a_{11} & ... & \Delta_{1n1n}/a_{11} \end{bmatrix}. \tag{2.3}$$

$S_1 = S_1(A)$ is called the Schur complement of a_{11}. The initial system of equations (2.1) is transformed into the equivalent system

$$a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = y_1 \tag{2.4}$$

and

$$S_1 \begin{pmatrix} x_2 \\ ... \\ x_n \end{pmatrix} = \begin{pmatrix} y_2 \\ ... \\ y_n \end{pmatrix} + y_1 \begin{pmatrix} -a_{21}/a_{11} \\ ... \\ -a_{n1}/a_{11} \end{pmatrix}. \tag{2.5}$$

The $n-1$ equations (2.5) are written more compactly as

$$S_1 \overline{\overline{x}}_{(1)} = \overline{\overline{y}}_{(1)} + y_1 \overline{\overline{l}}_{(1)}, \tag{2.6}$$

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where $\overline{\overline{x}}_{(1)}$ (respectively $\overline{\overline{y}}_{(1)}$) denotes the vector x (respectively, y) truncated of its first component, and $\overline{\overline{l}}_{(1)}$ the column-vector made of the last $n-1$ components of the first column of L_1.

LEMMA 2.1. *Let A be a nonsingular matrix such that a_{11} is nonzero. Then:*

- the leading minor of order k of A is equal to a_{11} times the leading minor of order $k-1$ of S_1 ($k = 2, ..., n$),

- the cth column of S_1^{-1} is obtained by deleting the first component of the $(c+1)$-th column of A^{-1}.

Proof. Because of the structure of matrix L_1, the leading minors of order k in A and L_1A are equal and, because of the structure of L_1A, this minor is a_{11} times the leading minor of order $k-1$ of S_1. Hence, the first statement follows.

Consider the solutions to $S_1\overline{\overline{x}} = \overline{\overline{e}}_c$, where $\overline{\overline{x}}$ and $\overline{\overline{e}}_c$ are vectors in R^{n-1}, $\overline{\overline{e}}_c$ being the cth unit vector. Let us extend $\overline{\overline{e}}_c$ into the $(c+1)$-th unit vector e_{c+1} of R^n by inserting a first component equal to zero. Relation $Ax = e_{c+1}$ is of the type (2.1) with $y = e_{c+1}$, therefore equality (2.6) holds with $\overline{\overline{x}}_{(1)} = \overline{\overline{x}}, \overline{\overline{y}}_{(1)} = \overline{\overline{e}}_c$ and $y_1 = 0$ and is reduced to $S_1\overline{\overline{x}} = \overline{\overline{e}}_c$. Therefore the solution $\overline{\overline{x}}$ to $S_1\overline{\overline{x}} = \overline{\overline{e}}_c$ derives from the solution x to $Ax = e_{c+1}$ by deleting the first component. As $\overline{\overline{x}} = S_1^{-1}\overline{\overline{e}}_c$ is the cth column of S_1^{-1}, and $x = A^{-1}e_{c+1}$ the $(c+1)$-th column of A^{-1}, we obtain the result. ◻

This transform is but the first step of an LU decomposition of the initial matrix A: the successive elimination of variables $x_1, ..., x_k$ from the first k equations is possible if the leading minors of A up to order k are nonzero. The operation amounts to premultiplying both members of the equality $Ax = y$ by some lower triangular matrix $L_{(k)} = L_k \cdots L_1$, with nonzero off-diagonal entries only in the first k columns. The system $Ax = y$ is then equivalent to a system written in two parts: in the first k equations, the jth equation ($j = 1, ..., k$) is written

$$u_{jj}x_j + u_{j,j+1}x_{j+1} + ... + u_{jn}x_n = y_j + l'_j(y_1, ..., y_{j-1}), \tag{2.7}$$

where $u_{1j} = a_{1j}$ and $l'_j(y_1, ..., y_{j-1})$ denotes some linear combination of $(y_1, ..., y_{j-1})$; the last $n-k$ equations are written in the matricial form

$$S_k\overline{\overline{x}}_{(k)} = \overline{\overline{y}}_{(k)} + \sum_{j=1}^{k} y_j\overline{\overline{l}}_{(k)j}, \tag{2.8}$$

where $S_k = S_k(A)$, the Schur complement of the leading minor of order k, is a square matrix of dimension $n-k$, $\overline{\overline{x}}_{(k)}$ and $\overline{\overline{y}}_{(k)}$ are the vectors x and y reduced to their last $n-k$ components, and $\overline{\overline{l}}_{(k)j}$ is the jth column of $L_{(k)}$ reduced to its last $n-k$ components. Clearly,

$$S_k(A) = S_1[S_1...S_1(A)]. \tag{2.9}$$

The step $k = n$ can be reached if all the leading minors of A are nonzero. The initial system $Ax = y$ is then transformed into an equivalent system in which the generic

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equation j is of the type (2.7) for $j = 1, ..., n$. This final system is written $Ux = L'y$, where U is an upper triangular matrix and L' a lower triangular matrix with 1s on the diagonal. The equivalence implies the matricial equality $A = LU$, where $L = L'^{-1}$ is a matrix of the same type as L'.

LEMMA 2.2. *Let A be a WHS matrix. Then A admits an LU decomposition and, for any k:*

- S_k is a WHS matrix,

- if A is inverse- (semi-) positive, so is S_k,

- if the last column of A^{-1} is (semi-) positive, so is the last column of S_k^{-1}.

Proof. These properties follow by induction from Lemma 2.1 and (2.9). □

3. Three characterizations of WHS. A first characterization of a WHS matrix is well known (Berman and Plemmons, [1]) but we remind the reader of the argument for the historical reasons detailed in section 7.

