Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 102-109, March 2008

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ON A MEAN VALUE THEOREM IN THE CLASS OF HERGLOTZ FUNCTIONS AND ITS APPLICATIONS*

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Abstract. Linear-fractional transformations of the pairs with J-property are considered. Extremal functions from an important subclass obtained in this way are expressed as mean values of extremal functions from another subclass of these linear-fractional transformations. Applications to some spectral and interpolation problems are discussed.

Key words. Linear-fractional transformation, J-property, Plus-condition, Mean value, Herglotz function, Extremal function, Spectral function.

AMS subject classifications. 15A99, 26E60, 30E05.

1. Introduction. We shall consider families of the linear-fractional transformations in the open upper half-plane $\mathbb{C}_+$:

$$\varphi(\lambda) = i\big(a(\lambda)R(\lambda) + b(\lambda)Q(\lambda)\big)\big(c(\lambda)R(\lambda) + d(\lambda)Q(\lambda)\big)^{-1}, \tag{1.1}$$

where a, b, c, d are fixed $n \times n$ matrix functions and each linear fractional transformation is induced by a pair of $n \times n$ matrix functions $\big(R(\lambda), Q(\lambda)\big)$. Such transformations are widely used in the interpolation and spectral theories (see, for instance, [2, 6, 10, 12, 13] and various references therein). Introduce the analytic matrix function $\mathfrak{A}$ and the matrix $J = J^* = J^{-1}$:

$$\mathfrak{A} := \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad J := \begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix}, \tag{1.2}$$

where I_n is the $n \times n$ identity matrix. As usual, we assume that $\mathfrak{A}$ maps J-nonnegative vectors $g \neq 0$ into J-positive vectors, that is, the relations

$$g \in \mathbb{C}^{2n}, \quad g \neq 0, \quad g^*Jg \geq 0 \tag{1.3}$$

imply

$$h^*Jh > 0 \quad \text{for} \quad h = \mathfrak{A}(\lambda)g, \quad \lambda \in \mathbb{C}_+. \tag{1.4}$$

*Received by the editors January 23, 2008. Accepted for publication February 26, 2008. Handling Editor: Harm Bart.

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Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 102-109, March 2008

Here $\mathbb{C}$ is the complex plane and $\mathbb{C}_+$ ($\mathbb{C}_-$) is the upper (lower) complex half-plane.

DEFINITION 1.1. When (1.3) implies (1.4), we say that $\mathfrak{A}$ satisfies the plus-condition. It is required that the pairs of meromorphic matrix functions (R, Q) are nonsingular and have J-property, that is,

$$(1.5) \qquad R(\lambda)^*R(\lambda)+Q(\lambda)^*Q(\lambda)>0, \quad \begin{bmatrix} R(\lambda)^* & Q(\lambda)^* \end{bmatrix} J \begin{bmatrix} R(\lambda) \\ Q(\lambda) \end{bmatrix} \geq 0.$$

We define the class $\mathcal{N}(\mathfrak{A})$ or simply $\mathcal{N}$ to be the class of all functions of the form (1.1), where (R, Q) is any nonsingular pair having the J-property and where $\mathfrak{A}$ is the analytic matrix function of the form (1.2) satisfying the plus-condition.

It is well-known that the matrix functions $v \in \mathcal{N}$ have the property $2\Im v = i(v^*-v) > 0$ for $\Im\lambda > 0$, i.e., they belong to the Herglotz (Nevanlinna) class. An important subset of $\mathcal{N}$ is generated by the pairs

$$(1.6) \qquad R(\lambda)\equiv\theta, \quad Q(\lambda)\equiv I_n, \quad \theta=\theta^*>0.$$