THEOREM 3.1. *A is a WHS matrix if and only if it admits a factorization $A = LU$, where L is a lower triangular matrix with unit diagonal entries and U is an upper triangular matrix with positive diagonal entries.*

Proof. A factorization $A = LU$ exists if all the principal minors are nonzero. By considering the first k rows and columns in the equality $A = LU$, it turns out that the successive diagonal elements of U are $u_{11} = a_{11}$, then the ratio of two consecutive leading minors of A. Therefore, A is of the WHS type if and only if the diagonal elements of U are positive. □

The next two characterizations of WHS matrices refer to systems of equations: Theorem 3.2 considers the system $Ax = y$, and Theorem 3.5 a linear complementarity problem. Let us consider the set

$$E_k = \{(x, y); (x, y) \neq (0, 0), Ax = y, y_1 = ... = y_{k-1} = 0 = x_{k+1} = ... = x_n\}. \tag{3.1}$$

THEOREM 3.2. *A is a WHS matrix if and only if the implication*

$$(x, y) \in E_k \Rightarrow x_k y_k > 0 \tag{3.2}$$

holds for any $k = 1, ..., n$.

Proof. Assume first that the k-th leading minor of A is zero: $\det \overline{A} = 0$. Then there exists a nonzero vector $\overline{x}$ of dimension k such that $\overline{A}\overline{x} = 0$. Let x be the vector $\overline{x}$ completed by $n-k$ zeroes, and $y = Ax$. Then $(x, y) \in E_k$ and $y_k = 0$, therefore the implication (3.2) does not hold.

On the contrary, if the leading minors of A are nonzero, matrix A admits an LU factorization. The system $LUx = y$ is equivalently written $Ux = L^{-1}y$. For $(x, y) \in E_k$, the k-th equation is reduced to $u_{kk}x_k = y_k$, therefore property (3.2) amounts to stating that the diagonal elements of U are positive. By Theorem 3.1, implication (3.2) holds if and only if A has the WHS property. □

DEFINITION 3.3. *(w, z) is said to be a simple solution to the linear complementarity problem LCP(q, A)*

$$w = Az + q \quad w \geqq 0, z \geqq 0, \widetilde{w}z = 0 \tag{3.3}$$

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if, for some minimal integer h *(*$h \in [0, n]$ *is called the height of the solution), the first* h *components of* w *and the last* $n-h$ *components of* z *are zero.*

After deletion of the last $n-h$ components of both w and z, the truncated vectors are such that $\overline{w} = 0$ and, by the minimality hypothesis, the last component of $\overline{z}$ is positive.

If A has the WHS property, $LCP(q, A)$ may have several simple solutions: for instance, for

$$q = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, A = \begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix}$$

two simple solutions, with respective heights $h_1 = 0$ and $h_2 = 2$, are

$$w_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, z_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}; w_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, z_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The question examined below is whether it is possible to have $h_2 - h_1 = 0$ or 1.

LEMMA 3.4. *The following properties are equivalent:*
- the leading principal minors of A *are all nonzero,*
- for any q*, two simple solutions of* $LCP\,(q, A)$ *have different heights.*

Proof. If the leading minor of order h of A is zero ($\det \overline{A} = 0$), let $\overline{z}_1$ be a positive vector of dimension h and $\overline{q} = -\overline{A}\overline{z}_1$. The problem $LCP\left(\overline{q}, \overline{A}\right)$ admits two solutions $(\overline{w}_1 = 0, \overline{z}_1)$ and $(\overline{w}_2 = 0, \overline{z}_2)$ of the same height h, where $\overline{z}_1$ and $\overline{z}_2$ are both positive and $\overline{A}(\overline{z}_2 - \overline{z}_1) = 0$. For $i = 1, 2$, these solutions are extended to non truncated vectors (q, z, w) by completing the last $n-h$ components of q by positive and large enough scalars, the last $n-h$ components of z_i by zeroes, and the last $n-h$ components of w_i by the corresponding positive components of $Az_i + q$. Two simple solutions (w_1, z_1) and (w_2, z_2) of $LCP\,(q, A)$ are thus obtained, with a common height h.

Conversely, let the leading minors of A be nonzero and consider two simple solutions to $LCP\,(q, A)$ with a common height h. Delete the last $n-h$ components of z_1, z_2, w_1, w_2, as well as the last $n-h$ rows and columns of A. The truncated vectors $(\overline{w}_1, \overline{z}_1)$ and $(\overline{w}_2, \overline{z}_2)$ are simple solutions to $LCP\left(\overline{q}, \overline{A}\right)$. As $\overline{w}_1 = \overline{w}_2 = 0$ and the solution $\overline{z}$ to $0 = \overline{A}\overline{z} + \overline{q}$ is unique (invertibility of $\overline{A}$), we have $\overline{z}_1 = \overline{z}_2$, hence $z_1 = z_2$ and the two solutions coincide. ∎

THEOREM 3.5. *Matrix* A *has the WHS property if and only if, for any* q*, the problem* $LCP\,(q, A)$ *does not admit simple neighboring solutions, i.e. with heights differing by zero or one.*