Another important subset of $\mathcal{N}$ is generated by the pairs

$$(1.7) \qquad R(\lambda)\equiv\theta, \quad Q(\lambda)\equiv iqI_n, \quad \theta=\theta^*>0, \quad q\in\mathbb{R},$$

where $\mathbb{R}$ is the real axis. We shall be interested in the case where the pair $\big(R(\lambda),\, Q(\lambda)\big)$ is given by either (1.6) or (1.7) for a given constant matrix θ and some scalar q and where the analytic matrix function $\mathfrak{A}$ is fixed. For the sequel we therefore assume that we are given a fixed constant matrix $\theta=\theta^*>0$ together with a fixed analytic matrix function $\mathfrak{A}$ satisfying the plus-condition. We then define

$$(1.8) \qquad v(\lambda)=i\big(a(\lambda)\theta+b(\lambda)\big)\big(c(\lambda)\theta+d(\lambda)\big)^{-1},$$

$$(1.9) \qquad v(q,\lambda)=i\big(a(\lambda)\theta+iqb(\lambda)\big)\big(c(\lambda)\theta+iqd(\lambda)\big)^{-1},$$

where $q=\overline{q}$ is a scalar. We shall be interested in how $v(\lambda)$ can be recovered from the collection $\big\{v(q,\lambda) : \, q\in\mathbb{R}\big\}$.

Notice, that the Fourier type transformations corresponding to $v(\lambda)$ have certain extremal properties [4, 5, 10]. In particular, the scalar products of the Fourier type transformations in the spaces $L^2(\tau)$, where τ are the weight functions in the Herglotz representations of $\varphi\in\mathcal{N}$, were considered in [10]. The maximal norms of the Fourier type transformations in these spaces are achieved for the weight function generated by v, when $\theta=I_n$ (Theorem 3 [10]). The functions $v(q,\lambda)$ generate orthogonal spectral functions and are also, in some sense, extremal [12]. Thus, it is of interest to study the connection between these extremal functions. We shall show that $v(\lambda)$ is the mean

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value of $v(q,\lambda)$, where q changes from $-\infty$ to ∞. This implies that the extremal properties of $v(\lambda)$ are directly connected with the extremal properties of $v(q,\lambda)$.

We shall understand $\int_{-\infty}^{\infty}$ as the principal value integral, that is, $\lim_{r\to\infty}\int_{-r}^{r}$.

THEOREM 1.2. *Let the matrix function* $\mathfrak{A}$ *of the coefficients of the linear fractional transformation (1.1) satisfy the plus-condition, and suppose that* $\det\big(c\theta - d\big) \neq 0$ *almost everywhere in* $\mathbb{C}_+$. *Then for all* $\lambda \in \mathbb{C}_+$ *the matrix function* v *admits representation*

$$v(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} v(q,\lambda)\frac{dq}{1+q^2}. \tag{1.10}$$

According to (1.9), the matrix functions $v(q,\lambda)$ are generated by the extremal pairs $R(\lambda) = \theta$, $Q = iqI_n$ such that $R^*Q + Q^*R = 0$. It is easy to see that

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{dq}{1+q^2} = 1, \tag{1.11}$$

i.e., for the measure $d\mu = \pi^{-1}(1+q^2)^{-1}dq$ we have $\mu(\mathbb{R}) = 1$.

Theorem 1.2 is closely related to the papers [7, 8]. For the subcase $n = 1$ and $ad - bc = 1$, a description of the functions $\varphi \in \mathcal{N}$ such that

$$\varphi(\lambda) = \int_{-\infty}^{\infty} v(q,\lambda)d\mu(q), \quad \int_{-\infty}^{\infty} d\mu(q) = 1, \tag{1.12}$$

where μ is a non-decreasing matrix function, was given in [7].

2. Proof of theorem.

Proof. **Step 1.** We shall need several inequalities. First, prove by contradiction that

$$\det\big(c(\lambda)R(\lambda) + d(\lambda)Q(\lambda)\big) \neq 0 \quad (\lambda \in \mathbb{C}_+) \tag{2.1}$$

for R and Q satisfying (1.5). In particular, it will imply that

$$\det(c\theta - izd) \neq 0 \quad (z \in \mathbb{C}_+ \cup \mathbb{R}). \tag{2.2}$$

Suppose that (2.1) is not true. That is, assume that for some $\lambda \in \mathbb{C}_+$ and $f \neq 0$ the equality $\big(c(\lambda)R(\lambda) + d(\lambda)Q(\lambda)\big)f = 0$ is valid. It is immediate that

$$\big(\mathfrak{A}(\lambda)g\big)^* J\mathfrak{A}(\lambda)g = 0, \quad g := \begin{bmatrix} R(\lambda) \\ Q(\lambda) \end{bmatrix} f. \tag{2.3}$$

On the other hand, by (1.5) we get

$$g^*Jg \geq 0, \quad g \neq 0. \tag{2.4}$$

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By the plus-condition the inequalities (2.4) yield $\big(\mathfrak{A}(\lambda)g\big)^* J\mathfrak{A}(\lambda)g > 0$, which contradicts (2.3).

By the plus-condition we have also

$$\det\, d(\lambda) \neq 0 \quad (\lambda \in \mathbb{C}_+). \tag{2.5}$$

Indeed, let $df = 0$, $f \neq 0$, and put $g = \mathrm{col}[0 \quad f]$, where col means column. It is immediate, that for this g the equality $g^*Jg = 0$ is valid, and so the inequality (1.4) holds. On the other hand, we have $h^*Jh = f^*(d^*b + b^*d)f = 0$. Thus, (2.5) is proved by contradiction.

Step 2. Let us consider $\lambda \in \mathbb{C}_+$ such that

$$\det(c(\lambda)\theta - d(\lambda)) \neq 0. \tag{2.6}$$

Now, omitting the variable λ in the notations, put

$$u := -id^{-1}c\theta, \quad X_1 := (ibu - a\theta)(I_n + u^2)^{-1}d^{-1},$$

$$X_2 := -iX_1 d, \quad X_3 := (ia\theta - X_2)(idu)^{-1}. \tag{2.7}$$

The matrix functions u and X_k $(k = 1, 2, 3)$ are well-defined. Indeed, the invertibility of d follows from (2.5) and the invertibility of $I_n + u^2$ follows from the inequalities $\det(u \pm iI_n) \neq 0$. Here, the inequality $\det(u + iI_n) \neq 0$ is true by the assumption (2.6). The relation $\det(u - iI_n) \neq 0$ is a particular case of (2.2). In view of (2.2) we get also that $\sigma(u) \subset \mathbb{C}_-$, where σ is spectrum. Thus $qI_n + u$ is invertible for all $q \in \mathbb{C}_- \cup \mathbb{R}$.

Next let us show that

$$\frac{v(q,\lambda)}{1+q^2} = \frac{qX_1(\lambda) + X_3(\lambda)}{1+q^2} + X_2(\lambda)\big(qI_n + u(\lambda)\big)^{-1}\big(id(\lambda)\big)^{-1}. \tag{2.8}$$

Indeed, taking into account (1.9) and the definition of u in (2.7) we obtain

$$v(q,\lambda)\big(id(\lambda)\big)\big(qI_n + u(\lambda)\big) = v(q,\lambda)\big(iqd(\lambda) + c(\lambda)\theta\big) = i\big(a(\lambda)\theta + iqb(\lambda)\big). \tag{2.9}$$

On the other hand, for the right-hand side

$$Z(\lambda) = \frac{qX_1(\lambda) + X_3(\lambda)}{1+q^2} + X_2(\lambda)\big(qI_n + u(\lambda)\big)^{-1}\big(id(\lambda)\big)^{-1} \tag{2.10}$$

of (2.8), the definitions of X_k in (2.7) imply

$$i(1+q^2)Zd\big(qI_n + u\big) = (1+q^2)X_2 + i(qX_1 + X_3)d(qI_n + u) = q^2(X_2 + iX_1 d)$$

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$$+iq(X_1du + X_3d) + X_2 + iX_3du = iq\big(X_1du + (a\theta + X_1d)u^{-1}\big) + ia\theta$$

$$= iq\big(X_1d(I_n + u^2)u^{-1} + a\theta u^{-1}\big) + ia\theta = -qb + ia\theta. \tag{2.11}$$

From (2.9)-(2.11) formula (2.8) is immediate.