Proof. Let A admit the WHS property and consider two simple solutions (w_1, z_1) and (w_2, z_2) of $LCP\,(q, A)$, with $h_2 > h_1$ (equality $h_1 = h_2$ is excluded by Lemma 3.4). After truncation of the last $n - h_2$ components, we have $\overline{w}_2 = 0$, the h_2-th component of $\overline{z}_2$ is positive, the first h_1 components of $\overline{w}_1$ are zeroes, and the h_2-th component of $\overline{z}_1$ is zero. $\overline{A}$ admits the decomposition $\overline{A} = LU$, U with a positive diagonal. Set $v_1 = L^{-1}\overline{w}_1$ and $p = L^{-1}\overline{q}$. From equalities $\overline{w}_i = \overline{A}\overline{z}_i + \overline{q} = LU\overline{z}_i + \overline{q}$ for $i = 1, 2$, there follows, after pre-multiplication by L^{-1},

$$v_1 = U\overline{z}_1 + p \tag{3.4}$$

$$0 = U\overline{z}_2 + p. \tag{3.5}$$

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Consider the (h_1+1)-th component in the vector equality (3.4). Since the first h_1 components of $\overline{w}_1$ are zeroes and $(L^{-1})_{h_1+1,h_1+1} = 1$, the (h_1+1)-th component of $\overline{w}_1$ coincides with that of $L^{-1}\overline{w}_1 = v_1$. Therefore, in the left-hand side, $(v_1)_{h_1+1}$ is a nonnegative scalar. In the right-hand side, the last h_2-h_1 components of $\overline{z}_1$ are zeroes, therefore the same properties holds for the last h_2-h_1 components of $U\overline{z}_1$, including $(U\overline{z}_1)_{h_1+1}$. We conclude that equality (3.4) implies that the (h_1+1)-th component of p is nonnegative. Similarly, consider the h_2-th component in the vector equality (3.5). It follows from the structure of U and $\overline{z}_2$ that the h_2-th component of $U\overline{z}_2$ is $u_{h_2,h_2}(\overline{z}_2)_{h_2} > 0$, therefore, from (3.5), the h_2-th component of p is negative. The overall conclusion is that $h_2 \neq h_1+1$, i.e. two simple solutions are not neighboring.

Conversely, assume that all the leading minors of matrix A are nonzero (otherwise, Lemma 3.4 applies), where the first h_1 minors are positive and the next one negative. Let $\overline{A}$ be the submatrix made of the first $h_2 = h_1+1$ rows and columns of A. The following construction defines a vector $\overline{q}$ of dimension h_2 such that the problem $LCP(\overline{q}, \overline{A})$ admits two solutions of heights h_2 and h_1, then extend these solutions to simple neighboring solutions to $LCP(q, A)$ for a certain vector q.

In the factorization $\overline{A} = LU$, we have $u_{ii} > 0$ for $i = 1, ..., h_1$ and $u_{h_2h_2} < 0$. For a given positive vector $\overline{z}_2$ of dimension h_2, we define successively the vector p by the equality (3.5), then the vector v_1 by $(v_1)_i = 0$ for $i = 1, ..., h_1$ and $(v_1)_{h_2} = (p)_{h_2} = -(U\overline{z}_2)_{h_2} > 0$, then the vector $\overline{z}_1$ by the equality (3.4). According to (3.4), the last component $(\overline{z}_1)_{h_2}$ of $\overline{z}_1$ is such that $(v_1)_{h_2} = u_{h_2h_2}(\overline{z}_1)_{h_2} + (p)_{h_2}$, therefore $(\overline{z}_1)_{h_2} = 0$ and vector $\overline{z}_1$ is orthogonal to v_1. Let e be the last unit vector of R^{h_2} and δ the last column of U^{-1}. By subtraction of equalities (3.4) and (3.5), we obtain $U(\overline{z}_1 - \overline{z}_2) = v_1 = (v_1)_{h_2}e$, therefore $\overline{z}_1 - \overline{z}_2 = (v_1)_{h_2}U^{-1}e = (v_1)_{h_2}\delta$. That is, $\overline{z}_1$ is obtained by adding to $\overline{z}_2$ a vector proportional to the last column of U^{-1}, in such a way that the last component of $\overline{z}_1$ is zero. Clearly, the positive vector $\overline{z}_2$, which has been chosen arbitrarily at the beginning of the construction, can be chosen in order that the other components of $\overline{z}_1$ are positive. Then, $(v_1, \overline{z}_1)$ and $(0, \overline{z}_2)$ are simple solutions to $LCP\,(p, U)$, with respective heights $h_1 = h_2 - 1$ and h_2.

Next, we define $\overline{w}_1 = Lv_1 = v_1, \overline{w}_2 = 0$ and $\overline{q} = Lp$. $(\overline{w}_1, \overline{z}_1)$ and $(\overline{w}_2, \overline{z}_2)$ are simple neighboring solutions to $LCP\left(\overline{q}, \overline{A}\right)$. Finally, let z_i be the vector $\overline{z}_i$ completed by $n - h_2$ zeros, q the vector $\overline{q}$ completed by positive and large enough components, and w_i $(i = 1, 2)$ the vector defined by equality $w_i = Az_i + q$. Then (w_1, z_1) and (w_2, z_2) are simple neighboring solutions to $LCP\,(q, A)$. ☐

Finally, a necessary condition is:

Theorem 3.6. *A WHS matrix A preserves the sign of some vector:*

$$\exists x \ \ \forall i \ \ x_i(Ax)_i > 0. \tag{3.6}$$

Proof. The proof is by induction on the dimension n of A. The result holds for $n = 1$. Let S_1 be the Schur complement of a_{11}. Since S_1 is a WHS matrix, the induction hypothesis implies the existence of $(n-1)$-column vectors $\overline{\overline{x}} = (x_2, ..., x_n)$ and $\overline{\overline{y}} = (y_2, ..., y_n)$ such that $\overline{\overline{y}} = S_1\overline{\overline{x}}$ and $x_iy_i > 0$ for $i = 2, ..., n$. These inequalities still hold in a neighborhood of $(\overline{\overline{x}}, \overline{\overline{y}})$ and, in particular, we can assume $a_{12}x_2 + ... + a_{1n}x_n \neq 0$ (except in the degenerate case $a_{12} = ... = a_{1n} = 0$ which can be

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studied separately). In that neighborhood, let us consider the vector $\overline{\overline{z}}$, whose i-th component is $y_{i+1} + \varepsilon(-a_{i+1,1}/a_{11})$, and the vector $\overline{\overline{x}}' = S_1^{-1}\overline{\overline{z}}$, whose components are denoted by $(x'_2, x'_3, \dots, x'_n)$. By construction, equality (2.5) holds for the $(n-1)$-vectors $\overline{\overline{x}}'$ and $\overline{\overline{y}}'$ and the scalar $y_1 = \varepsilon$. Let us define the scalar x'_1 by $a_{11}x'_1 = \varepsilon - a_{12}x'_2 - ... - a_{1n}x'_n \neq 0$, so that both equalities (2.4) and (2.5) hold for the n-vectors $x' = (x'_1, \overline{\overline{x}}')$ and $y' = (y'_1, \overline{\overline{y}})$, therefore $y' = Ax'$. We have $x'_i y'_i > 0$ for $i = 2, ..., n$ (continuity argument). As for the first components $(x'_1, y'_1 = \varepsilon)$, we choose ε small enough and such that $sign(\varepsilon) = sign(-a_{12}x_2 - ... - a_{1n}x_n)$, therefore $sign(x'_1) = sign(a_{11}x'_1) = sign(\varepsilon - a_{12}x'_2 - ... - a_{1n}x'_n) = sign(-a_{12}x_2 - ... - a_{1n}x_n)$ (this last equality by a continuity argument), hence $sign(x'_1) = sign(\varepsilon) = sign(y'_1)$. Sum, the n-vectors x' and y' are such that $x'_i y'_i > 0$ for any i and $y' = Ax'$, so that the n-vector x' is a solution to (3.6). □

4. WHS after reordering of columns. This section and the next are devoted to the study of a class of matrices introduced by Fujimoto and Ranade [6].

DEFINITION 4.1. *A square matrix is said to be of the FR type if it becomes of the WHS type after a suitable reordering ('permutation') of columns or, equivalently, if it is written as the product of a WHS matrix and a permutation matrix.*

Fujimoto and Ranade's Theorem 3.1 states that an inverse-semipositive matrix (that they call inverse-positive matrix) is of the FR type. The criterion considered in the following statement only refers to the last column of A^{-1}.

THEOREM 4.2. *Let A be a nonsingular matrix. If the last column of A^{-1} is positive (or if A^{-1} is semipositive), A is of the FR type.*

Proof. The proof is by induction on the dimension n of A. The result holds for $n = 1$, and we assume it for dimension $n-1$. If the last column of A^{-1} is positive, at least one element in the first row of A is positive. A permutation of columns moves it to position a_{11} (matrix A_1 is obtained). Since the rows of the inverse matrix are permuted, the last column of A_1^{-1} remains positive. Let us premultiply A_1 by the matrix L_1 defined by (2.2). The matrix S_1 defined in (2.3) appears. By the second assertion of Lemma 2.1, the last column of S_1 is positive and, by the induction hypothesis, the columns of S_1 can be reordered in such a way that the matrix becomes of the WHS type. By the first assertion of Lemma 2.1, the same reordering of columns 2 to n of A_1 transforms the initial matrix into a WHS matrix, hence the result follows.

Fujimoto and Ranade's result can be obtained by replacing everywhere, in the above argument, the positivity hypothesis on the last column of A^{-1} by the semipositivity hypothesis onthe matrix A^{-1}. □

A permutation matrix, denoted P_i $(i = 0, 1, ...)$, admits one entry equal to 1 in every row and every column, and 0s elsewhere so that no two 1's occupy the same row or column. Permutation matrices form a subgroup of themuliplicative group of orthogonal matrices $(\widetilde{P} = P^{-1})$. Pre-multiplying (resp. post-multiplying) a matrix by P_i amounts to reordering its rows (resp. columns). Let P_0 be the permutation matrix with 1s on the anti-diagonal $(\widetilde{P_0} = P_0^{-1} = P_0)$: pre- and post-multiplying by P_0 moves the i-th row and the i-th column of A to row and column $n+1-i$. A matrix is called an inverse-WHS matrix if the principal minors made up of the *last* k rows and columns are positive, for $k = 1, ..., n$. Combining pre- and post-multiplication by

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P_0 transforms an inverse-WHS matrix into a WHS matrix, and vice-versa. The WHS and the inverse-WHS properties are stable under transposition. As the decomposition $A = LU$ implies $A^{-1} = U^{-1}L^{-1}$, it turns out, by calculating the minor made up of the last k rows and columns of A^{-1} and using Theorem 3.1, it follows that the inverse of a WHS matrix is an inverse-WHS matrix, and vice-versa (the property also results from the Jacobi equality).

Let H denote a WHS matrix. The matrix $P_0H^{-1}P_0$ is a WHS matrix, and the same for $P_0\widetilde{H}^{-1}P_0$. An FR matrix is written $F = HP$, where H is a WHS matrix and P a permutation matrix. Equality $P_0\widetilde{F}^{-1} = P_0\widetilde{H}^{-1}P = (P_0\widetilde{H}^{-1}P_0)(P_0P)$ shows that $\gamma(F) = P_0\widetilde{F}^{-1}$ is also an FR matrix.

The following statement extends Theorem 4.2 to semipositivity hypotheses (Fujimoto and Ranade, [7]) and, more importantly, makes use of the transform γ to state a simple result based on matrix A itself: the second statement includes the cases where the first row of A is positive, or matrix A itself is semipositive.