As $\sigma(u) \subset \mathbb{C}_-$, it is evident that for the principal value integral we get

$$\frac{1}{i}\int_{-\infty}^{\infty} X_2\big(qI_n + u\big)^{-1} d^{-1}dq$$

$$= \frac{1}{i}X_2\left(\lim_{r\to\infty}\int_0^r \Big((qI_n + u)^{-1} - (qI_n - u)^{-1}\Big)dq\right)d^{-1}$$

$$= \frac{1}{2i}X_2\left(\lim_{r\to\infty}\int_{-r}^r \Big((qI_n + u)^{-1} - (qI_n - u)^{-1}\Big)dq\right)d^{-1} = \pi X_2 d^{-1}. \tag{2.12}$$

It is immediate also that

$$\int_{-\infty}^{\infty} \frac{qdq}{1+q^2}X_1 = 0, \tag{2.13}$$

and

$$\int_{-\infty}^{\infty} \frac{dq}{1+q^2}X_3 = \pi X_3. \tag{2.14}$$

By (2.8) and (2.12)-(2.14) we have

$$\frac{1}{\pi}\int_{-\infty}^{\infty} v(q,\lambda)\frac{dq}{1+q^2} = X_2d^{-1} + X_3. \tag{2.15}$$

From (2.7) it follows that

$$X_2 = (bu + ia\theta)(I_n + u^2)^{-1},$$

$$X_3 = a\theta u^{-1}d^{-1} + i(bu + ia\theta)(I_n + u^2)^{-1}u^{-1}d^{-1}. \tag{2.16}$$

Hence, using (2.16) we derive

$$X_2d^{-1} + X_3 = \big(bu^2 + ia\theta u + a\theta(I_n + u^2) + ibu - a\theta\big)(I_n + u^2)^{-1}u^{-1}d^{-1}$$

$$= (a\theta + b)(u^2 + iu)(I_n + u^2)^{-1}u^{-1}d^{-1} = i(a\theta + b)(d + idu)^{-1}. \tag{2.17}$$

From the definitions of u and v and formula (2.17) we obtain

$$X_2d^{-1} + X_3 = i(a\theta + b)(c\theta + d)^{-1} = v. \tag{2.18}$$

Formulas (2.15) and (2.18) yield (1.10).

By the theorem's conditions, assumption (2.6) is true everywhere in $\mathbb{C}_+$, excluding, perhaps, isolated points. Let λ_0 be such an isolated point. Then, (1.10) is valid in some neighborhood $\mathcal{O}(\lambda_0) \setminus \lambda_0$. According to (2.1) and (2.5), the functions $v(q,\lambda)$ are bounded, uniformly in q and in λ, in some neighborhood $\mathcal{O}(\lambda_0) \subset \mathbb{C}_+$. Now, it is immediate that (1.10) at $\lambda = \lambda_0$ holds as the limit of equalities (1.10), where λ tends to λ_0. Thus, (1.10) holds everywhere in $\mathbb{C}_+$. □

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3. Examples.

3.1. Canonical system. Consider the well-known canonical system

$$dW(x,\lambda)/dx = i\lambda JH(x)W(x,\lambda), \quad 0 \le x \le l < \infty, \quad W(0,\lambda) = I_m, \tag{3.1}$$

where $H(x) \ge 0$ and $m = 2n$. Its Weyl functions are defined [13] by (1.1), where

$$\mathfrak{A}(x,\lambda) := W(x,\overline{\lambda})^*, \quad \mathfrak{A}(\lambda) := \mathfrak{A}(l,\lambda). \tag{3.2}$$

By (3.1) we have

$$\int_0^l \mathfrak{A}(x,\lambda)H(x)\mathfrak{A}(x,\overline{z})^* dx = i\,\frac{\mathfrak{A}(\lambda)J\mathfrak{A}(\overline{z})^* - J}{\lambda - z}. \tag{3.3}$$

The “positivity type” condition:

$$\int_0^l H(x)dx > 0 \tag{3.4}$$

is often assumed [9] to be true. Then, according to [9], p.249 we get

$$\int_0^l \mathfrak{A}(x,\lambda)H(x)\mathfrak{A}(x,\lambda)^* dx = \int_0^l W(x,\overline{\lambda})^* H(x)W(x,\overline{\lambda})dx > 0. \tag{3.5}$$

From (3.3) and (3.5) it follows that $\mathfrak{A}(\lambda)^* J\mathfrak{A}(\lambda) > J$ for $\lambda \in \mathbb{C}_+$, and therefore the plus-condition is satisfied.