THEOREM 4.3. *Let A be a nonsingular matrix. Then A is of the FR type if one of the following two sufficient conditions are met:*

- the last nonzero element in every row of A^{-1} is positive,

- the first nonzero element in every column of A is positive.

If the last column of A^{-1} or the first row of A is semipositive, the columns of A can be reordered in such a way that the leading minors are all positive or null.

Proof. Under the first hypothesis, any column j of A^{-1} can be transformed into a semipositive column by adding to it some positive combination of columns $j+1$ to n. The operation amounts to post-multiplying A^{-1} by some lower triangular matrix L with 1s on the diagonal. As matrix $L^{-1}A$ admits a semipositive inverse, Theorem 4.2 applies to it, and we can therefore write an equality $L^{-1}A = HP$, hence $A = (LH)P$. Since the leading minors of LH coincide with those of H, LH is a WHS matrix and A is an FR matrix.

Under the second hypothesis, let $B = \gamma(A) = P_0\widetilde{A}^{-1}$. Since the $(n+1-j)$-th column of $B^{-1} = \widetilde{A}P_0$ is the jth row of A, the last nonzero element in every row of B^{-1} is positive. According to the above result, B is of the FR type. Hence, the same for $\gamma(B) = P_0\widetilde{B}^{-1} = A$.

Under the final hypotheses, A can be approximated by a sequence of matrices to which the previous results apply. Therefore A is the limit of an infinite sequence of FR matrices. As the number of permutation matrices is finite, there exists a permutation matrix P and a subsequence of WHS matrices H_t such that $A = \lim H_tP$. Hence, the property follows. □

If either the last column of A^{-1} or the first row of A has no positive entry (the polar hypotheses symmetrical to those retained in Theorems 4.2 and 4.3), A cannot be of the FR type. If the last column of A^{-1} (respectively, the first row of A) is semipositive instead of positive, A is not necessarily of the FR type, as shown by the example

$$A = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \text{ with } A^{-1} = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$$

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(the argument which does not extend in the proof of Theorem 4.2 is that, if the last column of A^{-1} is semipositive, it is not guaranteed that the first row of A admits a positive entry).

To take a more abstract view of some arguments used in the proofs of Theorems 4.2 and 4.3, an interpretation in terms of a group of transforms is useful. The basic idea is that the family $\mathcal{F}$ of the FR matrices $F = HP$ is stable under three types of transforms:

- transform α is the pre-multiplication by a lower triangular matrix with a positive diagonal: $\alpha_L(F) = LF$ (this set $\mathcal{L}$ of matrices is a subgroup under multiplication of matrices);

- transform β is the post-multiplication by a permutation matrix or, more generally, by a matrix Q with one positive element in every row and every column and zeroes elsewhere: $\beta_Q(F) = FQ$ (this set $\mathcal{Q}$ of matrices is a subgroup for the multiplication of matrices);

- transform γ is the involutive transform: $\gamma(F) = P_0\widetilde{F}^{-1}$.

Any combination of operations of the types α, β or γ transforms a matrix in $\mathcal{F}$ into another matrix in the same family: in other words, $\mathcal{F}$ is stable by the group $\mathcal{G}$ of transforms generated by these operations. It is therefore natural to study the group $\mathcal{G}$ and the set $\mathcal{G}(M)$, called the orbit of M.

THEOREM 4.4. *Let M be a nonsingular square matrix. The orbit $\mathcal{G}(M)$ is the set of matrices N which are written either $N = LMQ$ (L lower triangular matrix with a positive diagonal, Q with one positive element in every row and column and zeroes elsewhere) or $N = L'\widetilde{M}^{-1}Q$ (L' with a positive anti-diagonal and zeroes above it). If M is of the FR type, so is any matrix N in $\mathcal{G}(M)$.*

Proof. By definition, $\mathcal{G}(M)$ is the set of matrices which can be written as $N = \Delta(M)$, where $\Delta = \delta_m \circ \delta_{m-1} \circ ... \circ \delta_1$ for some natural integer m, δ_i being any of the transforms α, β or γ. For an invertible matrix R, we have $\gamma \circ \alpha_L(R) = P_0\widetilde{(LR)}^{-1} = P_0\widetilde{L}^{-1}\widetilde{R}^{-1} = (P_0\widetilde{L}^{-1}P_0)(P_0\widetilde{R}^{-1}) = \alpha_{L_1} \circ \gamma(R)$, with $L_1 = P_0\widetilde{L}^{-1}P_0 \in \mathcal{L}$. Therefore the identity $\gamma \circ \alpha_L = \alpha_{L_1} \circ \gamma$ holds. Similarly, $\gamma \circ \beta_Q(R) = P_0\widetilde{(RQ)}^{-1} = P_0\widetilde{R}^{-1}\widetilde{Q}^{-1} = \beta_{Q_1} \circ \gamma(R)$ with $Q_1 = \widetilde{Q}^{-1} \in \mathcal{Q}$, hence the identity $\gamma \circ \beta_Q = \beta_{Q_1} \circ \gamma$. Moreover, a transform of type α commutes with a transform of type β. These properties imply that, in the sequence of transforms defining Δ, the transforms of type α can be written first, then those of type β, finally the transforms γ. As the product of α-transforms is an α-transform, a product of β-transforms is a β-transform, and γ is involutive ($\gamma \circ \gamma = Id$), Δ is reduced to either $\Delta = \alpha_L \circ \beta_Q$ or $\Delta = \alpha_L \circ \beta_Q \circ \gamma$ for some adequate matrices L and Q. Therefore, N is written as $N = LMQ$, or as $N = LP_0\widetilde{M}^{-1}Q = L'\widetilde{M}^{-1}Q$. □

The usefulness of the approach in terms of transformation groups is illustrated by the following alternative proof of Theorem 4.2, once Fujimoto and Ranade's initial result is admitted. Their result can be stated as: the set $\mathcal{F}_1$ of inverse-semipositive matrices belongs to $\mathcal{F}$. It follows from Theorem 4.4 that $\mathcal{F}$ also contains the matrices of the type $N = LM$, with $M \in \mathcal{F}_1$, i.e. the nonsingular matrices N such that $N^{-1}L$ is semipositive for some lower triangular matrix $L \in \mathcal{L}$. It is easily seen that this property holds as soon as the last column of N^{-1} is positive (choose positive and

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large enough coefficients in the last row of L), hence Theorem 4.2; similarly, Theorem 4.3 results from the application of transform γ. In technical terms, Theorems 4.2 and 4.3 may be viewed as the completion of Fujimoto and Ranade's initial result by the group $\mathcal{G}$.

Theorems 4.2 and 4.3 state sufficient criteria for a matrix to be of the FR type. These properties are not necessary: for $n = 3$, matrix

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

and any matrix close to it (the zeroes are inessential) is of the WHS type, but both the first row of A and the last column of A^{-1} have negative elements. The lesson is that a further extension of the above results requires identifying another subset $\mathcal{F}_2$ for which the FR property also holds. Then the property will automatically hold for its completion $\mathcal{G}(\mathcal{F}_2)$.

5. The simple algorithm. Beyond the existence results, the question examined here concerns the effective determination of a reordering of columns which transforms an FR matrix into a WHS matrix (if one knows that the initial matrix is indeed of the FR type) or the determination of the type of the matrix (if it is *a priori* unknown): how can we find a suitable permutation, or identify the type, without having to check each of the possible $n!$ substitutions of columns? We do not know a general answer to the question, but we define a specific algorithm and study its convergence.

The *simple algorithm,* applied to a given square matrix A_0, is defined as follows: in the first row of A_0, pick any positive element a_{1j} (for instance, the one for which j is minimum) and permute the j-th column with the first. Matrix A_1 is obtained, and the choice of the first column is definitive. Next, we look for a column j ($j \geq 2$) such that the 2×2 minor Δ_{121j} is positive. Once such a column is found, we permute it with the second column of A_1 and obtain a new matrix A_2. The choice of the second column is definitive. At step k, the first $k-1$ columns of A_{k-1} are given, we look for a new column which gives us a positive $k \times k$ leading minor and put it in the k-th position in A_k; and so forth until $k = n$, when the algorithm stops (in fact, it stops at step $n-1$, as no choice remains for the last column).

Starting from an arbitrary matrix, the simple algorithm fails if, at some step, it is impossible to complete the actual $(k-1) \times (k-1)$ leading minor and obtain a positive $k \times k$ leading minor. But there are two possible causes of failure: (i) it may be the case that A_0 is not of the FR type, or (ii) though A_0 is of the FR type, it is the algorithm itself which goes in a wrong direction, as it is the case for

$$A_0 = \begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix}.$$

At the first step (if one follows the $\min j$ rule), the first column remains in its position, hence $A_1 = A_0$ and the reordering is finished, but $\det(A_1) < 0$. However, the initial matrix is of the FR type (permute the two columns). The example shows the difference

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between the existence result and the successfulness of the simple algorithm, hence the question: for which type(s) of FR matrices are we sure that the simple algorithm ends up with a suitable reordering of columns? By 'success' of the simple algorithm, we mean that it ends up with a convenient reordering of columns, independent of the secondary rule relative to the choice of the new column (e.g., the $\min j$ rule), i.e. independent of chance.

THEOREM 5.1. *Let a nonsingular matrix be such that its inverse admits a positive last column. Then the simple algorithm works, which is not always the case if the matrix has a positive first row.*

Proof. The above numerical example illustrates the second statement. Assume that the last column of A^{-1}, of components α_j $(1 \leq j \leq n)$, is positive and that, for a given k $(0 \leq k \leq n-2)$, the columns of $A = A_0$ have already been permuted (matrix A_k is obtained) in such a way that the matrix $\overline{A}_k$ made of the first k rows and columns of A_k is of the WHS type. Consider the n vectors $\overline{a}_j \in R^{k+1}$ made by the columns of A_k, truncated to their first $(k+1)$ components. The vector equality $\sum_{j=1}^{n} \alpha_j \overline{a}_j = 0$ holds, component by component. Therefore:

$$\begin{aligned}\sum_{j=k+1}^{n} \alpha_j \det(\overline{a}_1, ..., \overline{a}_k, \overline{a}_j) &= \det(\overline{a}_1, ..., \overline{a}_k, \sum_{j=k+1}^{n} \alpha_j \overline{a}_j) \\ &= -\det(\overline{a}_1, ..., \overline{a}_k, \sum_{j=1}^{k} \alpha_j \overline{a}_j),\end{aligned}$$

hence

$$\sum_{j=k+1}^{n} \alpha_j \det(\overline{a}_1, ..., \overline{a}_k, \overline{a}_j) = 0. \tag{5.1}$$

As the α_js are positive and not all determinants $\det(\overline{a}_1, ..., \overline{a}_k, \overline{a}_j)$ are zero (otherwise, the first $k+1$ rows of A_k, and therefore of A, would be linearly dependent), at least one $\det(\overline{a}_1, ..., \overline{a}_k, \overline{a}_j)$ is positive for $k+1 \leq j \leq n$. By moving any column j of this type to the $(k+1)$-th position, it turns out that the matrix A_{k+1} thus obtained has positive principal minors up to order $k+1$ (permute accordingly the components α_j and renumber them). By repeating the argument from $k = 0$ to $k = n-2$, the columns of A can be reordered in such a way that all the leading minors of A_{n-1} up to order $n-1$ are positive. The argument for the last step relies on equality $\alpha_n = \det A_{n-1} / \det \overline{A}_{n-1}$: since α_n is positive by hypothesis and $\det \overline{A}_{n-1}$ by construction, so is $\det A_{n-1}$. Therefore, the simple algorithm transforms the initial matrix into the WHS matrix A_{n-1}. ∎

This proof shows that the last step is treated separately, and suggests that the signs of the intermediate principal minors can be chosen arbitrarily. To state a simple result, we avoid the complications, due to the presence of zeroes, studied in Theorem 4.3.