According to (3.1) we get $\mathfrak{A}(x,0) = I_m$, and so, by partitioning (1.2), we have $\det\big(c(0)\theta - d(0)\big) = \det(-I_n) \ne 0$. Thus, the inequality $\det\big(c(\lambda)\theta - d(\lambda)\big) \ne 0$ holds almost everywhere. So, under the “positivity type” condition (3.4) the conditions of Theorem 1.2 are fulfilled, and Weyl functions of the canonical systems satisfy (1.10).

Canonical systems include Dirac type systems, matrix Schrödinger equations, and matrix string equations [11, 13]. For the scalar string equation a non-orthogonal spectral function is represented in [3] as the convex continuous linear combination (with respect to the measure $d\mu = \pi^{-1}(1+q^2)^{-1}dq$ from (1.10)) of the orthogonal spectral functions.

3.2. Nevanlinna matrices. The 2×2 matrix function

$$N(\lambda) = \begin{bmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{bmatrix}$$

is called Nevanlinna matrix [1] if its entries are entire transcendental functions such that $AD - BC \equiv 1$ and for each $q = \overline{q}$ the function

$$\phi(q,\lambda) := -\frac{A(\lambda)q - B(\lambda)}{C(\lambda)q - D(\lambda)} \tag{3.6}$$

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satisfies the inequalities

$$\Im\phi(q,\lambda)/\Im\lambda > 0 \quad (\Im\lambda \neq 0). \tag{3.7}$$

For the Nevanlinna matrices corresponding to the moment problem treated in [1], the functions ϕ admit representations

$$\phi(q,\lambda) = \int_{-\infty}^{\infty} \frac{d\sigma(q,x)}{x-\lambda}, \tag{3.8}$$

where $\sigma(q,x)$ are bounded functions, which do not decrease with respect to x. The functions σ constructed in this way are called N-extremal in [1]. There is a simple connection between the functions $\phi(q,\lambda)$ and the functions $v(q,\lambda)$, which are considered in Theorem 1.2. Namely, putting

$$\mathfrak{A}(\lambda) = \begin{bmatrix} i & 0 \\ 0 & 1 \end{bmatrix} N(\lambda) \begin{bmatrix} -i & 0 \\ 0 & 1 \end{bmatrix} J = \begin{bmatrix} iB(\lambda) & A(\lambda) \\ D(\lambda) & -iC(\lambda) \end{bmatrix} \tag{3.9}$$

and putting $\theta = 1$ in (1.9), we get $\phi(q,\lambda) = v(-q,\lambda)$.

To show that $\mathfrak{A}$ satisfies the plus-condition, consider functions $\phi(\tau,\lambda)$ $(\Im\tau \geq 0)$ given by the formula (3.6) after we substitute τ instead of q. When $\lambda \in \mathbb{C}_+$ and $\Im\tau \geq 0$ or $\tau = \infty$, one can show that the inequality (3.7) implies $C(\lambda)\tau - D(\lambda) \neq 0$, $C(\lambda) \neq 0$ and $\Im\phi(\tau,\lambda) > 0$. In other words, we have

$$[\overline{\tau} \quad -1]N(\lambda)^*\widetilde{J}N(\lambda)\begin{bmatrix} \tau \\ -1 \end{bmatrix} = i|C(\lambda)\tau - D(\lambda)|^2(\overline{\phi}(\tau,\lambda) - \phi(\tau,\lambda)) > 0, \tag{3.10}$$

$$[1 \quad 0]N(\lambda)^*\widetilde{J}N(\lambda)\begin{bmatrix} 1 \\ 0 \end{bmatrix} > 0, \quad \widetilde{J} := \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}. \tag{3.11}$$

By (3.10) and (3.11) the inequalities $g^*\widetilde{J}g \geq 0$, $g \neq 0$ imply $g^*N(\lambda)^*\widetilde{J}N(\lambda)g > 0$ for $\lambda \in \mathbb{C}_+$. So, in view of (3.9) it follows that $\mathfrak{A}$ satisfies the plus-condition, and we can apply Theorem 1.2. Therefore, the mean of the extremal functions $\phi(q,\lambda)$ equals $v(\lambda)$.

Acknowledgment. L.A. Sakhnovich is grateful to Leipzig University for electing him Leibniz Professor and for their warm hospitality.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 102-109, March 2008

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