THEOREM 5.2. *Let a simple profile of signs be a sequence of n signs + or -, the last two signs being identical. We consider the family of nonsingular matrices with*

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no zeroes in the last column of the inverse matrix. For a matrix A of this type, the following two properties are equivalent:

- the last column of A^{-1} is positive,

- the simple algorithm succeeds in reordering the columns in such a way that the sequence of the leading minors of A follows any predetermined simple profile of signs.

Proof. Let the last column of A^{-1} be positive, and assume that the first k columns $(0 \leq k \leq n-2)$ have been reordered in such a way that the signs of the first k principal minors follow the beginning of the predetermined sequence of signs. Equality (5.1) with all α_js positive shows that at least two of the determinants have opposite signs, therefore it is possible to follow the profile one step further. At the last step $(k = n-1)$ however, there is no room for reordering and $\det A$ has the sign of $\det \overline{A}_{n-1}$.

Conversely, assume that the last column of A^{-1} has at least one negative component, that can be moved to the last position α_n. Once this is done, for the sequence of profiles corresponding to that of $(\det \overline{A}_1, \det \overline{A}_2, ..., \det \overline{A}_{n-1})$, the simple algorithm completed by the $\min j$ rule does not permute the columns of A, but the sequence cannot be completed as a simple profile since $\det A / \det \overline{A}_{n-1} = \alpha_n < 0$. □

6. WHS after general reorderings. We now allow for reorderings of both rows and columns of the initial matrix. A matrix which transforms some positive vector (notation: $x > 0$) into a positive vector is usually called a Stiemke matrix [16].

Theorem 6.1. *Let A be a nonsingular matrix such that*

$$\exists x > 0 \quad Ax > 0 \quad \textit{or} \quad \widetilde{x}A > 0. \tag{6.1}$$

After suitable reorderings of rows and columns, A is transformed into a matrix with positive leading principal minors.

Proof. The proof is by induction on the dimension n of the matrix. The result being obvious for $n = 1$, we assume it for any k $(k \leq n-1)$ and extend it to $n = \dim(A)$. We retain the hypothesis that A transforms some positive column-vector into a positive column-vector (otherwise, transpose A): $Ax = y > 0$. If A admits a semipositive inverse, the conclusion holds by Fujimoto and Ranade's result. If not, there exists a positive vector y' such that the vector $x' = A^{-1}y'$ is not positive and not proportional to x. Therefore, some convex combination v of x and x' is semipositive but not positive, and its image by A is positive. Let us reorder the components of v and, accordingly, the columns of A (vector w and matrix B are obtained), in order that the first k components of w are positive (they represent a vector $\overline{w} \in R^k_{++}$), while the last $n-k$ $(n-k \geq 1)$ are zero. By construction, Bw is positive. Note also that the first k columns of B, being extracted from A, have maximal rank.

Let M be the matrix obtained by replacing the first column of B by the vector Bw, i.e., by a positive combination of the first k columns of B. According to the second statement of Theorem 4.3 applied to $\widetilde{M}$, the rows of M can be reordered in such a way that its leading minors become positive: M is transformed into N, and the same reordering of the rows of B gives the matrix C; N and C coincide, except that the first column of N is a positive combination of the first k columns of C. In particular, since all the leading minors of N of order greater than or equal to k are

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positive and since, up to a positive factor, these minors are those of C, it turns out that all the leading minors of C of order greater than or equal to k are positive.

Vector Cw, being obtained by reordering the components of Bw, is positive. In particular, the sub-matrix $\overline{C}$ made of the first k rows and columns of C is such that $\overline{C}\overline{w} > 0$ (because the components $k+1$ to n of w are zero, so that vector $\overline{C}\overline{w}$ coincides with the first k components of Cw) and $\det \overline{C} > 0$ (because it is the leading minor of order k of C). By the induction hypothesis applied to $\overline{C}$, there exists a reordering of the first k rows and columns of C (call this new matrix D) such that the leading minors of D of order smaller than or equal to k are positive. These last reorderings alter the sign of the leading minors of order greater than or equal to k by a common factor ± 1, which however is $+1$ since the sign of the leading minor of order k is positive in both C and D.

The conclusion is that all leading minors of D, be they of order smaller, equal or greater than k, are positive, where D is deduced from A by reorderings of rows and columns. □

The matrix

$$S = \begin{bmatrix} -1 & -2 & 2 \\ -2 & -3 & 2 \\ 2 & -3 & 2 \end{bmatrix}$$

is a Stiemke matrix ($Sx > 0$ for $\widetilde{x} = (1, 1, 3)$) with a positive determinant. In an attempt to obtain a WHS matrix after a reordering of columns only, the third column must be put in the first position (to have a positive 1×1 leading minor), followed by a permutation of the other two columns (to preserve the sign of the determinant), but the leading minor of order 2 is then negative. A similar experiment on the rows shows that reorderings of both rows and columns are required in the statement of Theorem 6.1.

As a simple application of Theorem 6.1 to economics, consider a square linear model of production: given n goods, a multiple-product method is described by an input vector and an output vector, both semipositive column-vectors of dimension n, where the input vector represents investment (labor can be ignored for the present purpose) and the output vector the corresponding gross product. A square joint production system is obtained by stacking n input vectors (representing n methods) as columns of an input matrix A and n output vectors as columns of an output matrix B (Leontief [9], Sraffa [15]). For activity levels of the various methods represented by a nonnegative vector x, it is assumed (linear model hypothesis) that the overall input vector in the economy is Ax, while the overall product is Bx. When applied to matrix $B - A$, the Stiemke hypothesis means that the economy is productive, i.e. there exist activity levels such that the overall physical net product $Bx - Ax$ is positive. Theorem 6.1 then asserts that it is possible to reorder the methods (i.e., the columns) and the commodities (i.e., the rows), in such a way that the net product matrix $B - A$ has the WHS property. This is a generalization of the result obtained in the traditional case of single-product systems, when method i only produces one unit of commodity i: then B is the identity matrix, and the productivity hypothesis combined with the

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'Ostrowski-Hawkins-Simon theorem' (see comments on this reference below) implies that $B - A$ has the WHS property. The economic interpretation of the dual hypothesis $\widetilde{x}(B - A) > 0$ is that all methods are profitable at the prices represented by the row-vector $\widetilde{x}$.

7. Historical note. Maurice Potron (1872-1942) is a French mathematician and a Jesuit, whose economic works have been recently rediscovered and published [14]. As soon as 1911, with no connection with the economists of his time, Potron built an economic model which anticipates input-output analysis. He considered a single-product system and applied the recently published Perron-Frobenius theorem [3, 4, 11] to prove the existence of (semi-) positive prices and wages (in order to state general results, Potron anticipated Frobenius' later extension [5] to decomposable matrices). In his views, the solution defines the 'just' prices and the 'just' wages referred to by the scholastic doctrine of the Church, as updated by pope Leo XIII in the encyclical *Rerum Novarum* (1891). The semipositivity property of prices and wages follows from the fact that matrix $(I - A)^{-1}$ (the 'Leontief inverse', in modern parlance) is semipositive when the scalar 1 is greater than the dominant eigenvalue of the semipositive input-output matrix A.

How can it be checked that a given scalar ρ is greater than the dominant eigenvalue α of A, when α is not precisely known? A necessary condition is that $\det(\rho I - A)$ is positive (because the sign of $\det(xI - A)$ does not change for $x > \alpha$ and is positive for x large enough), more generally all principal minors of $\rho I - A$ are positive (because the dominant root of A is at least equal to that of the extracted principal matrices). In 1913, Potron [12] stated the converse statement that, today, most mathematicians (e.g. [1], 1994, chapter 6) attribute to Ostrowski ([10], 1937), and most economists to Hawkins and Simon ([8], 1949). Potron's proof is based on the observation that the first derivative of $\det(xI - A)$ is equal to the sum of the principal minors of order $n - 1$; more generally, its k-th derivative is a positive combination of the principal minors of order $n - k$. Therefore, if all principal minors are nonnegative for $x = \rho$, a Taylor expansion of $\det(xI - A)$ at point ρ shows that the characteristic polynomial is positive for $x > \rho$, hence $\rho \geq \alpha$.

Potron was not totally satisfied with this result, because he had in mind numerical applications of his model: the number of principal minors is 2^n, and Potron considered $n = 10^6$ as a realistic magnitude for the number of goods and services. In March and April 1937, he gave a series of six lectures at the Catholic Institute of Paris [13]. In his fifth lecture, Potron showed that it suffices to check the positivity of the leading principal minors of $\rho I - A$. This is the criterion we have referred to as the WHS criterion. Potron's argument is that, if the leading minors are positive, the matrices L and U in the LU factorization of $\rho I - A$ have positive diagonal elements and nonpositive off-diagonal elements (hint: just proceed to the calculation of the entries of L and U). Then U^{-1} is semipositive (hint: solve the system $Ux = y$ for $y > 0$), as well as L^{-1} (same argument), and so is the product $U^{-1}L^{-1} = (\rho I - A)^{-1}$, therefore ρ is greater than α. This WHS criterion reduces the number of positivity conditions from 2^n to n. In spite of this significant improvement, the value $n = 10^6$ remains beyond any scope, hence Potron's final appraisal: "By making us touch the theoretical difficulties

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of the problem, the mathematical science gives us a new reason to repeat to our Father in Heaven the traditional prayer: Give us today our daily bread". Potron's priority in the statement of both 'Ostrowski-Hawkins-Simon' criteria and the elegance of his proofs should be acknowledged, and many other features of his economic model are very innovative as well (Bidard *et al.*, [2]).

8. Summary. We have first given three characterizations of WHS matrices, one in terms of the LU factorization, the second in terms of the solutions to a linear system, the third in terms of a linear complementarity problem. Fujimoto and Ranade's sufficient condition for ensuring that a matrix meets the WHS criterion after a suitable reordering of columns considers the signs of the n^2 entries of the inverse matrix, whereas the WHS condition itself requires to check n signs only. In this respect, the sufficient criteria we propose (positivity of the first row of A or the last column of the inverse matrix) are parsimonious. An important idea is that, once it has been shown that the matrices belonging to some family are of the WHS type, the result holds for the extended family obtained by considering a group of transforms. If the last column of the inverse matrix is positive, a simple algorithm determines a convenient permutation of columns. If permutations of the rows and columns are both allowed, a nonsingular Stiemke matrix can be transformed into a matrix of the WHS type.

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